Calculus II
Spring 2004
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Midterm 2 (take home)

Instructions

- The test is due in class on **Tuesday, April 20**.
- Write the answers and all relevant work (**neatly!**) on separate sheets of paper, as you do with homework. Please use enough space on the page to make it **readable**—if you have to use more paper, Greenpeace will forgive you, just this once.
- Show all relevant work, and **circle** (or box or highlight, etc.) **your final answers**. I will take off a point for every answer that I have to make an effort to find. I will not give credit to any answers that appear without supporting work.
- You **may** use your textbook or any other materials available to you.
- You **may** use a calculator to check your answers, but you must also show me that you know how to get the answers without one.
- You **may not** discuss your work with others in the class until after you hand it in.

Have fun!

1. [6 points each] Compute each integral (or if it is improper and divergent, say so):

   (a) $\displaystyle\int_1^\infty \frac{\ln x}{x^3}\,dx$ (Hint: you'll need more than one trick here. Try starting with a substitution.)

   (b) $\displaystyle\int_{-2}^3 \frac{dx}{x^3}$

   (c) $\displaystyle\int_4^5 \frac{x}{\sqrt{x^2-16}}\,dx$

2. [6 points] Compute $\int e^{4x}\cos x\,dx$. Hint: I'm aware of two possible ways to do this—one of them uses the fact that $e^{4x}\cos x$ is the real part of a rather simple complex-valued function.

3. [5 points each] For each sequence, find the limit, or state if the sequence diverges. (**Achtung!** These are *sequences*, not series.)

   (a) $\displaystyle\left(1+\frac{1}{n}\right)^{n/2}$ (b) $\displaystyle\frac{n^2}{2^n}$

4. [5 points each] Compute the sum of each series, or state if it diverges.

   (a) $\displaystyle\sum_{n=0}^\infty \frac{(-5)^n}{4^{n+1}}$ (b) $\displaystyle\sum_{n=1}^\infty \frac{1}{n^2+n}$ (c) $\displaystyle\sum_{n=1}^\infty \frac{1-2^n}{3^{2n}}$

5. [5 points each] State whether each series converges absolutely, converges conditionally or diverges. Justify.

   (a) $\displaystyle\sum_{k=2}^\infty \frac{2}{k(\ln k)^2}$ (b) $\displaystyle\sum_{n=1}^\infty \frac{(-1)^n}{\sqrt{n(n+1)}}$ (c) $\displaystyle\sum_{n=1}^\infty \frac{(-1)^n(n-1)}{n}$

6. This problem demonstrates a clever way to compute the series $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{2^k k}$.

   (a) [5 points] Define a function $f(x)$ as the power series

   $$f(x) = \sum_{k=1}^{\infty} \frac{1}{2^k k} x^k.$$

   What is the interval of convergence for this series, i.e. for what range of x does the series converge? For what x is the convergence *absolute*, and when is it *conditional*?

   (b) [5 points] Write down $f'(x)$ as a power series. What is the interval of convergence for *this* series? For what x is the convergence absolute?

   (c) [5 points] If you did part (b) correctly, you might notice that the series for $f'(x)$ is geometric. Use this fact to write down an explicit formula for $f'(x)$. (In this context, "explicit" means "not involving any infinite series".)

   (d) [5 points] Integrate the expression from part (c) to obtain an explicit formula for $f(x)$. Be careful with the constant (it's not arbitrary!).

   (e) [4 points] Knowing what you now know, compute $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{2^k k}$.

7. What do you really *know* about the number e?

   From a mathematical perspective, one only needs to know the basic ideas of differential calculus in order to prove that there is a number (call it a for the moment) such that the function $f(x) = a^x$ is its own derivative—and indeed, there is only one. We define e to be that number. One can do this without having any idea that the number in question is approximately 2.718. Knowing a little more calculus, one can then apply Taylor's formula and derive the expansion $e^x = 1 + x + x^2/2! + \ldots$, which produces a formula for e in the form

   $$e = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

   This can be used to approximate e, but it offers little obvious insight beyond that.

   The goal of this problem is to prove that e is an *irrational* number. Recall that a number is called *rational* if it can be expressed as a fraction p/q, where both p and q are integers; otherwise, it's called irrational. So we want to prove that any attempt to write e precisely as a fraction is doomed.

   In order to do this, we look at the partial sums of the series above:

   $$S_q = \sum_{n=0}^{q} \frac{1}{n!}.$$

   Since the series is known to converge, we know that the sequence of numbers S_q gets closer to e as $q \to \infty$; put another way, we know that the positive numbers $e - S_q$ shrink toward 0 as q gets larger. We can make this more precise in the following way.

   (a) [6 points] Let's write $e - S_q$ as an infinite series in the form $e - S_q = \sum_{n=q+1}^{\infty} \frac{1}{n!}$. Use this to prove that $q!(e - S_q) < 1/q$ for all positive integers q. Hint: it may help you to remove the sigma-notation and write out the first several terms of the series above, each term multiplied by $q!$. Then think about how you might compare this to some geometric series.

   (b) [6 points] We now prove that if $e = p/q$, with p and q both positive integers, then something impossible is true. Show first that $q!S_q$ is always a positive integer. Conclude from this that if $e = p/q$, then $q!(e - S_q)$ is also a positive integer. Why does this contradict the result of part (a)?

   Conclusion: there is no pair of integers p and q such that $p/q = e$. Irrational numbers are here to stay!