Notes on Convergence of Power Series

Chris Wendl

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1 Introduction

The question is this: given an infinite series of the form $\sum_{n=0}^{\infty} a_n(x-x_0)^n$, for what values of x does it converge? This is an important thing to know, as it tells us, for instance, when we can expect a Taylor expansion like $\ln(1+x) = x - x^2/2 + x^3/3 - x^4/4 + \dots$ to provide information that is either useful or relevant or correct: if the series *doesn't* converge, then the expansion, however nice it looks, is nonsense. The general pattern is that we can examine the coefficients a_n to determine a certain *interval of convergence*, that is, a range of values that x must fall into in order for the series to converge. This problem can be attacked in a number of different ways, and the methods used in your textbook are different from those that have been introduced in class. The purpose of these notes is to describe both methods and clarify the differences and similarities between them. We'll start with the method discussed in class, and then move on the approach taken by your textbook, using the so-called "ratio test".

2 A Theorem for Power Series

The following theorem is quite simple to use once you get used to it, but unfortunately it does not appear in your textbook:

Theorem 1. *Given a power series $\sum_{n=0}^{\infty} a_n x^n$, suppose $|a_{n+1}/a_n|$ converges to some number L as $n \to \infty$ (L could also be ∞). Then define $R = 1/L$ (or if $L = 0$ set $R = \infty$, and if $L = \infty$ set $R = 0$). This number R is called the* radius of convergence *of the power series, and it has the following properties:*

1. *The series converges absolutely for all x with $|x| < R$.*
2. *The series diverges for all x with $|x| > R$.*

If $R = \infty$, this means the series converges for all x.

Note that the theorem makes no conclusion about what happens when $|x| = R$: the cases $x = R$ and $x = -R$ must be investigated separately. Also, those who've already read and absorbed the book's method to be described below, pay close attention: the ratios $|a_{n+1}/a_n|$ being examined here are *not* ratios of the actual terms in the series, *they involve only the coefficients*! (A ratio of actual terms would look like $|(a_{n+1}x^{n+1})/(a_n x^n)|$ rather than $|a_{n+1}/a_n|$.) Before presenting the proof, let's look at an example.

Example 1. *Consider the power series $\frac{x}{2} - \frac{x^2}{2\cdot 2^2} + \frac{x^3}{3\cdot 2^3} - \frac{x^4}{4\cdot 2^4} + \dots$. This can be written as $\sum_{n=0}^{\infty} a_n x^n$ where $a_n = \frac{(-1)^{n-1}}{n2^n}$, thus*

$$\frac{|a_{n+1}|}{|a_n|} = \frac{1/[(n+1)2^{n+1}]}{1/(n2^n)} = \frac{n2^n}{(n+1)2^{n+1}} = \frac{1}{2}\frac{n}{n+1} \to \frac{1}{2} \quad \text{as} \quad n \to \infty.$$

Thus we set $L = 1/2$ and $R = 1/L = 2$, to conclude that the series converges for $|x| < 2$ and diverges for $|x| > 2$. We must now investigate the cases $x = \pm 2$ separately: when $x = 2$, the series becomes $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$; this is the alternating harmonic series, which we know is conditionally convergent. At $x = -2$, we have $-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \dots$, which diverges (it's minus the harmonic series). So we conclude that the interval of convergence is $-2 < x \le 2$.

It goes without saying that a series of the form $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ can always be handled in the same way: we examine the coefficients to derive a radius of convergence R and conclude that the series converges for $|x-x_0| < R$ and diverges for $|x-x_0| > R$. Again the two points where $|x-x_0| = R$ must be checked separately. So for instance, $\frac{x+2}{2} - \frac{(x+2)^2}{2\cdot 2^2} + \frac{(x+2)^3}{3\cdot 2^3} - \frac{(x+2)^4}{4\cdot 2^4} + \ldots$ converges for $-2 < x+2 \leq 2$, in other words $-4 < x \leq 0$.

The theorem is proven by comparing the power series for a given value of x with a certain geometric series, whose convergence or divergence we can easily check. The essential point is quite simple but may seem obscured in the details, so you might want to skip the proof on a first reading.

Proof of Theorem 1. Assume $|a_{n+1}/a_n|$ converges to a finite number $L \geq 0$, and set $R = 1/L$. (We'll ignore for the moment the case where $L = \infty$; it requires a slightly different argument and is after all not very interesting, as it yields a power series that only converges at $x = 0$.) We'll first prove that the power series is absolutely convergent if $|x| < R$. First, we already know it converges if $x = 0$, so pick any $x \neq 0$ with $|x| < R$. The fact that $|x| < R = 1/L$ and thus $1/|x| > L$ allows us to choose a small positive number ϵ such that $0 < \epsilon < 1/|x| - L$. (If $L = 0$ we can do this for *any* $x \neq 0$, so we define $R = \infty$ to mean that there's no constraint on the size of x.) Since $|a_{n+1}/a_n| \to L$, we can find some large integer N so that

$$L - \epsilon < \frac{|a_{n+1}|}{|a_n|} < L + \epsilon \tag{1}$$

whenever $n \geq N$. In particular, $|a_{n+1}| < (L+\epsilon)|a_n|$ for all $n \geq N$. Now let p be some positive integer, and observe that since every integer from N to $N+p$ is greater than or equal to N, we can apply the inequality $|a_{n+1}| < (L+\epsilon)|a_n|$ repeatedly and obtain:

$$|a_{N+p}| < (L+\epsilon)|a_{N+p-1}| < (L+\epsilon)^2|a_{N+p-2}| < (L+\epsilon)^3|a_{N+p-3}| < \ldots < (L+\epsilon)^p|a_N|.$$

Setting $n = N + p$, we rewrite this as

$$|a_n| < (L+\epsilon)^{n-N}|a_N| = \frac{|a_N|}{(L+\epsilon)^N}(L+\epsilon)^n. \tag{2}$$

We want to prove that $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely, but of course it would suffice to prove that $\sum_{n=N}^{\infty} a_n x^n$ converges absolutely, which by definition would mean that $\sum_{n=N}^{\infty} |a_n x^n| = \sum_{n=N}^{\infty} |a_n||x|^n < \infty$. Since we're now summing only terms with $n \geq N$, we can use Inequality 2 for comparison:

$$\sum_{n=N}^{\infty} |a_n||x|^n < \sum_{n=N}^{\infty} \frac{|a_N|}{(L+\epsilon)^N}(L+\epsilon)^n|x|^n = \frac{|a_N|}{(L+\epsilon)^N} \sum_{n=N}^{\infty} \left[(L+\epsilon)|x|\right]^n.$$

The sum on the right is simply a geometric series. Does it converge? The answer is yes, due to the very special condition we placed on ϵ at the beginning: we required that $0 < \epsilon < 1/|x| - L$, and with a little bit of algebra this translates into $(L+\epsilon)|x| < 1$. Thus the geometric series converges, and by comparison, so does the power series (absolutely).

Much of this same reasoning can be recycled and adapted to prove that the series will diverge whenever $|x| > R = 1/L$. In this case we can choose a number ϵ such that $0 < \epsilon < L - 1/|x|$. Again there is some large integer N such that Inequality 1 holds whenever $n \geq N$, and in particular, $|a_{n+1}| > (L-\epsilon)|a_n|$. We can again choose a positive integer p and apply this inequality repeatedly to obtain

$$|a_{N+p}| > (L-\epsilon)|a_{N+p-1}| > (L-\epsilon)^2|a_{N+p-2}| > (L-\epsilon)^3|a_{N+p-3}| > \ldots > (L-\epsilon)^p|a_N|,$$

or setting $n = N + p$,

$$|a_n| > (L-\epsilon)^{n-N}|a_N| = \frac{|a_N|}{(L-\epsilon)^N}(L-\epsilon)^n. \tag{3}$$

Remember that in order for $\sum_{n=0}^{\infty} a_n x^n$ to converge, it is necessary that the sequence of terms $a_n x^n$ must approach 0. We can now prove that this won't happen: when $n \geq N$ Inequality 3 tells us

$$|a_n x^n| = |a_n||x|^n > \frac{|a_N|}{(L-\epsilon)^N}(L-\epsilon)^n|x|^n = \frac{|a_N|}{(L-\epsilon)^N}\left[(L-\epsilon)|x|\right]^n.$$

Since $\epsilon < L - 1/|x|$, it turns out that $(L-\epsilon)|x| > 1$, so the term on the right diverges to ∞ as $n \to \infty$. Therefore $a_n x^n$ cannot possibly converge to 0, and the power series cannot converge. $\square$

This theorem can be applied to most, but not all power series. Its greatest success story is the exponential series, $e^x = \sum_{n=0}^{\infty} x^n/n!$. Here the coefficients are $a_n = 1/n!$, thus

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1/(n+1)!}{1/n!} = \frac{n!}{(n+1)!} = \frac{1}{n+1} \to 0,$$

so we set $L = 0$ and conclude that the radius of convergence $R = 1/L = \infty$, or in more pedestrian terms, the series converges absolutely for all x.

Here's an example of a series for which the theorem is not applicable:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

Notice that all the odd powers of x in this series are missing, so in order to express it in the usual form as $\sum_{n=0}^{\infty} a_n x^n$, we'd have to define the sequence of coefficients $\{a_n\}_{n=0}^{\infty}$ as $\{1, 0, -1/2!, 0, 1/4!, 0, -1/6!, 0, \dots\}$. Now the ratios $|a_{n+1}/a_n|$ don't converge to anything at all, in fact every other term is undefined (has a 0 in the denominator). But all is not lost: all we have to do is compare this series with the result we already obtained for the exponential series. Namely, the exponential series converges absolutely, so

$$1 + |x| + \frac{|x|^2}{2!} + \frac{|x|^3}{3!} + \frac{|x|^4}{4!} + \frac{|x|^5}{5!} + \frac{|x|^6}{6!} + \dots < \infty$$

for all x. Obviously then,

$$1 + \frac{|x|^2}{2!} + \frac{|x|^4}{4!} + \frac{|x|^6}{6!} + \dots < \infty,$$

since every term in this series is also included in the one above. Therefore the series for $\cos x$ converges absolutely.

3 The Ratio Test

I'm tempted to recommend that you not read the rest of this—the ratio test can be a dangerous weapon in the hands of an inexperienced student. It seems so simple to use that there's always a temptation to use it too much, often in situations where it really isn't appropriate. Nevertheless, you may have already noticed that your textbook employs the ratio test quite frequently, particularly in dealing with power series. So it's only fair to include some discussion of it here, if only for the sake of completeness.

In some sense, Theorem 1 above is an application of the ratio test, which can be used to test the convergence of a wide variety of infinite series, not just power series. Here's how it works:

Theorem 2 (The Ratio Test). *Consider an infinite series $\sum_{n=1}^{\infty} b_n$, and suppose $|b_{n+1}/b_n|$ converges to some number L as $n \to \infty$. Then,*

1. *If $L < 1$, the series $\sum_{n=1}^{\infty} b_n$ converges absolutely.*
2. *If $L > 1$ (or $L = \infty$), the series $\sum_{n=1}^{\infty} b_n$ diverges.*

As will surely not surprise you by now, no conclusion results if $L = 1$. There are obvious similarities between this and Theorem 1, and they are not coincidental. However, we should take note of some differences: most importantly, where Theorem 1 dealt with ratios of the coefficients in a power series, Theorem 2 uses ratios of the actual terms. It wouldn't make sense for Theorem 2 to talk about coefficients, since it doesn't even make any explicit mention of a power series: $\sum_{n=1}^{\infty} b_n$ could be any infinite series at all.

Example 2. *$\sum_{n=0}^{\infty} \frac{1}{n!}$ is not a power series, but we can use the ratio test to prove that it converges. The series can be expressed as $\sum_{n=0}^{\infty} b_n$ if $b_n = 1/n!$, then*

$$\left|\frac{b_{n+1}}{b_n}\right| = \frac{1/(n+1)!}{1/n!} = \frac{n!}{(n+1)!} = \frac{1}{n+1} \to 0,$$

so since $0 < 1$, the ratio test tells us that this series converges.

The ratio test can be used to test the convergence of all the same power series for which Theorem 1 is applicable, though since we're now using a principle that isn't explicitly designed for power series, the method is a little bit less direct. Let's repeat some previous examples in this new context to see how it works.

Example 3. *We want to find out for what values of x the series $\frac{x}{2} - \frac{x^2}{2\cdot 2^2} + \frac{x^3}{3\cdot 2^3} - \frac{x^4}{4\cdot 2^4} + \dots$ converges. We write this as $\sum_{n=0}^{\infty} b_n$ where $b_n = \frac{(-1)^{n-1}x^n}{n2^n}$, then*

$$\frac{|b_{n+1}|}{|b_n|} = \frac{|x|^{n+1}/[(n+1)2^{n+1}]}{|x|^n/(n2^n)} = \frac{n2^n|x|^{n+1}}{(n+1)2^{n+1}|x|^n} = \frac{|x|}{2}\frac{n}{n+1} \to \frac{|x|}{2} \quad \text{as} \quad n \to \infty.$$

So we have $L = |x|/2$, and according to Theorem 2, the series will converge absolutely if $|x|/2 < 1$, i.e. $|x| < 2$, and diverge if $|x|/2 > 1$, i.e. $|x| > 2$. Since no conclusion can be drawn from the ratio test if $L = 1$, which in this case means $|x| = 2$, we must again check explicitly whether the series converges for $x = \pm 2$; we would do this step exactly the same way is in Example 1, and conclude as before that the power series converges for $-2 < x \leq 2$.

Example 4. *Let's apply the ratio test to $\sum_{n=0}^{\infty} \frac{x^n}{n!}$. Here $b_n = x^n/n!$, so*

$$\frac{|b_{n+1}|}{|b_n|} = \frac{x^{n+1}/(n+1)!}{x^n/n!} = \frac{n!x^{n+1}}{(n+1)!x^n} = \frac{x}{n+1} \to 0 \quad \text{as} \quad n \to \infty.$$

It may seem confusing at first that the limit $L = 0$ doesn't depend on x. Actually this makes the problem very easy: since $L < 1$ regardless of what value of x we may choose, we conclude that this series will converge for all x, or in other words, that the radius of convergence of the series is ∞.

As yet I've made no attempt to convince you that Theorem 2 is true. As a test for convergence of series in general, it's quite easy to use, but not nearly as intuitive as the comparison test, or even the alternating series test. This, and the fact that in practice it's rarely needed except when dealing with power series, is why we haven't discussed it explicitly in class. The book states and proves a version of Theorem 2 in Section 11.3 (page 654), so I will not repeat the proof here. Suffice it to say that the proof is quite similar to our proof of Theorem 1, and based on the same principle: comparing the series with some geometric series that we can easily see either converges or diverges.

It's worth noting though, that if we take Theorem 1 as given, we can construct a very easy alternative proof of the ratio test: assume we have a series $\sum_{n=0}^{\infty} b_n$ with $|b_{n+1}/b_n| \to L$, and note that the sum of this series is the same as the sum of the power series $\sum_{n=0}^{\infty} b_n x^n$ when $x = 1$. Theorem 1 then tells us that the power series has radius of convergence $R = 1/L$. If $L < 1$, then $R > 1$ and 1 is inside the interval of convergence, therefore $\sum_{n=0}^{\infty} b_n$ converges absolutely. If $L > 1$, $R < 1$ so 1 is outside the interval of convergence and $\sum_{n=0}^{\infty} b_n$ diverges.

4 Which method is better?

The fact is, at least as far as power series are concerned, every problem that can be solved by Theorem 1 can also be solved by the ratio test, and vice versa. Which method you choose to employ is a matter of personal preference. But beware: if you do decide that the ratio test is your friend, *don't get carried away!* As a test of absolute convergence for infinite series in general (not just power series), the ratio test can be quite powerful, but it won't be helpful as often as you may be tempted to think. Let it be the last method you try: after the integral test, the comparison test and the limit comparison test.