V63.0122.003, Calculus 2
Instructor: Chris Wendl
Fall 2001, Homework 11 Solutions

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10. $x = 2\cos\theta$, $y = \sin^2\theta \Rightarrow \frac{x^2}{4} + y = \cos^2\theta + \sin^2\theta = 1$. So the points on this curve lie on the parabola given by $y = 1 - \frac{x^2}{4}$. To see what portion of the parabola is represented, observe that $x = 2\cos\theta$ oscillates over the range of values $-2 \le x \le 2$. So the parametrized point $(x(t), y(t))$ moves continually back and forth along the parabola between the two endpoints $(-2, 0)$ and $(2, 0)$ (see Figure 1).

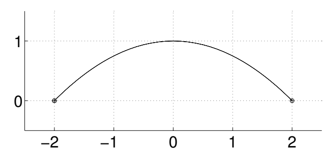

Figure 1

12. $x = \ln t$, $y = \sqrt{t} \Rightarrow t = e^x \Rightarrow y = \sqrt{e^x} = (e^x)^{1/2} = e^{x/2}$. This represents part of an exponential growth curve: as t increases from 1 to ∞, x moves from 0 to ∞, thus we graph the curve $y = e^{x/2}$ only for $x \ge 0$ (Figure 2).

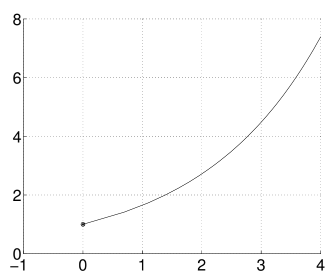

Figure 2

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20. $x = t^3 - 3t^2$, $y = t^3 - 3t \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2-3}{3t^2-6t} = \frac{t^2-1}{t^2-2t} = \frac{(t-1)(t+1)}{t(t-2)}$. Thus the curve has horizontal tangents at $t = -1$ and 1, and vertical tangents at $t = 0$ and 2. We plug these into the equations for x and y to determine the corresponding points on the graph: these are indicated with dots in Figure 3. Note that $t = -1$ and $t = 2$ both represent the point $(x, y) = (-4, 2)$: this is because the curve passes through that point twice at different times in different directions. Now note that $\frac{d}{dt}\frac{dy}{dx} = \frac{d}{dt}\frac{t^2-1}{t^2-2t} = \frac{2t(t^2-2t)-(t^2-1)(2t-2)}{(t^2-2t)^2} = -\frac{2(t^2-t+1)}{(t^2-2t)^2} < 0$ for all t (this is true because $t^2 - t + 1$ is always positive, as you can check with a little algebra). We conclude that the slope is always decreasing as t increases, which means that the curve will appear concave down whenever it is being traced to the right (i.e. $dx/dt > 0$), and concave up when traced to the left (i.e. $dx/dt < 0$). With these facts in mind, we connect the dots to form the graph in Figure 3.

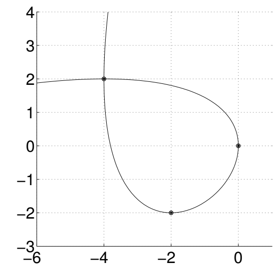

Figure 3

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6. $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta) \Rightarrow s = \displaystyle\int_0^\pi \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$
$= \displaystyle\int_0^\pi \sqrt{[a(-\sin\theta + \theta\cos\theta + \sin\theta)]^2 + [a(\cos\theta + \theta\sin\theta - \cos\theta)]^2}\, d\theta = \int_0^\pi a\sqrt{\theta^2\cos^2\theta + \theta^2\sin^2\theta}\, d\theta =$
$a\displaystyle\int_0^\pi \theta\, d\theta = \boxed{\frac{\pi^2 a}{2}}$.

10. $x = 3t - t^3$, $y = 3t^2 \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{6t}{3-3t^2} = \frac{2t}{1-t^2}$, so the graph has a horizontal tangent when $t = 0$, and a vertical tangent at $t = 1$ (there's also a vertical tangent at $t = -1$ but we can ignore it since we're only concerned with the interval $0 \le t \le 2$. Along with the points corresponding to $t = 0$ and $t = 1$, we plot the endpoint of the interval, where $t = 2$: these three points are $(0, 0)$, $(2, 3)$ and $(-2, 12)$ respectively, as shown by dots in Figure 4. The slope changes according to $\frac{d}{dt}\frac{2t}{1-t^2} = \frac{2(1-t^2)-2t(-2t)}{(1-t^2)^2} = \frac{2(t^2+1)}{(1-t^2)^2} > 0$ for all t, so the slope is always increasing as t increases, as shown in Figure 4. The length of the curve is $s = \displaystyle\int_0^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^2 \sqrt{(3-3t^2)^2 + (6t)^2}\, dt =$
$\displaystyle\int_0^2 \sqrt{9 + 18t^2 + 9t^4}\, dt = \int_0^2 (3 + 3t^2)\, dt = \boxed{14}$.

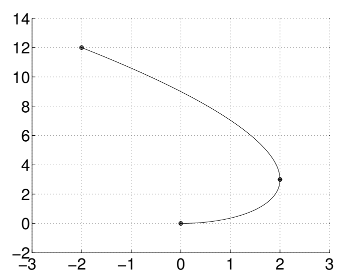

Figure 4

24. $x = 3t - t^3$, $y = 3t^2 \Rightarrow A = \displaystyle\int_0^1 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^1 2\pi(3t^2)\sqrt{(3-3t^2)^2 + (6t)^2}\, dt =$
$6\pi\displaystyle\int_0^1 t^2(3 + 3t^2)\, dt = 18\pi\int_0^1 (t^2 + t^4)\, dt = 18\pi\frac{8}{15} = \boxed{\frac{48\pi}{5}}$.

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8. $A = \int_0^{\pi/4} \frac{1}{2} r^2 \; d\theta = \int_0^{\pi/4} \frac{1}{2} \sin^2 4\theta \; d\theta = \frac{1}{8} \int_0^{\pi} \sin^2 u \; du = \frac{1}{16} \int_0^{\pi} (1 - \cos 2u) \; du = \boxed{\frac{\pi}{16}}$.

48. $r = \theta \Rightarrow s = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \; d\theta = \int_0^{2\pi} \sqrt{\theta^2 + 1} \; d\theta = \int_0^{\arctan 2\pi} \sec^3 t \; dt,$
where we've used the trigonometric substitution $\theta = \tan t$. Stewart demonstrates in Section 7.2, Example 8 that $\int \sec^3 t \; dt = \frac{1}{2}(\sec t \tan t + \ln|\sec t + \tan t|)$, so plugging this in and using the facts that $\tan(\arctan x) = x$ and $\sec(\arctan x) = \sqrt{1+x^2}$ (you can use the identity $1 + \tan^2 x = \sec^2 x$ to prove this; try it!), we get $s = \int_0^{\arctan 2\pi} \sec^3 t \; dt = \frac{1}{2}(\sec t \tan t + \ln|\sec t + \tan t|)\Big|_0^{\arctan 2\pi} = \frac{1}{2}\left[2\pi\sqrt{1+4\pi^2} + \ln\left(\sqrt{1+4\pi^2} + 2\pi\right)\right] = \boxed{\pi\sqrt{1+4\pi^2} + \frac{1}{2}\ln\left(\sqrt{1+4\pi^2} + 2\pi\right)}$.