V63.0122.003, Calculus 2
Instructor: Chris Wendl
Fall 2001, Midterm 2 Solutions

1. *Compute the area of the surface obtained by rotating the curve $y = \sqrt{x}$, $2 \le x \le 6$ about the x-axis.*

   Since $y = \sqrt{x}$ defines a one-to-one function when $2 \le x \le 6$, we have the choice of setting this up as an integral over either x or y. Either way we imagine the surface of revolution cut into slices that form thin annuli centered around the x-axis: if such a slice has radius r and thickness $\Delta\ell$, then its surface area is $\Delta A = 2\pi r \Delta\ell$. In the situation at hand, the radius r is simply y, which we must reexpress as $\sqrt{x}$ if we want to set up the integral with respect to x. Then we also must write $\Delta\ell$ in terms of x and Δx like so:

   $$\Delta\ell = \sqrt{1 + (dy/dx)^2}\ \Delta x = \sqrt{1 + \left(\frac{1}{2\sqrt{x}}\right)^2}\ \Delta x = \sqrt{1 + \frac{1}{4x}}\ \Delta x.$$

   The integral must be computed from $x = 2$ to $x = 6$, so the total surface area is

   $$A = \int_2^6 2\pi\sqrt{x}\sqrt{1 + \frac{1}{4x}}\ dx = 2\pi \int_2^6 \sqrt{x + \frac{1}{4}}\ dx$$
   $$= 2\pi \int_{9/4}^{25/4} \sqrt{u}\ du = 2\pi \frac{2}{3} u^{3/2}\Big|_{9/4}^{25/4} \frac{4\pi}{3}\left(\frac{125}{8} - \frac{27}{8}\right) = \boxed{\frac{49\pi}{3}}.$$

   Here we've used the substitution $u = x + 1/4$.

   If we choose instead to integrate over y, we write $x = y^2$ and $\Delta\ell = \sqrt{1 + (dx/dy)^2}\ \Delta y = \sqrt{1 + (2y)^2}\ \Delta y = \sqrt{1 + 4y^2}\ \Delta y$. We must also alter the limits of integration: as x goes from 2 to 6, y goes from $\sqrt{2}$ to $\sqrt{6}$. Thus

   $$A = \int_{\sqrt{2}}^{\sqrt{6}} 2\pi y \sqrt{1 + 4y^2}\ dy.$$

   This is easily solved by substituting $u = 1 + 4y^2$, and the answer obtained is once again $49\pi/3$—you can work out the details for yourself.

2. *Consider the differential equation $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = 0$.*

   (a) *For what value of a constant k will the function $x(t) = e^{kt}$ solve this equation?*

   We'll use the abbreviations $\dot{x}$ and $\ddot{x}$ to mean $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$ respectively. Assume $x = e^{kt}$ where k is some as yet undetermined constant. Then $\dot{x} = ke^{kt}$, and $\ddot{x} = k^2e^{kt}$, so $\ddot{x} + 2\dot{x} + x = k^2e^{kt} + 2ke^{kt} + e^{kt} = (k^2 + 2k + 1)e^{kt} = (k+1)^2e^{kt}$. If $x(t)$ is to solve the differential equation $\ddot{x} + 2\dot{x} + x = 0$, then the above expression must be 0 for all t, which means $k + 1 = 0$ (since e^{kt} can never be 0). Therefore $\boxed{k = -1}$.

   (b) *Which of the following functions is also a solution?*
   (i) $x(t) = \sin t$ *(ii)* $x(t) = te^{-t}$

   Let $x = \sin t$, then $\dot{x} = \cos t$ and $\ddot{x} = -\sin t$. The differential equation $\ddot{x} + 2\dot{x} + x = 0$ then requires that $-\sin t + 2\cos t + \sin t = 2\cos t = 0$ *for all* t. This last stipulation is important: we can easily find specific values of t for which $2\cos t = 0$, but this is not enough—by definition, $x(t)$ is a solution if it satisfies the differential equation for *all* t, not just for some. So $\sin t$ is <u>not a solution</u>.

   Now let $x = te^{-t}$. Then $\dot{x} = -te^{-t} + e^{-t} = (1-t)e^{-t}$ and $\ddot{x} = -(1-t)e^{-t} - e^{-t} = (t-2)e^{-t}$. Plugging into the differential equation,

   $$\ddot{x} + 2\dot{x} + x = (t-2)e^{-t} + 2(1-t)e^{-t} + te^{-t} = (t - 2 + 2 - 2t + t)e^{-t} = 0.$$

   Since this relation is satisfied for *all* t, te^{-t} <u>is a solution</u>.

3. *Compute each sum.*

(a) $$\sum_{n=1}^{\infty} \frac{3^n + 2^n}{6^n} = \sum_{n=1}^{\infty} \frac{3^n}{6^n} + \sum_{n=1}^{\infty} \frac{2^n}{6^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n + \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$$

$$= \frac{1}{2}\left[1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \dots\right] + \frac{1}{3}\left[1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 \dots\right] = \frac{1}{2}\left(\frac{1}{1-1/2}\right) + \frac{1}{3}\left(\frac{1}{1-1/3}\right) = \boxed{\frac{3}{2}}.$$

Here we've used the geometric series formula $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$, which is applicable since both $1/2$ and $1/3$ have absolute values less than 1.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+5n+6} = \sum_{n=1}^{\infty} \frac{1}{(n+2)(n+3)} = \sum_{n=1}^{\infty} \left(\frac{1}{n+2} - \frac{1}{n+3}\right)$, using the partial fractions decomposition of $\frac{1}{n^2+5n+6}$. This is what's known as a "telescoping series", and it's easy to see what the sum is just by writing out the first several terms:

$$\sum_{n=1}^{\infty} \left(\frac{1}{n+2} - \frac{1}{n+3}\right) = \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \frac{1}{5} - \frac{1}{6} + \dots = \boxed{\frac{1}{3}},$$

since all but the first term in the sum cancel out.

For a more precise and airtight argument, we really should look at the partial sums:

$$s_N = \sum_{n=1}^{N} \frac{1}{n^2+5n+6};$$

by definition, the sum of our series is the limit of the sequence s_N as $N \to \infty$. We can use the partial fraction decomposition above to obtain an explicit formula for s_N:

$$s_N = \sum_{n=1}^{N} \left(\frac{1}{n+2} - \frac{1}{n+3}\right) = \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \frac{1}{5} - \frac{1}{6} + \dots \frac{1}{N+1} - \frac{1}{N+2} + \frac{1}{N+2} - \frac{1}{N+3}$$
$$= \frac{1}{3} - \frac{1}{N+3}.$$

Clearly as $N \to \infty$, $s_N = \frac{1}{3} - \frac{1}{N+3} \to \frac{1}{3}$, thus confirming the result we obtained by somewhat more informal methods above.

4. *Determine whether each series converges absolutely, converges conditionally or diverges.*

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges conditionally. We can see by the alternating series test that it converges, since $1/\sqrt{n}$ is a decreasing sequence that approaches 0 as $n \to \infty$. However, $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{\sqrt{n}}\right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \infty$, which can be proven either explicitly by the integral test (since $\int_1^{\infty} \frac{dx}{\sqrt{x}}$ diverges), or simply by stating the well-known result that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges when $p \leq 1$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1+\sqrt{n}}{n}$ diverges. There are a few ways to prove this: the simplest is probably to note that $\frac{1+\sqrt{n}}{n} > \frac{1}{n}$ for all n, so $\sum_{n=1}^{\infty} \frac{1+\sqrt{n}}{n} > \sum_{n=1}^{\infty} \frac{1}{n} = \infty$.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos 2n}{n^3}$ converges absolutely. In fact, the easiest way to prove that this series converges, is to prove that $\sum_{n=1}^{\infty} \left|\frac{\cos 2n}{n^3}\right|$ converges. Note that $|\cos 2n| \leq 1$ always, thus $\sum_{n=1}^{\infty} \left|\frac{\cos 2n}{n^3}\right| = \sum_{n=1}^{\infty} \frac{|\cos 2n|}{n^3} \leq \sum_{n=1}^{\infty} \frac{1}{n^3} < \infty$. This last result comes from the integral test: $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges because so does the integral $\int_1^{\infty} \frac{dx}{x^3}$.

5. *Let* $f(x) = \dfrac{1}{2-x}$.

(a) *Find the Taylor series expansion of this function about 0.*

As with many things in life, there's an easy way to do this problem, and a hard way. The hard way is to compute all the derivatives of $f(x)$ and plug them into Taylor's formula $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$. The first few derivatives look like this:

$$\begin{aligned} f'(x) &= \frac{1}{(2-x)^2} \\ f''(x) &= \frac{2}{(2-x)^3} \\ f'''(x) &= \frac{3 \cdot 2}{(2-x)^4} \\ f^{(4)}(x) &= \frac{4 \cdot 3 \cdot 2}{(2-x)^5} \\ &\cdots \end{aligned}$$

It becomes clear that for general n, $f^{(n)}(x) = \frac{n!}{(2-x)^{n+1}}$, and in particular $f^{(n)}(0) = \frac{n!}{2^{n+1}}$. (Note that these formulas are equally valid for $n = 0$, since by definition $f^{(0)}(x) = f(x)$ and $0! = 1$; not all examples work out so cleanly.) Plugging this into the Taylor formula, we have

$$\frac{1}{2-x} = \sum_{n=0}^{\infty} \frac{n!}{2^{n+1} n!} x^n = \boxed{\sum_{n=0}^{\infty} \frac{x^n}{2^{n+1}}} = \frac{1}{2} + \frac{x}{4} + \frac{x^2}{8} + \frac{x^3}{16} + \ldots$$

That was the hard way, now here's the easy way: recall the formula for a geometric series, $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ when $|r| < 1$. It turns out that with a little algebra we can make $\frac{1}{2-x}$ look very much like this, namely,

$$\frac{1}{2-x} = \frac{1}{2}\left(\frac{1}{1-(x/2)}\right) = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n = \sum_{n=0}^{\infty}\frac{x^n}{2^{n+1}}.$$

We've just used the geometric series formula, replacing r by $x/2$. As a bonus we can even deduce from this the answer to part (b): we know that the standard geometric series converges only when $|r| < 1$, which in this case means $|x/2| < 1$, or $|x| < 2$. Nice, no?

(b) *Find the interval of convergence for this series.*

Assuming you didn't think of the geometric series trick outlined in part (a), you can still treat this Taylor series the same as any other power series to find its interval of convergence: we start by finding the radius of convergence, and then investigate whether the series converges at the boundary of the resulting interval. Recall our method for finding radius of convergence: if the series can be written as $\sum_{n=0}^{\infty} a_n x^n$ and we find that the ratios of coefficients $|a_{n+1}/a_n|$ converge to a number L as $n \to \infty$, then the radius of convergence is $1/L$. In the present problem, $a_n = 1/2^{n+1}$, so

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1/2^{n+2}}{1/2^{n+1}} = \frac{2^{n+1}}{2^{n+2}} = \frac{1}{2},$$

which obviously converges to $1/2$ as $n \to \infty$. (If this statement seems confusing, keep in mind that in most problems the ratios $|a_{n+1}/a_n|$ form a sequence with terms that depend on n, it's only by coicidence that in this problem all of those terms turned out to be $1/2$. That is, what we're looking at is a *sequence* of the form $\{1/2, 1/2, 1/2, \ldots\}$, and the limit of this sequence certainly could not be anything other than $1/2$). So we conclude that the radius of convergence is 2, which means the series converges for $|x| < 2$ and diverges for $|x| > 2$.

It remains to be determined whether the series converges for $x = \pm 2$. For $x = 2$, the series becomes $\sum_{n=0}^{\infty} \frac{2^n}{2^{n+1}} = \sum_{n=0}^{\infty} \frac{1}{2}$. *Do not be tempted to think that this series converges to* $\frac{1}{2}$. It diverges to infinity, as you will easily see if you try writing out the terms of the series,

$$\sum_{n=0}^{\infty} \frac{1}{2} = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots = \infty.$$

Thus our Taylor series diverges for $x = 2$. When $x = -2$, we have

$$\sum_{n=0}^{\infty} \frac{(-2)^n}{2^{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{2^{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2} = \frac{1}{2} - \frac{1}{2} + \frac{1}{2} - \frac{1}{2} + \ldots$$

This series doesn't diverge to infinity, but it certainly doesn't converge either: the partial sums just oscillate between $1/2$ and 0, and there can be no limit. Another way to see this is with the so-called *test for divergence* based on the following theorem: if $\sum_{n=0}^{\infty} b_n$ converges, then it must be that $b_n \to 0$ as $n \to \infty$. We usually apply this by saying that if b_n does not approach 0, the sum $\sum_{n=0}^{\infty} b_n$ can't possibly converge. In the present case, we have $b_n = \frac{(-1)^n}{2}$, which certainly doesn't converge to 0. In fact we could just as easily have applied this test to the series for $x = 2$ above, where $b_n = 1/2$.

Having tested the boundary points, we conclude that the Taylor series converges only when $\boxed{-2 < x < 2}$.

(c) *Now write down the Taylor series about 0 for $g(x) = -\ln(2 - x)$. Note that $-\ln(2 - x) = \int \frac{dx}{2-x}$, so you can use the result of part (a) and avoid Taylor's formula altogether.*

One could do this in a few different ways, but the most straightforward is to use the fact that $-\ln(2 - x) = \int \frac{dx}{2-x}$ together with the power series expression for $\frac{1}{2-x}$ we obtained in part (a). Just as a power series can be differentiated term by term, it can similarly be integrated, and the result is another power series:

$$-\ln(2 - x) = \int \frac{dx}{2 - x} = \int \sum_{n=0}^{\infty} \frac{x^n}{2^{n+1}}\, dx = \sum_{n=0}^{\infty} \int \frac{x^n}{2^{n+1}}\, dx = \left(\sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)2^{n+1}} \right) + C$$

There's a slightly subtle point here: *the constant matters!* Notice that the power series on the right hand side actually contains no constant term: the lowest power of x present here is x^1. But in theory every Taylor series may have a nonzero constant term (usually it's the one we write first), and the Taylor formula tells us exactly what it is: $\frac{g^{(0)}(0)}{0!} x^0 = g(0)$. Since C is the only term in our series that doesn't depend on x, we deduce that $C = g(0) = -\ln 2$.

We can also see this without making any reference to the Taylor formula: we simply note that if the expression

$$-\ln(2 - x) = C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)2^{n+1}}$$

is valid anywhere at all, it ought to be valid for $x = 0$. Plugging this into both sides yields the result $-\ln 2 = C$. Thus

$$-\ln(2 - x) = -\ln 2 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)2^{n+1}} = \boxed{-\ln 2 + \sum_{n=1}^{\infty} \frac{x^n}{n 2^n}}.$$

(d) *Find the interval of convergence for the series from part (c). Is it the same as the answer to part (b)?*

As far as convergence is concerned, we can ignore the constant term and concentrate on $\sum_{n=1}^{\infty} \frac{x^n}{n2^n}$. This can be written as $\sum_{n=1}^{\infty} a_n x^n$ with $a_n = \frac{1}{n2^n}$, thus

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{1/[(n+1)2^{n+1}]}{1/(n2^n)} = \frac{n2^n}{(n+1)2^{n+1}} = \frac{n}{n+1} \frac{1}{2} \to \frac{1}{2} \quad \text{as } n \to \infty.$$

Thus the radius of convergence is 2. When $x = 2$, the series becomes $\sum_{n=1}^{\infty} \frac{2^n}{n2^n} = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$, so 2 is not part of the interval of convergence. However for $x = -2$, we have

$$\sum_{n=1}^{\infty} \frac{(-2)^n}{n2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{n2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n},$$

which converges, by the alternating series test. So the interval of convergence is $\boxed{-2 \leq x < 2}$. It is *almost* the same as the answer to part (b): the radius of convergence is the same, the only difference here is the one point on the boundary of the interval. This reflects a general fact: if a given series has a certain radius of convergence, then its derivative or antiderivative will always have the same radius of convergence—but what happens at the boundary of the interval is anyone's guess.

6. *Many important first-order initial value problems can be written in the form*

$$\begin{cases} \frac{dx}{dt} = F(x) \\ x(t_0) = x_0 \end{cases}$$

where t_0 and x_0 are some given numbers, and $F(x)$ is a given function. An important theorem in the study of ordinary differential equations states that whenever the function $F(x)$ has certain "nice" properties at x_0, this problem has a unique solution. One problem that violates *uniqueness is the following:*

$$\begin{cases} \frac{dx}{dt} = \sqrt{|x|} \\ x(0) = 0 \end{cases}$$

(a) *Use separation to find a solution $x(t)$ for $t \geq 0$. (Hint: you'll lose nothing by assuming $x(t) \geq 0$.)*
Assuming that $x(t) \geq 0$ means we can drop the absolute value and just look for a function $x(t)$ that satisfies $\frac{dx}{dt} = \sqrt{x}$. Separation yields the following:

$$\frac{dx}{\sqrt{x}} = dt \quad \Rightarrow \quad \int \frac{dx}{\sqrt{x}} = \int dt \quad \Rightarrow \quad 2\sqrt{x} = t + C \quad \Rightarrow \quad x = \left(\frac{t+C}{2}\right)^2.$$

To satisfy the initial condition $x(0) = 0$, we need $0 = x(0) = (C/2)^2$, which is only satisfied if $C = 0$. Thus our solution is $\boxed{x(t) = \frac{1}{4}t^2}$.

Of course we're not done until we've verified that this really is a solution. To that end, we observe that it obviously satisfies $x(0) = 0$, and its derivative $\frac{dx}{dt} = \frac{1}{2}t = \sqrt{\left|\frac{1}{4}t^2\right|} = \sqrt{|x|}$ for all $t \geq 0$, as required. So far so good.

(b) *Now find* another *solution, by guessing! (Hint: write down the simplest function you can possibly think of.)*
You'll kick yourself for not figuring this out. Actually there are an infinite number of other solutions, but the simplest by far is $\boxed{x(t) = 0}$. Indeed, it satisfies $x(0) = 0$ and $\frac{dx}{dt} = 0 = \sqrt{|x|}$, for all t.

(c) *As an educated guess, why do you think it might be that this problem fails to have a unique solution? Put another way, in what sense is the function $\sqrt{|x|}$ "not so nice"?*
In fact $\sqrt{|x|}$ is usually very nice, except at one point, and that point happens to be the value of x that we picked for the initial condition: 0. The theorem says that

$$\begin{cases} \frac{dx}{dt} = F(x) \\ x(t_0) = x_0 \end{cases}$$

is guaranteed to have a unique solution if $F(x)$ is "slightly more than continuous" at x_0. The technical name for this condition is *Lipschitz continuity*, and it basically amounts to the requirement that the derivative of $F(x)$ near x_0 should be bounded. The function $\sqrt{|x|}$ is not differentiable at 0, but this in itself is not enough to violate uniqueness (e.g. the equation $\frac{dx}{dt} = |x|$ always has unique solutions, even though $|x|$ isn't differentiable at 0). What matters is not just that the derivative doesn't exist, but that it's infinite, i.e. the graph of this function has a vertical tangent at 0.

It should be noted that since 0 is the only point where $\sqrt{|x|}$ has this problem, we do get uniqueness if we ask for any initial condition *other than* $x(t_0) = 0$; so for example there is a unique function that satisfies $\frac{dx}{dt} = \sqrt{|x|}$ and $x(0) = 1$. (You can find it by separation, if you're curious. If you're really enthusiastic, you can then spend hours trying to *guess* another solution, but since you'd be doomed to failure, I wouldn't recommend it.)