**V63.0140, Fall 2003**
**Linear Algebra**
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# Study Questions

The following questions address some of the important concepts that you may want to think about in preparation for the final. It's great to be able to compute things (and you should review homework problems in the textbook to remind yourself how to do that), but one should also understand *why* linear algebra works the way it does.

1. Let $\mathbf{A}$ be an $m \times n$ matrix.
   (a) What vector space is $\operatorname{Col}\mathbf{A}$ a subspace of? What about $\operatorname{Nul}\mathbf{A}$?
   (b) Given a nonzero vector $\mathbf{b} \in \mathbb{R}^m$, what condition on $\mathbf{b}$ will guarantee that the equation $\mathbf{A}\mathbf{x} = \mathbf{b}$ is consistent? What does this have to do with $\operatorname{Col}\mathbf{A}$?
   (c) Does the solution set of $\mathbf{A}\mathbf{x} = \mathbf{b}$ form a vector space? (Remember that $\mathbf{b} \neq 0$.)
   (d) If there is a solution, what condition will guarantee that it is unique? Does this condition depend on $\mathbf{b}$, $\mathbf{A}$ or both? What does this have to do with the equation $\mathbf{A}\mathbf{x} = \mathbf{0}$, or the subspace $\operatorname{Nul}\mathbf{A}$?

2. Let $\mathbf{A}$ be an $m \times n$ matrix and let $\mathbf{B}$ be its reduced row echelon form.
   (a) It's clear on inspection that the pivot rows of $\mathbf{B}$ form a basis for $\operatorname{Row}\mathbf{B}$. Why is it true that they also form a basis for $\operatorname{Row}\mathbf{A}$? Why is it *not* true that the pivot columns of $\mathbf{B}$ span $\operatorname{Col}\mathbf{A}$? What *do* the pivot columns of $\mathbf{B}$ tell you about $\operatorname{Col}\mathbf{A}$, and why?
   (b) Prove that $\operatorname{Row}\mathbf{A}$ is the orthogonal complement of $\operatorname{Nul}\mathbf{A}$. What relation does this imply between the dimensions of $\operatorname{Row}\mathbf{A}$ and $\operatorname{Nul}\mathbf{A}$?
   (c) If $m \neq n$, then $\operatorname{Row}\mathbf{A}$ and $\operatorname{Col}\mathbf{A}$ are subspaces of different vector spaces. . . yet their dimensions are related to each other. How? Can you prove the relation? (Think about pivots.)

3. (a) Find a square matrix $\mathbf{A}$ that is invertible but not diagonalizable.
   (b) Find one that is diagonalizable but not invertible.
   (c) If $\mathbf{A}$ is diagonalizable, what does this mean about its eigenvectors?
   (d) What condition on the eigenvalues of $\mathbf{A}$ suffices to guarantee that $\mathbf{A}$ is diagonalizable? Do *all* diagonalizable matrices satisfy this condition?
   (e) Find an $n \times n$ matrix that has only one eigenvalue (of multiplicity $n$) but is diagonalizable. Can you conclude anything in general about such matrices? (You should be able to write down all of them once you know the eigenvalue.)

4. Find an example to prove that $\det(\mathbf{A}+\mathbf{B}) \neq \det\mathbf{A}+\det\mathbf{B}$ in general. How can the formula $\det(\mathbf{A}\mathbf{B}) = \det\mathbf{A} \cdot \det\mathbf{B}$ be interpreted in terms of linear transformations and areas?

5. (a) Let $\mathbf{u}$ and $\mathbf{v}$ be column-vectors in $\mathbb{R}^2$, and consider the $2 \times 2$ matrix $\mathbf{A} = \begin{bmatrix}\mathbf{u} & \mathbf{v}\end{bmatrix}$. What is $|\det\mathbf{A}|$ the area of?
   (b) Let $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ be row-vectors in $\mathbb{R}^3$. If $\det\begin{bmatrix}\mathbf{v}_1\\ \mathbf{v}_2\\ \mathbf{v}_3\end{bmatrix} = c$, then what is $\det\begin{bmatrix}\mathbf{v}_2\\ \mathbf{v}_3\\ \mathbf{v}_1\end{bmatrix}$? Now switch to column vectors and compare $\det\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3\end{bmatrix}$ with $\det\begin{bmatrix}\mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_1\end{bmatrix}$. Any difference?
   (c) Suppose $\mathbf{U}$ is a $2 \times 2$ rotation matrix. Prove by a purely geometric argument that $\det\mathbf{U} = \pm 1$. (Think about areas.)
   (d) Recall that rotation matrices are always *orthogonal*, i.e. $\mathbf{U}^{-1} = \mathbf{U}^T$. Prove that $n \times n$ orthogonal matrices always have determinant $\pm 1$. Hint: how are $\det\mathbf{U}$ and $\det\mathbf{U}^T$ related?

6. Let $V$ and $W$ be a pair of $n$-dimensional vector spaces, with $\beta = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ a basis of $V$ and $\gamma = \{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ a basis of $W$. Suppose we have a linear transformation $T : V \to W$ such that $T(\mathbf{v}_j) = \mathbf{w}_j$ for each $j = 1, \ldots, n$.
   (a) Are there any other linear transformations that act the same way on these basis vectors, or is $T$ the only one?
   (b) Is $T$ onto? Is it one-to-one? Is it an isomorphism?
   (c) Using the coordinates defined by our two bases on $V$ and $W$, we can define another linear transformation which takes any coordinate vector $[\mathbf{x}]_\beta \in \mathbb{R}^n$ to the corresponding coordinate vector $[T(\mathbf{x})]_\gamma \in \mathbb{R}^n$. This transformation maps $\mathbb{R}^n$ to $\mathbb{R}^n$, so it can be defined by some $n \times n$ matrix. What is this matrix? Does your answer make sense in light of part (b)? (If not, try again!)
   (d) Make a slight change to $T$: suppose it takes $\mathbf{v}_j$ to $\mathbf{w}_j$ for $j = 1, \ldots, n-1$, but let $T(\mathbf{v}_n) = \mathbf{w}_{n-1}$ instead of $\mathbf{w}_n$. Now the answers to parts (b) and (c) should change. How?

7. Let $\mathbf{A}$ be an $m \times n$ matrix with $m > n$.
   (a) How do you know that for some (indeed, for most) choices of $\mathbf{b} \in \mathbb{R}^m$, the equation $\mathbf{Ax} = \mathbf{b}$ will be inconsistent?
   (b) Let $\mathbf{P}$ be the $m \times m$ matrix which projects vectors in $\mathbb{R}^m$ orthogonally onto the subspace $\operatorname{Col}\mathbf{A}$. Given any $\mathbf{b} \in \mathbb{R}^m$, how would you compute the distance between $\mathbf{b}$ and $\operatorname{Col}\mathbf{A}$?
   (c) How do you know that the equation $\mathbf{Ax} = \mathbf{Pb}$ has a solution? In what sense is this an *approximate* solution to the (probably unsolvable) equation $\mathbf{Ax} = \mathbf{b}$?
   (d) Why might you want to solve the equation $\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b}$? Is there always a solution? What condition on $\mathbf{A}$ will guarantee uniqueness?

8. The following steps essentially constitute a proof of the relation between the eigenvalues of a matrix and its determinant and trace. This relation is especially useful in the $2 \times 2$ case; see the last two parts of question 10.
   (a) Assume $\mathbf{A}$ is a diagonalizable matrix. Use the factorization $\mathbf{A} = \mathbf{S\Lambda S}^{-1}$ to prove that $\det\mathbf{A}$ is the product of the eigenvalues of $\mathbf{A}$, counted with multiplicity. (It's true also when $\mathbf{A}$ is not diagonalizable, but less simple to prove.)
   (b) Recall that the *trace* of a square matrix is defined to be the sum of its diagonal entries:
$$\operatorname{tr}\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} = a_{11} + a_{22} + \ldots + a_{nn}.$$
   Is it true that for all $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$, $\operatorname{tr}(\mathbf{A} + \mathbf{B}) = \operatorname{tr}\mathbf{A} + \operatorname{tr}\mathbf{B}$?
   (c) Find an example to prove that $\operatorname{tr}(\mathbf{AB}) \neq \operatorname{tr}\mathbf{A} \cdot \operatorname{tr}\mathbf{B}$ in general.
   (d) It does however turn out to be true always that $\operatorname{tr}(\mathbf{AB}) = \operatorname{tr}(\mathbf{BA})$, even if $\mathbf{AB} \neq \mathbf{BA}$ (i.e. in fancy language, $\mathbf{A}$ and $\mathbf{B}$ don't *commute*). If you're feeling adventurous, you can prove this by calculation: write the entries of the matrices $\mathbf{A}$, $\mathbf{AB}$, $\mathbf{B}$ and $\mathbf{BA}$ as $A_{ij}$, $(AB)_{ij}$ etc., then the definition of matrix multiplication is
$$(AB)_{ij} = A_{i1}B_{1j} + A_{i2}B_{2j} + \ldots + A_{in}B_{nj} = \sum_{k=1}^{n} A_{ik}B_{kj}.$$
   Stare at this formula long enough to convince yourself that it's true, then use it to prove that $\mathbf{AB}$ and $\mathbf{BA}$ have the same trace, i.e.
$$\sum_{i=1}^{n}(AB)_{ii} = \sum_{i=1}^{n}(BA)_{ii}.$$
   If you get stuck, just try to prove it in the $2 \times 2$ case.

(e) Using the formula $\operatorname{tr}(\mathbf{AB}) = \operatorname{tr}(\mathbf{BA})$, prove that any two matrices have the same trace if they are similar. (Recall: $\mathbf{P}$ and $\mathbf{Q}$ are called *similar* if we can write $\mathbf{P} = \mathbf{SQS}^{-1}$.)

(f) Assuming once more that $\mathbf{A}$ is diagonalizable, use the factorization $\mathbf{A} = \mathbf{S\Lambda S}^{-1}$ to prove that $\operatorname{tr}\mathbf{A}$ is the *sum* of the eigenvalues of $\mathbf{A}$, counted with multiplicity. (This is also true for non-diagonalizable $\mathbf{A}$, but again the proof is more complicated.)

9. Define a quadratic form $Q(x, y) = xy$.

   (a) Reexpress this by writing $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ and finding a symmetric matrix $\mathbf{A}$ such that $Q(\mathbf{v}) = \mathbf{v} \cdot \mathbf{Av}$.

   (b) Find orthonormal eigenvectors of $\mathbf{A}$ and use them to define new coordinates $(\xi, \eta)$ in which $Q$ has the form $Q(\xi, \eta) = \lambda_1\xi^2 + \lambda_2\eta^2$.

   (c) If you've done this correctly so far, you should conclude that $Q$ is an *indefinite* form, i.e. its values are sometimes positive, sometimes negative, and its graph is saddle-shaped. How do you conclude this from the eigenvalues? Could you have concluded this more quickly by looking at the determinant and/or trace of $\mathbf{A}$?

10. Let $\mathbf{A}$ be a $2\times2$ matrix with real eigenvalues $\lambda_1$ and $\lambda_2$. Consider the differential equation $\dot{\mathbf{x}}(t) = \mathbf{Ax}(t)$, where as usual $\dot{\mathbf{x}}(t)$ means $\frac{d}{dt}\mathbf{x}(t)$.

    (a) Suppose $\mathbf{v}_1$ and $\mathbf{v}_2$ are nonzero vectors such that $\mathbf{Av}_1 = \lambda_1\mathbf{v}_1$ and $\mathbf{Av}_2 = \lambda_2\mathbf{v}_2$. What are the solutions of $\dot{\mathbf{x}} = \mathbf{Ax}$ with initial conditions $\mathbf{x}(0) = \mathbf{v}_1$ and $\mathbf{x}(0) = \mathbf{v}_2$? Verify that your answers are correct by plugging them into the differential equation.

    (b) What is the solution of $\dot{\mathbf{x}} = \mathbf{Ax}$ with initial condition $\mathbf{x}(0) = 3\mathbf{v}_1 - 2\mathbf{v}_2$? Again, verify that your answer is correct.

    (c) The long-term behavior of any solution depends on the eigenvalues $\lambda_1$ and $\lambda_2$. There are always two things we want to know: first, does $\mathbf{x}(t)$ expand outward to infinity, shrink inward to the origin, or neither? Secondly, as $t \to \pm\infty$, does $\mathbf{x}(t)$ get closer to being a multiple of $\mathbf{v}_1$ or $\mathbf{v}_2$, or neither? Consider both of these questions for the solution from part (b) in each of the following cases. (You may want to try drawing sample pictures of the trajectories.)

       i. $\lambda_1 > \lambda_2 > 0$
       ii. $\lambda_1 > 0$ and $\lambda_2 < 0$
       iii. $\lambda_2 < \lambda_1 < 0$
       iv. $\lambda_1 > 0$ and $\lambda_2 = 0$
       v. $\lambda_1 < 0$ and $\lambda_2 = 0$

    (d) Assume now that $\lambda_1 = \lambda_2 = \lambda$, and $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent. Can you now simplify the solution from part (b)? Consider its long-term behavior in the cases where $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$.

    (e) Recall that a dynamical system $\dot{\mathbf{x}} = \mathbf{Ax}$ is called *stable* if the origin is an attractor, i.e. all solutions $\mathbf{x}(t)$ shrink to $\mathbf{0}$ as $t \to +\infty$. In which of the cases considered above is the system stable? What conditions on the eigenvalues $\lambda_1$ and $\lambda_2$ guarantee stability?

    (f) In the cases where the system is stable, are the determinant and trace of $\mathbf{A}$ positive or negative?

    (g) Consider the system

    $$\begin{aligned} \dot{x}_1 &= -2x_1 + x_2 \\ \dot{x}_2 &= x_1 - 3x_2 \end{aligned}$$

    Is it stable? Don't solve it, don't compute the eigenvalues, just answer the question. *Do this one in your head!*