V63.0005.002, Spring 2002
Mathematical Thinking
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Take-home quiz 9.5: The Cardinal Sins of Algebra

This is a homework assignment to be handed in by Friday, May 10. You can either give it to me in class on Monday or bring it to my office (720 WWH) or mailbox (on the ground floor of Warren Weaver Hall).

Each of the following examples begins with an algebraic manipulation which is *wrong*. (Many of these are taken from or inspired by your own quizzes!) For each example we explain what is wrong and then how to correct the problem. Read through and understand the examples before doing the exercises that follow.

Alert! The following contains many equations that are woefully incorrect, and some that aren't. Pay attention so that you always know which is which.

Example 1.

$$\frac{4 \pm \sqrt{48}}{4} = 1 \pm \sqrt{48}$$

If you want to make cancellation happen, you have to apply it to every relevant term in the expression; here $\sqrt{48}$ has been ignored. Note that $\sqrt{48} = \sqrt{16 \cdot 3} = \sqrt{16} \cdot \sqrt{3} = 4\sqrt{3}$. Cancellation in an example like this works because of the simple fact that $\frac{4}{4} = 1$: doing things more carefully, the result would be

$$\frac{4 \pm \sqrt{48}}{4} = \frac{4 \pm 4\sqrt{3}}{4} = \frac{4(1 \pm \sqrt{3})}{4} = \frac{4}{4} \cdot (1 \pm \sqrt{3}) = 1 \pm \sqrt{3}.$$

A slightly different but equally good approach would be to take apart the fraction before cancelling:

$$\frac{4 \pm \sqrt{48}}{4} = \frac{4 \pm 4\sqrt{3}}{4} = \frac{4}{4} \pm \frac{4\sqrt{3}}{4} = 1 \pm \sqrt{3}.$$

Example 2.

$$\frac{4 \pm \sqrt{48}}{4} = \frac{4 \pm 4\sqrt{3}}{4} = \frac{4}{4} \pm \frac{4\sqrt{3}}{4} = \pm\sqrt{3}$$

The correct answer is again $1 \pm \sqrt{3}$, but we've somehow lost the 1. This error seems usually to result from the following thought process: "I have a 4 on top and a 4 on the bottom, so they cancel each other and that term disappears." Yes, they cancel, but it doesn't mean anything disappears: they cancel in the sense that $\frac{4}{4}$ is simplified to 1, which cannot be discarded because we need to *add* it to the other term. (If we were *multiplying* it, that would be another story.)

Example 3.

$$\begin{aligned}(x+y)^2 &= x^2 + y^2 \\ (x+y)^3 &= x^3 + y^3 \\ &\text{etc.}\end{aligned}$$

Repeat after me: *THOU SHALT NOT DISTRIBUTE THE EXPONENT!* If you're ever tempted to believe that $(x+y)^2 = x^2 + y^2$, just plug in $x = 3$ and $y = 2$: you get $(3+2)^2 = 3^2 + 2^2 \quad \Rightarrow \quad 25 = 13$. Hopefully you can see why this is problematic. To expand $(x+y)^2$ correctly, you can either remember the formula we learned early in the semester, or when all else fails, you can foil it:

$$(x+y)^2 = (x+y)(x+y) = x^2 + xy + yx + y^2 = x^2 + 2xy + y^2.$$

Try it now with $x = 3$ and $y = 2$: you'll see that it works out (the extra $2xy$ makes all the difference). Similarly,

$$\begin{aligned}(x+y)^3 = (x+y)^2 \cdot (x+y) = (x^2+2xy+y^2)(x+y) &= (x^2+2xy+y^2)x + (x^2+2xy+y^2)y \\ &= x^3 + 2x^2y + y^2x + x^2y + 2xy^2 + y^3 = x^3 + 3x^2y + 3xy^2 + y^3\end{aligned}$$

and you can expand $(x+y)^4$, $(x+y)^5$ etc. in this same way, though it takes a lot more work when the exponent gets larger. Nobody ever said life was easy.

Example 4.

$$\sqrt{x+y} = \sqrt{x} + \sqrt{y}$$

This is really just another case of distributing the exponent: we could rephrase the statement as $(x+y)^{1/2} = x^{1/2} + y^{1/2}$. It's false either way: plug in $x = 4$ and $y = 9$, then we have $\sqrt{4+9} = \sqrt{4} + \sqrt{9} \quad \Rightarrow \quad \sqrt{13} = 2 + 3 = 5$. The square root of 13 is emphatically not 5, it's not even a rational number! Is there any correct way to expand $\sqrt{x+y}$? Actually, **NO**, the best thing to do is leave it as it is. But see the next example.

Example 5.

$$\sqrt{4x^4 + x^2} = \sqrt{4x^4} + \sqrt{x^2} = 2x^2 + x$$

This is catastrophically wrong, for the same reason as in Example 4. (The second step is fine, but by that point the damage is already done.) This time we can at least do *something* to simplify the expression in a correct fashion:

$$\sqrt{4x^4 + x^2} = \sqrt{x^2(4x^2+1)} = \sqrt{x^2}\sqrt{4x^2+1} = x\sqrt{4x^2+1}$$

That's as simple as it will get. Here we've used the useful identify $\sqrt{ab} = \sqrt{a}\sqrt{b}$. This one follows simply from the rules of exponents, but don't ever be tempted to confuse it with the untrue statement $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$!

Example 6.

$$\begin{aligned}
&x^2 + 4x = 7 \quad \text{(complete the square)} \\
&\quad \Rightarrow \quad x^2 + 4x + 4 = 7 \\
&\quad \Rightarrow \quad (x+2)^2 = 7 \quad \Rightarrow \quad x + 2 = \pm\sqrt{7} \quad \Rightarrow \quad x = -2 \pm \sqrt{7}.
\end{aligned}$$

Here we're trying to solve a quadratic equation by completing the square, but there's a problem in passing from the first line to the second: we added 4 to the left hand side, but not to the right. Remember, always when manipulating equations, *DO UNTO THE RIGHT HAND SIDE AS YOU WOULD DO UNTO THE LEFT.* If you change one side without the other, or change both sides but in different ways, then you end up with an equation that has no actual relation to the one you started with. It would be similarly wrong to do the following:

$$\begin{aligned}
&x^2 + 4x = 7 \\
&\quad \Rightarrow \quad x^2 + 4x + 2^2 = 7 + 2 \\
&\text{etc.}
\end{aligned}$$

Here we added 4 to the left side but 2 to the right. Philosophically, the way to solve an equation in algebra almost always involves modifying the equation in some way so that we get a new (hopefully simpler) equation which we know has the same solutions as the old one. The numbers obtained above, $-2 \pm \sqrt{7}$ are *correct solutions to the equation* $x^2 + 4x + 4 = 7$, i.e. the equation in the second line. But this *has nothing to do with the original problem*, which is to solve $x^2 + 4x = 7$. The proper way to go about it would have been:

$$\begin{aligned}
&x^2 + 4x = 7 \quad \text{(complete the square)} \\
&\quad \Rightarrow \quad x^2 + 4x + 4 = 7 + 4 = 11 \\
&\quad \Rightarrow \quad (x+2)^2 = 11 \quad \Rightarrow \quad x + 2 = \pm\sqrt{11} \quad \Rightarrow \quad x = -2 \pm \sqrt{11}.
\end{aligned}$$

Example 7.

$$\begin{aligned}
&-x^2 + 2x = 3 \quad \text{(multiply by } -1) \\
&\quad \Rightarrow \quad x^2 - 2x = 3
\end{aligned}$$

It's a quadratic equation, and we don't like having a negative coefficient in front of the x^2 term. So we multiply by -1, what's wrong with that? WE ONLY CHANGED THE LEFT HAND SIDE! We should have changed the other side too:

$$\begin{aligned} & -x^2 + 2x = 3 \\ & \quad \Rightarrow \quad x^2 - 2x = -3 \end{aligned}$$

This is fine; another thing we could have done is to move the 3 to the other side first:

$$\begin{aligned} & -x^2 + 2x = 3 \\ & \quad \Rightarrow \quad -x^2 + 2x - 3 = 0 \\ & \quad \Rightarrow \quad x^2 - 2x + 3 = 0 \end{aligned}$$

In the last step here we multiplied by -1. But wait a minute, didn't we make the same mistake again—didn't we forget to change the right hand side? Actually no, we did multiply the right side by -1, but it didn't change anything since $-1 \cdot 0 = 0$. This is something we often do when dealing with various sorts of equations: we can get away with multiplying one side by whatever we want and leaving the other side unchanged, but only if the other side happens to be 0.

Example 8.

$$x^3 - 2x + 3 - (2x^2 + 3x - 2) = x^3 - 2x + 3 - 2x^2 + 3x - 2 = x^3 - 2x^2 + x + 1$$

BE CAREFUL TO KEEP TRACK OF SIGNS! We muddled some details when the parentheses were removed: we should have distributed the minus sign (remember, it's the same thing as multiplying by -1):

$$x^3 - 2x + 3 - (2x^2 + 3x - 2) = x^3 - 2x + 3 - 2x^2 - 3x + 2 = x^3 - 2x^2 - 5x + 5$$

Example 9.

$$\frac{(x^2y^3)^4}{xy} = (xy^2)^4 = x^4y^8$$

Oh dear. The idea here was to cancel the x^2 on top with the x on the bottom, and similarly the y^3 with the y. But the exponent outside the parentheses causes complications: in fact the quantities in the numerator are *not* actually x^2 and y^3 (see the solution below), and we shouldn't be doing any cancellation until we really know what we're dealing with. The solution is to eliminate the parentheses before cancelling:

$$\frac{(x^2y^3)^4}{xy} = \frac{(x^2)^4(y^3)^4}{xy} = \frac{x^8y^{12}}{xy} = x^7y^{11}$$

Note that we've used here the rules $(ab)^r = a^rb^r$ and $(a^r)^s = a^{rs}$.

Example 10.

$$\begin{aligned} & \sqrt{x-1} = x + 4 \quad \text{(square both sides)} \\ & \quad \Rightarrow \quad x - 1 = x^2 + 16 \end{aligned}$$

Can you hear me saying "thou shalt not distribute the exponent"? If you square the right side of the equation, it is not the same thing as squaring *each term* individually. Let's try this again:

$$\begin{aligned} & \sqrt{x-1} = x + 4 \\ & \quad \Rightarrow \quad (\sqrt{x-1})^2 = (x+4)^2 \\ & \quad \Rightarrow \quad x - 1 = x^2 + 8x + 16 \end{aligned}$$

Ahh, that's better.

Exercises

Now it's time for *you* to correct *my* work. Each of the following equations is printed along with one or more solutions, each of which is wrong. For each problem, circle the step in each printed solution where a mistake is made (for example, in the relation $3a - (2b+2) = 3a - 2b + 2$, you would circle the last "2" since it has the wrong sign). Then write up a corrected solution on a separate sheet of paper. Note that these mistakes may not always be exactly analogous to those in the examples above.

1. $(x-4)^2 + (x+2)(x+1) = 18.$

   Solution:

$$(x-4)^2 + (x+2)(x+1) = x^2 - 16 + x^2 + 3x + 2 = 2x^2 + 3x - 14 = 18$$

$$\Rightarrow \quad 2x^2 + 3x - 32 = 0 \quad \Rightarrow \quad x = \frac{-3 \pm \sqrt{9 - 4(2)(-32)}}{4} = \boxed{\frac{-3 \pm \sqrt{265}}{4}}$$

2. $\dfrac{x+2}{x+1} - 2 = \dfrac{1}{x+1}.$

   Solution A:

$$\frac{x+2}{x+1} - 2 = \frac{1}{x+1} \quad \Rightarrow \quad \left(\frac{x}{x} + \frac{2}{1}\right) - 2 = (1+2) - 2 = 1 = \frac{1}{x+1} \quad \text{(multiply by } x+1\text{)}$$

$$\Rightarrow \quad x + 1 = 1 \quad \Rightarrow \quad \boxed{x = 0}$$

   Solution B:

$$\frac{x+2}{x+1} - 2 = \frac{1}{x+1} \quad \text{(multiply by } x+1\text{)}$$

$$\Rightarrow \quad (x+2) - 2 = 1 \quad \Rightarrow \quad \boxed{x = 1}$$

   Solution C:

$$\frac{x+2}{x+1} - 2 = \frac{1}{x+1}$$

$$\Rightarrow \quad -2 = \frac{1}{x+1} - \frac{x+2}{x+1} = \frac{1 - x + 2}{x+1} = \frac{3-x}{x+1} \quad \text{(multiply by } x+1\text{)}$$

$$\Rightarrow \quad -2(x+1) = 3 - x \quad \Rightarrow \quad -2x - 2 = 3 - x \quad \Rightarrow \quad -5 = x \quad \Rightarrow \quad \boxed{x = -5}$$

3. $\sqrt{y+7} - \sqrt{y+2} = 1.$

   Solution A:

$$\sqrt{y+7} - \sqrt{y+2} = \sqrt{y} + \sqrt{7} - (\sqrt{y} + \sqrt{2}) = \sqrt{7} - \sqrt{2}$$

$$\Rightarrow \quad \sqrt{7} - \sqrt{2} = 1 \quad \Rightarrow \quad \boxed{\text{no solution}}$$

   Solution B:

$$\sqrt{y+7} - \sqrt{y+2} = 1 \quad \text{(square both sides)}$$

$$\Rightarrow \quad (y+7) - (y+2) = y + 7 - y - 2 = 5 = 1 \quad \Rightarrow \quad \boxed{\text{no solution}}$$

4. $(2t-3)^2 - 9(2t-3) = -20.$

Solution A:

$$(2t-3)^2 - 9(2t-3) = 4t^2 - 12t + 9 - 18t - 27 = -20$$
$$\Rightarrow \quad 4t^2 - 30t + 2 = 0 \quad \text{(divide by 2)}$$
$$\Rightarrow \quad 2t^2 - 15t + 2 = 0 \quad \Rightarrow \quad t = \frac{15 \pm \sqrt{225 - 4(2)(2)}}{4} = \boxed{\frac{15 \pm \sqrt{209}}{4}}$$

Solution B:

$$(2t-3)^2 - 9(2t-3) = -20$$
$$\Rightarrow \quad (2t-3)^2 - 9(2t-3) + 20 = 0 \quad \Rightarrow \quad 2t - 3 = \frac{9 \pm \sqrt{81 - 4(1)(20)}}{2} = \frac{9 \pm 1}{2}$$
$$\Rightarrow \quad 2t = 3 + \frac{9 \pm 1}{2} \quad \Rightarrow \quad t = \frac{3}{2} + \frac{9 \pm 1}{2} = \frac{3 + 9 \pm 1}{2} = \boxed{\frac{13}{2} \text{ or } \frac{11}{2}}$$

5. $(3y+2)^2 - (3y-2)^2 = 24$.

Solution A:

$$(3y+2)^2 - (3y-2)^2 = 9y^2 + 4 - (9y^2 - 4) = 8 \quad \Rightarrow \quad 8 = 24 \quad \Rightarrow \quad \boxed{\text{no solution}}$$

Solution B:

$$(3y+2)^2 - (3y-2)^2 = 9y^2 + 12y + 4 - 9y^2 - 12y + 4 = 8 \quad \Rightarrow \quad 8 = 24 \quad \Rightarrow \quad \boxed{\text{no solution}}$$

Logic Questions

6. Suppose I try to convince you that the identity $(a-b)^2 = a^2 - b^2$ is true. As proof, I offer the fact that if you plug in $a = 3$ and $b = 0$, you get $(3-0)^2 = 3^2 - 0^2 \quad \Rightarrow \quad 9 = 9$, which is true. Are you convinced by this argument? Explain why it doesn't actually prove anything.

7. The following statement is **false**:

$$\text{If } a > b \text{ then } a^2 > b^2.$$

Prove this is false by finding two numbers a and b for which $a > b$ but $a^2 \leq b^2$. (Hint: try negative numbers.)

8. Consider the inequality $\sqrt{x-2} > -3$.

(a) We could try to solve this as follows:

$$\sqrt{x-2} > -3 \quad \text{(square both sides)}$$
$$\Rightarrow \quad x - 2 > 9 \quad \Rightarrow \quad \boxed{x > 11}.$$

But this is wrong. Why doesn't this method yield the right solution, i.e. what's wrong with our logic here? (Hint: what does it have to do with Exercise 7?)

(b) Can you solve the inequality anyway? (Actually it's very easy if you step back from it for a moment. What's the lowest number that $\sqrt{x-2}$ can possibly equal?)