What do these questions have in common?

1. **Topology**
   Given a closed manifold $M$, is it the boundary of a compact manifold?

2. **Global analysis**
   Given two (almost) complex manifolds $W$ and $W'$, what is the structure of the space of *holomorphic maps* $W \to W'$?
   Is it smooth? Is it compact? Is its topology interesting?

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   Given $H(q_1, p_1, \ldots, q_n, p_n)$, does $H^{-1}(c)$ contain periodic orbits of
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Basic notions

- **Symplectic structures** \((\dim W = 2n)\)
- **Contact structures** \((\dim M = 2n - 1)\)

The following answer to Question 3 may serve as motivation:

**Theorem (Rabinowitz-Weinstein '78)**

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$$\omega = dp_1 \wedge dq_1 + \ldots + dp_n \wedge dq_n.$$  

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An example from complex geometry

A **Stein manifold** is a complex manifold \((W, J)\) with a proper holomorphic embedding

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(W, J) \hookrightarrow (\mathbb{C}^N, i).
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(Grauert) \(\iff\) \((W, J)\) admits an exhausting **plurisubharmonic** function \(f : W \to \mathbb{R}\), meaning

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\omega_J := \frac{i}{2} \partial \bar{\partial} f = -d(df \circ J) \text{ is symplectic (on all complex submanifolds)}. 
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Then \(\omega_J\) is dilated by \(\nabla f\), so \(W_c := f^{-1}((-\infty, c])\) is symplectic with convex boundary \(M_c := f^{-1}(c)\).
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is then a **contact structure** on \( M_c \), i.e. it is maximally nonintegrable.

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Some problems in contact topology

1. Classification of contact structures
Given $\xi_1, \xi_2$ on $M$, is there a diffeomorphism $M \to M$ taking $\xi_1$ to $\xi_2$?

2. Weinstein conjecture
Do Hamiltonian flows on compact contact hypersurfaces always have periodic orbits?

3. Fillings and cobordisms
What are all the symplectic fillings of $(M, \xi)$?
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Rigidity and flexibility

Gromov (ICM 1986): “soft” vs. “hard” symplectic geometry
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\[ \text{SYMPLECTIC GEOMETRY} \]

\[ \text{flexible} \]
\[ \text{rigid} \]

**Insight:** The interesting questions are on the borderline.
Flexibility

Flexibility ("soft") comes from the h-principle,

Examples of symplectic flexibility

- **Existence** of symplectic structures on open manifolds [Gromov 1969]:
  \[
  \begin{align*}
  \{\text{sympl. forms}\} & \xrightleftharpoons{1:1} \{\text{almost } \mathbb{C}\text{-structures}\} \\
  \text{deformation} & \quad \text{homotopy}
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- There is a flexible class of Stein structures: two such structures are Stein homotopic \(\Leftrightarrow\) homotopic as almost complex structures. [Cieliebak-Eliashberg 2012]
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Two “overtwisted” contact structures $\xi_1, \xi_2$ are isotopic $\iff$ they are homotopic. [Eliashberg 1989] + [Borman-Eliashberg-Murphy 2014]
Rigidity ("hard") comes from invariants: Gromov-Witten, Floer homology, symplectic field theory (SFT), Seiberg-Witten. . .

Examples of symplectic rigidity

- \((S^3, \xi_{\text{std}})\) has a unique Stein filling up to deformation. [Gromov 1985], [Eliashberg 1989]
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- The 3-torus admits an infinite sequence of contact structures that are homotopic but not isotopic. [Giroux 1994]
  Only the first is fillable [Eliashberg 1996], and its filling is unique. [W. 2010]
The middle ground: quasiflexibility

*Stein* is generally **more rigid** than *symplectic*, e.g. Ghiggini ’05 proved

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\text{Stein}(W) \to \text{Symp}^{\text{convex}}(W)
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is **not always surjective** on \( \pi_0 \).

**Open question**

Is there a manifold with two Stein structures that are symplectomorphic but **not** Stein homotopic?

**Main theorem (Lisi, Van Horn-Morris, W. ’17)**

Suppose \( \dim_{\mathbb{R}} W = 4 \), \( J_0 \) and \( J_1 \) are Stein structures on \( W \), and \( J_0 \) admits a compatible **Lefschetz fibration of genus 0**. Then

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Chris Wendl (HU Berlin) — When is a Stein manifold merely symplectic? — November 28, 2017
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in local complex coordinates.

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Lefschetz fibrations with boundary (over $\mathbb{D}^2$)

$\partial W = \partial v W \cup \partial h W$, where

$\partial v W := \pi^{-1}(\partial \mathbb{D}^2)$ fibration $\rightarrow \partial \mathbb{D}^2 = S^1$,

$\partial h W := \bigcup_{z \in \mathbb{D}^2} \partial (\pi^{-1}(z)) \sim = \biguplus (S^1 \times \mathbb{D}^2)$

$\Rightarrow$ $\partial W$ inherits an open book decomposition.

Chris Wendl (HU Berlin)
When is a Stein manifold merely symplectic?

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Lemma (Lisi, Van Horn-Morris, W.)

Suppose $\pi : W \to \mathbb{D}^2$ has no closed components in its singular fibers (i.e. $\pi$ is “allowable”). Then $W$ admits a canonical deformation class of Stein structures such that the fibers are holomorphic curves, and the contact structure on $\partial W$ is supported (in the sense of Giroux) by the induced open book decomposition.
Fundamental lemma of symplectic topology (Gromov '85)

On every symplectic manifold \((W, \omega)\), there is a contractible space of “tamed” almost complex structures

\[
\{ J : TW \to TW \mid J^2 = -1 \text{ and } \omega(X, JX) > 0 \text{ for all } X \neq 0 \}.
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Given a Riemann surface \((\Sigma, j)\), a map \(u : \Sigma \to W\) is called \(J\)-holomorphic if it satisfies the nonlinear Cauchy-Riemann equation:

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Tu \circ j = J \circ Tu
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\(\Leftrightarrow\) in local coordinates \(s + it\),

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This is a first-order elliptic PDE.
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Lemma (W. '10)

Suppose $(W^4, \omega_{\tau})$ is a 1-parameter family of symplectic fillings of $(M^3, \xi)$, where $\xi$ is supported by a planar open book (i.e. its fibers have genus zero).

Choose a generic family $J_{\tau}$ of $\omega_{\tau}$-tame almost complex structures on the symplectic completion $(\hat{W}, \hat{\omega}_{\tau})$.

Then the open book extends to a smooth family of Lefschetz fibrations

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with $J_{\tau}$-holomorphic fibers, and they are allowable if $\omega_{\tau}$ is exact for any $\tau$. 

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Remark: This does not work with higher-genus open books. Curves have index $2 - 2g$. 

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Proof of main theorem:
Symplectic deformation
\[\implies\] isotopy of Lefschetz fibrations
\[\implies\] homotopy of Stein structures.
Conclusion

Even rigid structures can be...
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Some questions for the future

- Is there quasiflexibility in higher dimensions?
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- Is there quasiflexibility in **higher dimensions**?
- Is there a quasiflexible class of **contact structures** in dimension 3? *(planar?)*
Thank you for your attention!

Pictures of contact structures by Patrick Massot:

https://www.math.u-psud.fr/~pmassot/exposition/gallerie_contact/