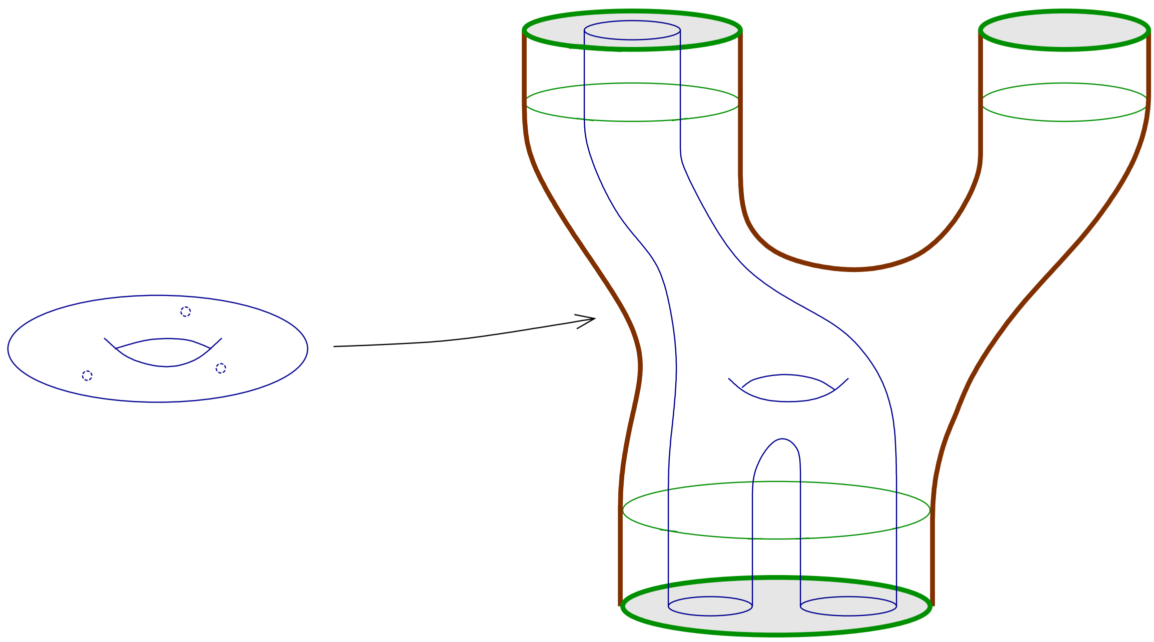
On Contact Topology, Symplectic Field Theory and the PDE That Unites Them



Chris Wendl

University College London

Slides available at:

http://www.homepages.ucl.ac.uk/~ucahcwe/publications.html#talks

How are the following related?

Problem 1 (dynamics):
If $H(q_1, p_1, \ldots, q_n, p_n)$ is a time-independent Hamiltonian and $H^{-1}(c)$ is convex, does

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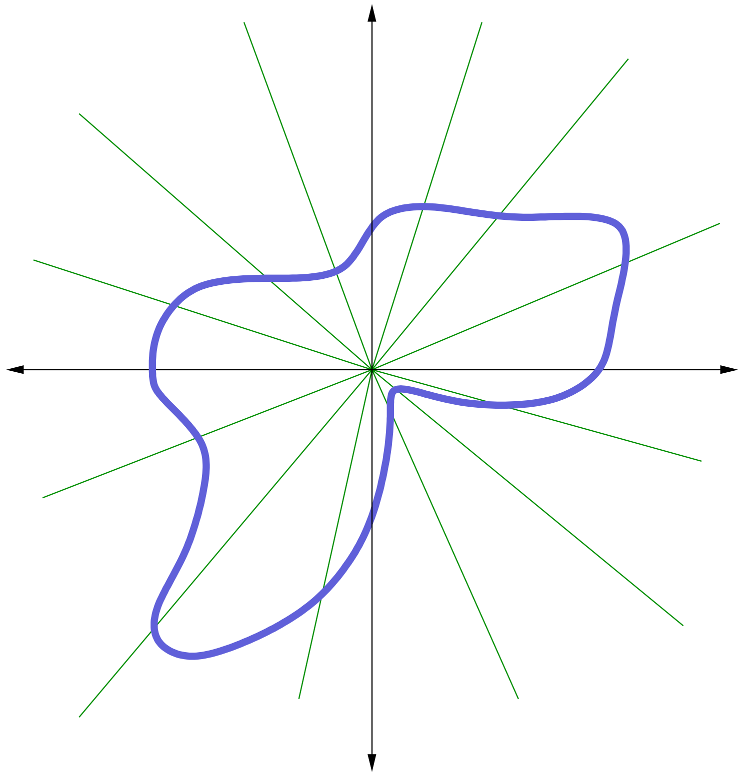
Problem 4 (mathematical physics):
How trivial is my TQFT?

Theorem (Rabinowitz-Weinstein '78)**.**
Every **star-shaped** *hypersurface in* $\mathbb{R}^{2n}$ *admits a periodic orbit.*



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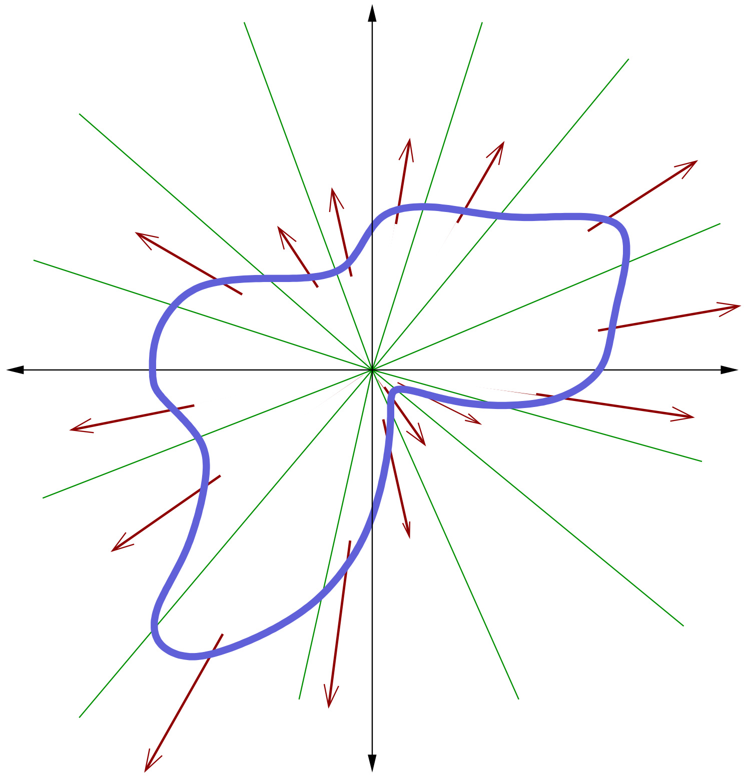

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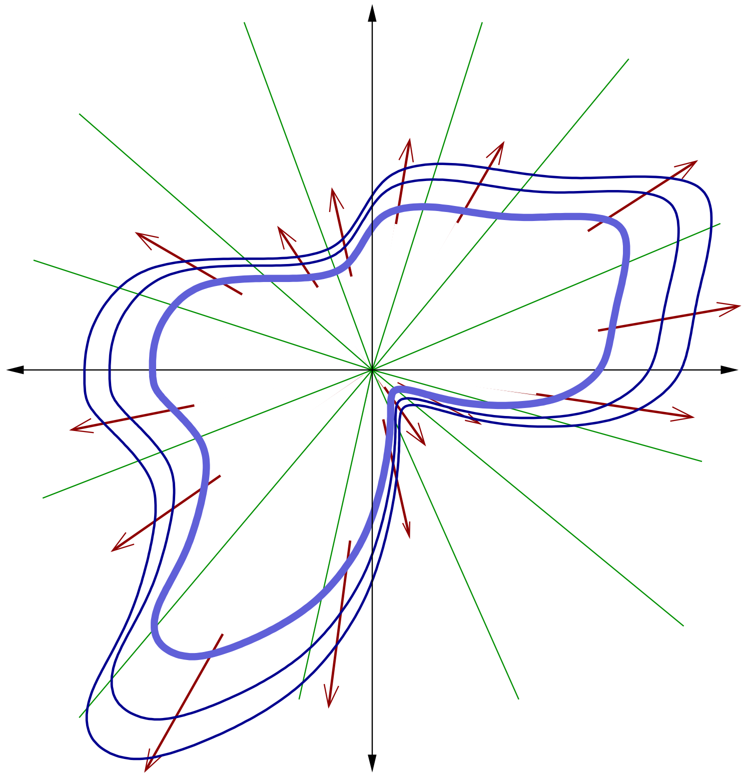

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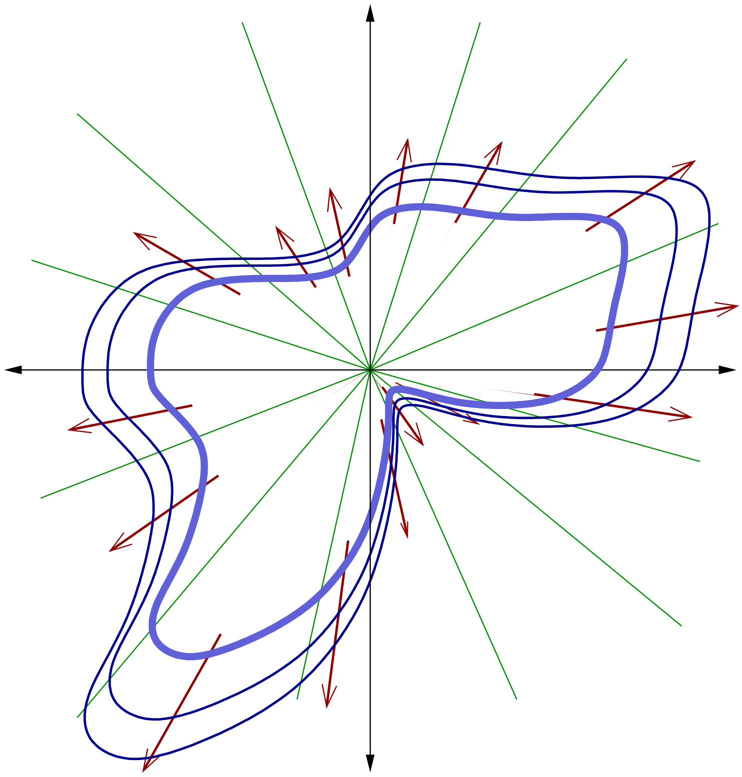

Definition. A **symplectic structure** on a $2n$-dimensional manifold W is a system of local coordinate systems $(q_1, p_1, \ldots, q_n, p_n)$ in which Hamilton's equations are invariant.

It carries a natural volume form:

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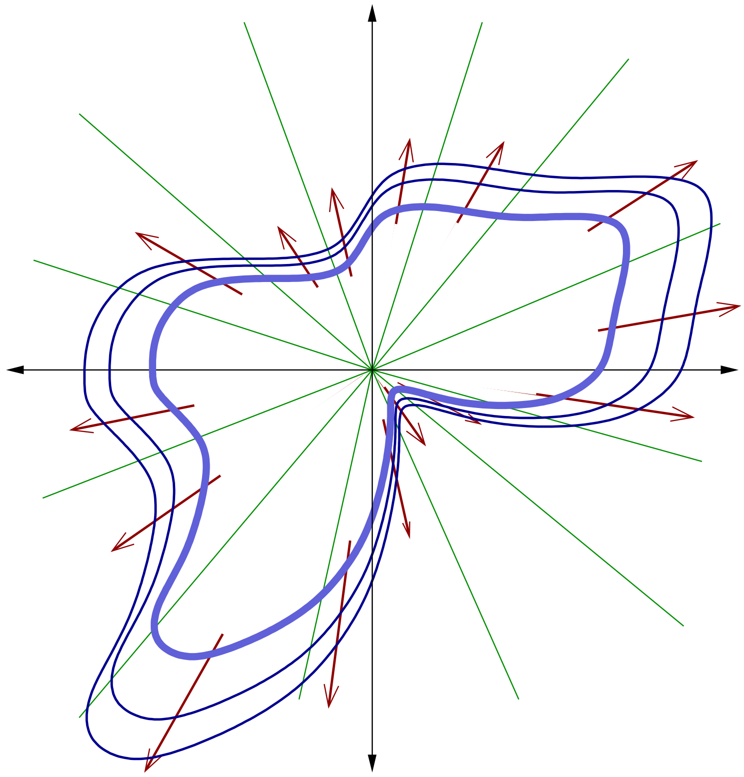

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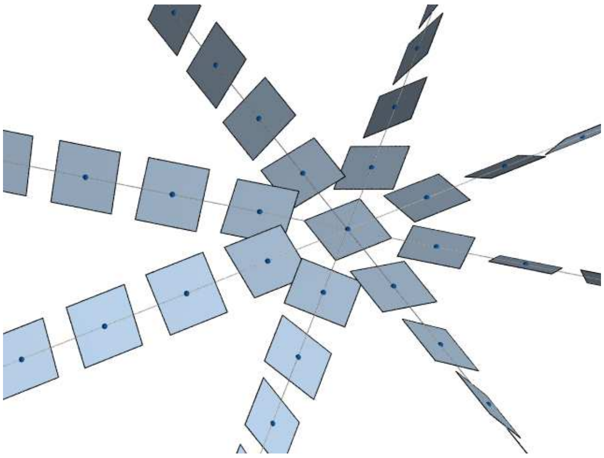
∂W is **convex** if it is transverse to a vector field Y that *dilates* the symplectic structure.

$M := \partial W$ convex $\rightsquigarrow$ **contact structure**

$$\xi \subset TM,$$

a field of tangent hyperplanes that are "locally twisted" (*maximally nonintegrable*),



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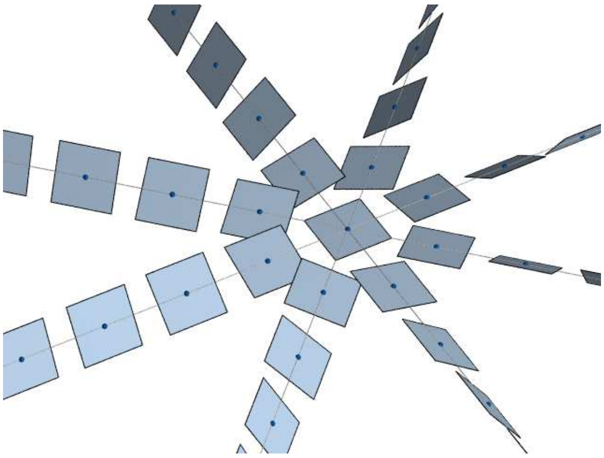

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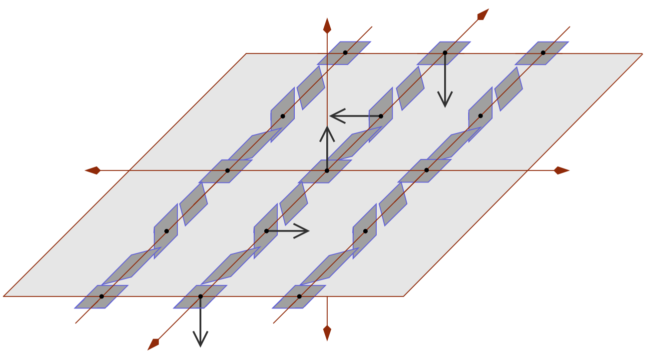
Example: $T^3 := S^1 \times S^1 \times S^1$



$= \text{boundary of } T^2 \times \mathbb{D} = D^*T^2 \subset T^*T^2.$

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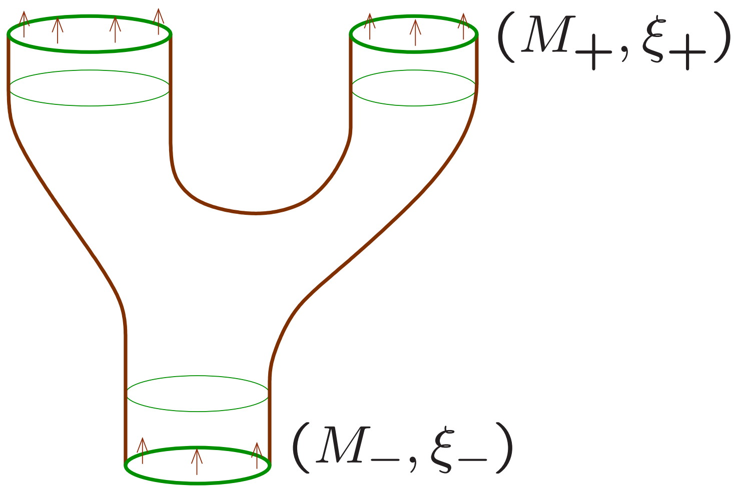

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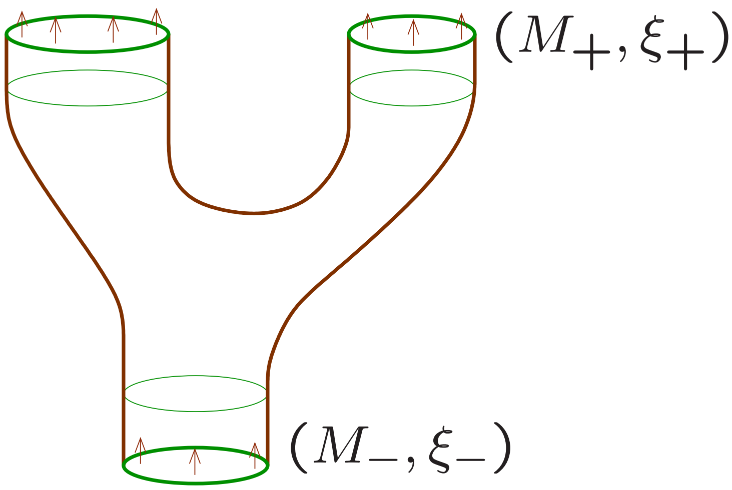
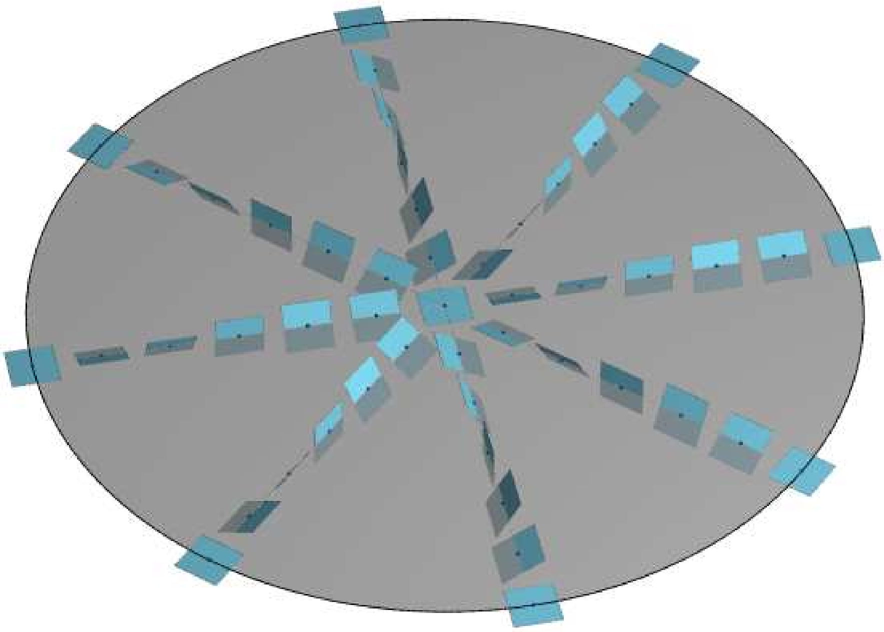


When is $(M_-, \xi_-) \prec (M_+, \xi_+)$?
When is $\emptyset \prec (M, \xi)$? (Is it *fillable*?)

Overtwisted vs. tight

Theorem (Eliashberg '89).
If ξ_1 and ξ_2 are both overtwisted, then
$(M, \xi_1) \cong (M, \xi_2) \Leftrightarrow \xi_1$ *and* ξ_2 *are homotopic.*



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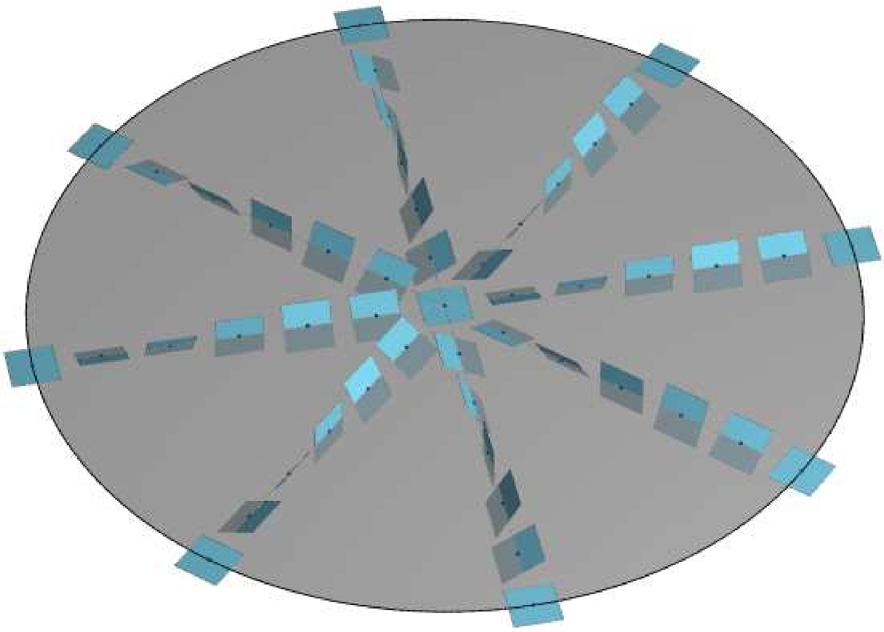

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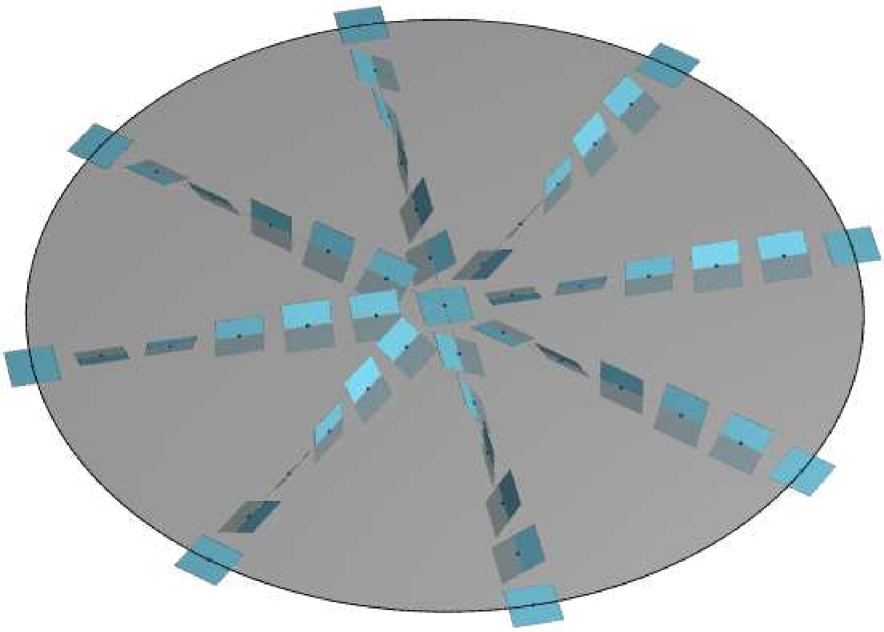

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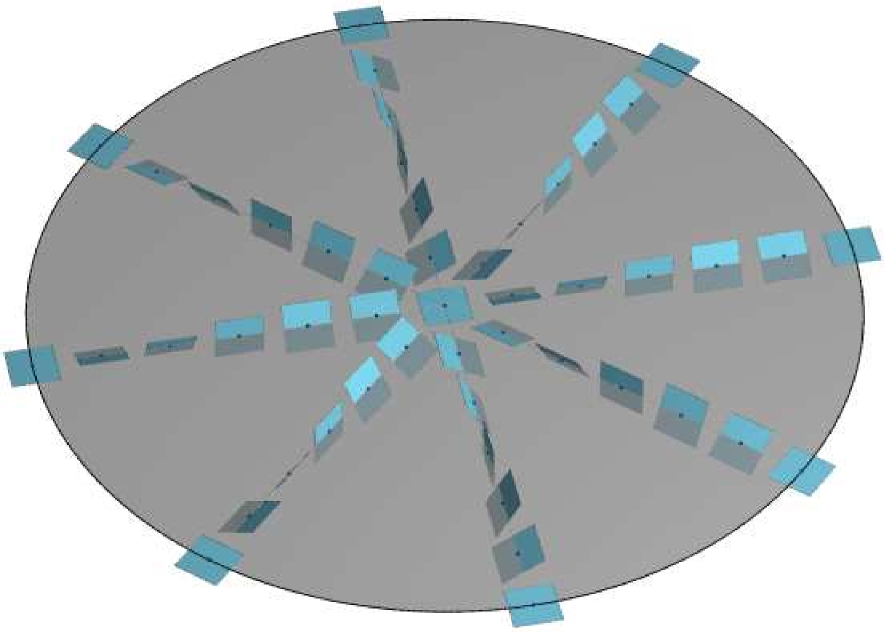

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Non-overtwisted contact structures are called **"tight"**.

They are not fully understood.

Conjecture.

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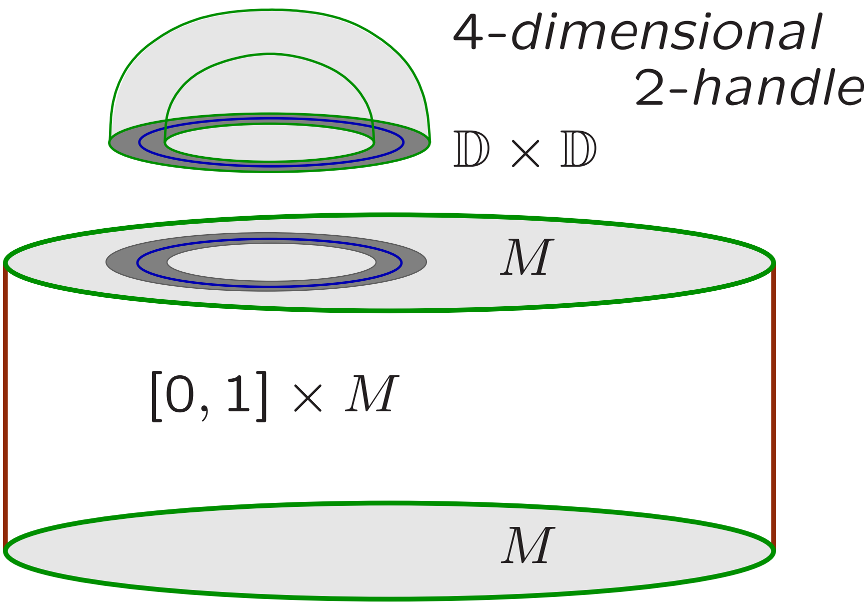

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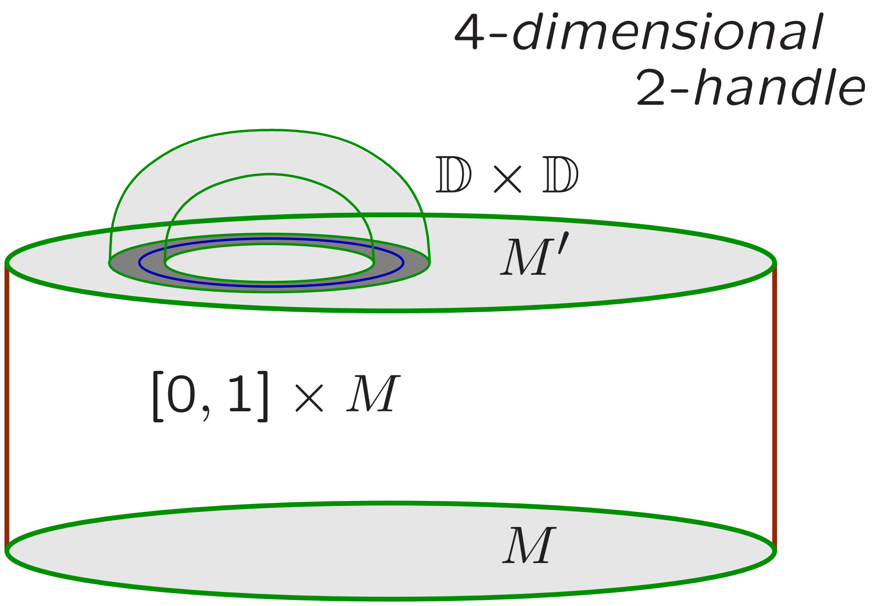

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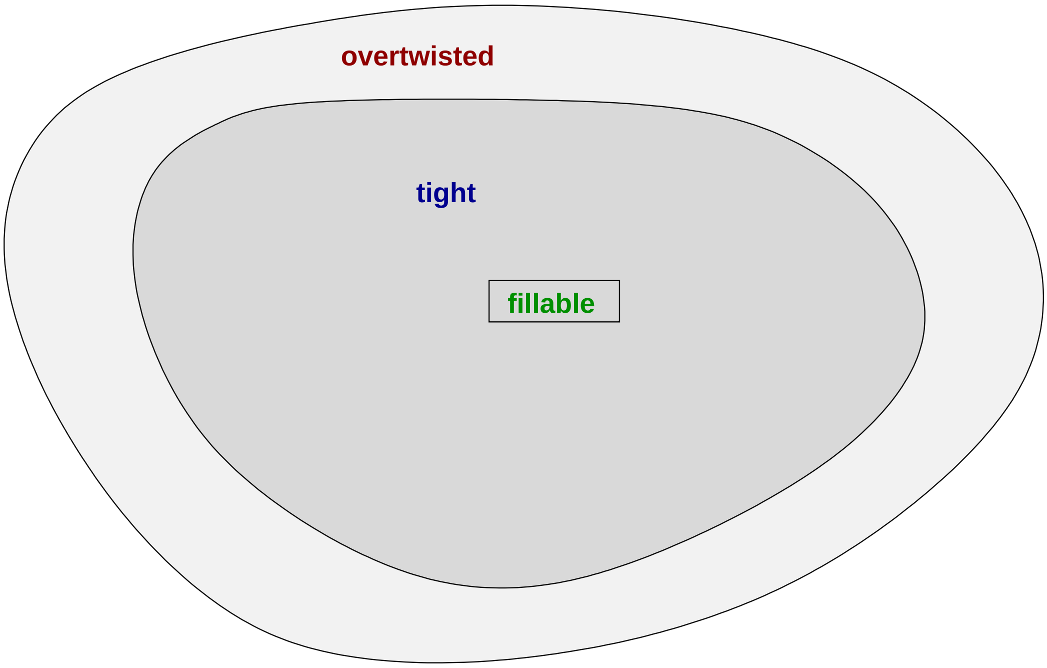
Recent results: $\exists$ "degrees of tightness".

Theorem (Latschev-W. 2010)**.**



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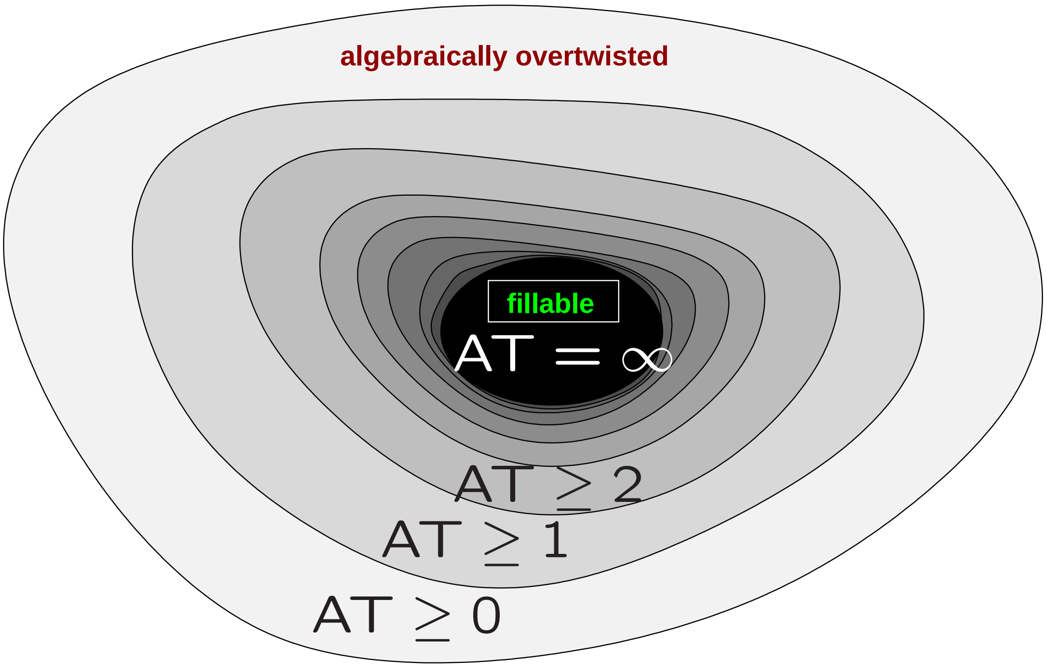

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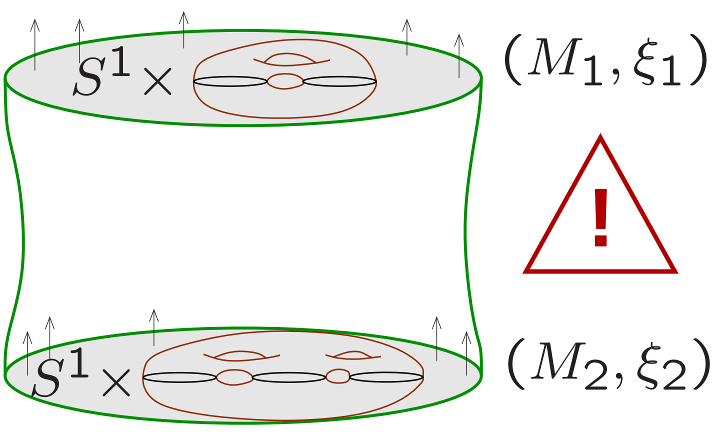

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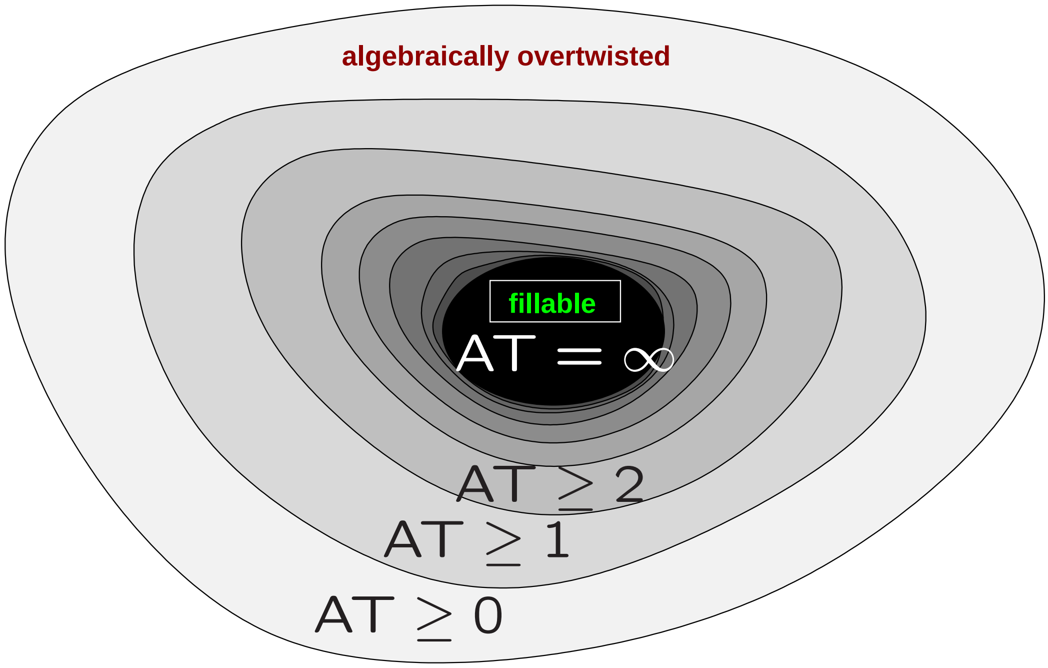

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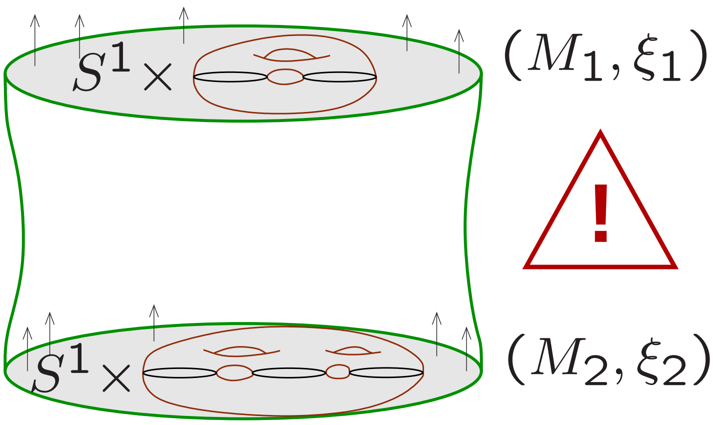

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(Eliashberg-Givental-Hofer '00 + Cieliebak-Latschev '09)

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"Theorem": There exists $\mathcal{H} \in \mathcal{W}$ with $\mathcal{H}^2 = 0$ such that $D_{\mathcal{H}}(1) = 0$ and

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Symplectic cobordism $(M_-, \xi_-) \prec (M_+, \xi_+)$
$\rightsquigarrow$ **natural map**

$$H_*^{\mathrm{SFT}}(M_+, \xi_+) \to H_*^{\mathrm{SFT}}(M_-, \xi_-)$$

preserving elements of $\mathbb{R}[[\hbar]]$.

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all "interesting" contact invariants vanish:

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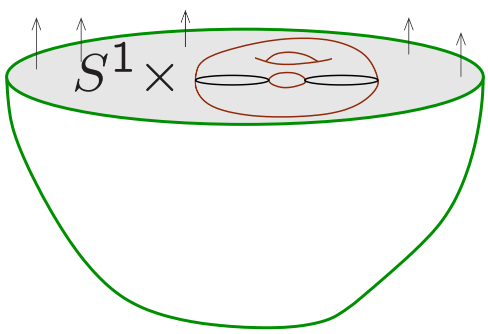

$$\begin{tikzcd}[\hbar] = 0 \in H_*^{\mathrm{SFT}}(M,\xi) \arrow[d] \\ [\hbar] \neq 0 \in H_*^{\mathrm{SFT}}(\emptyset)\end{tikzcd}$$

A beautiful idea (Witten '82 + Floer '88):

(X, g) Riemannian manifold, $f : X \to \mathbb{R}$ generic Morse function. Then singular homology

$$H_*(X; \mathbb{Z}) \cong H_*\left(\mathbb{Z}^{\#\,\mathrm{Crit}(f)}, d_f\right),$$

where d_f counts rigid gradient flow lines,

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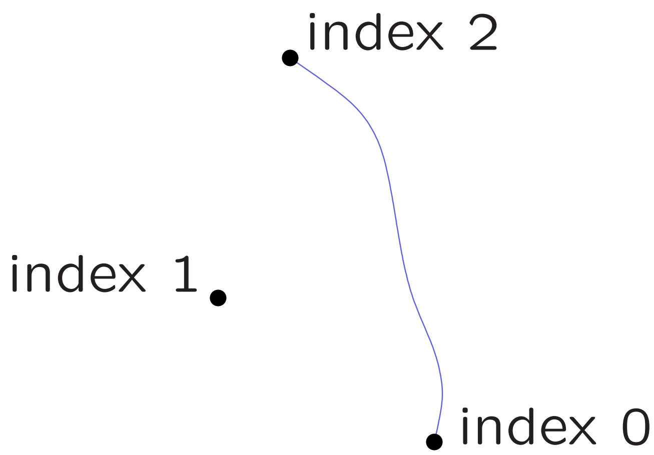

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$$H_*(X;\mathbb{Z}) \cong H_*\left(\mathbb{Z}^{\#\,\mathrm{Crit}(f)}, d_f\right),$$

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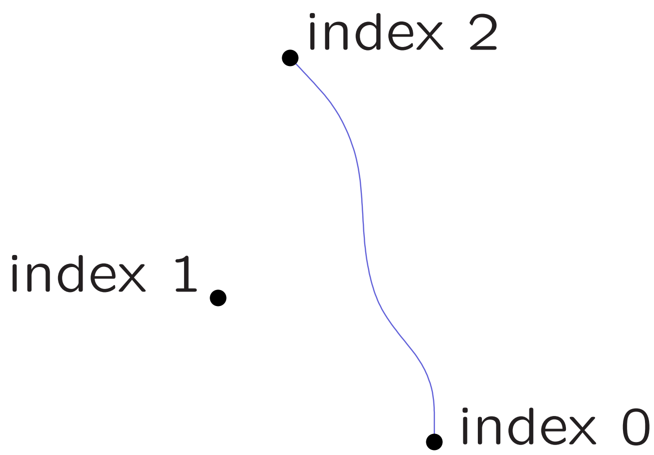

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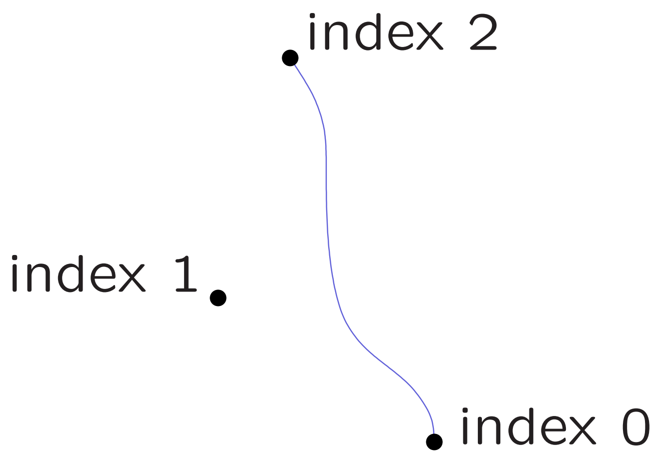

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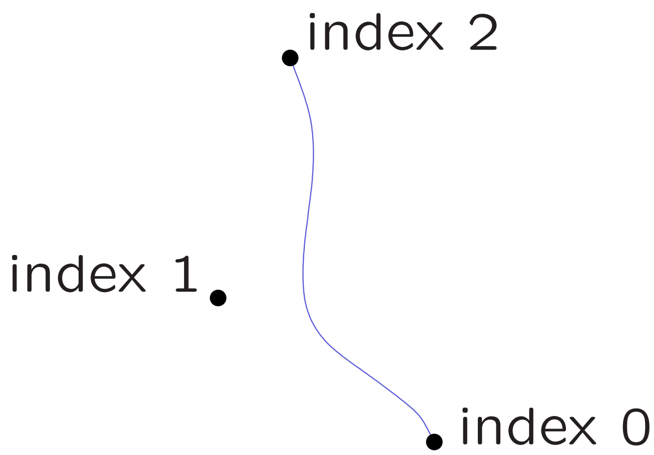

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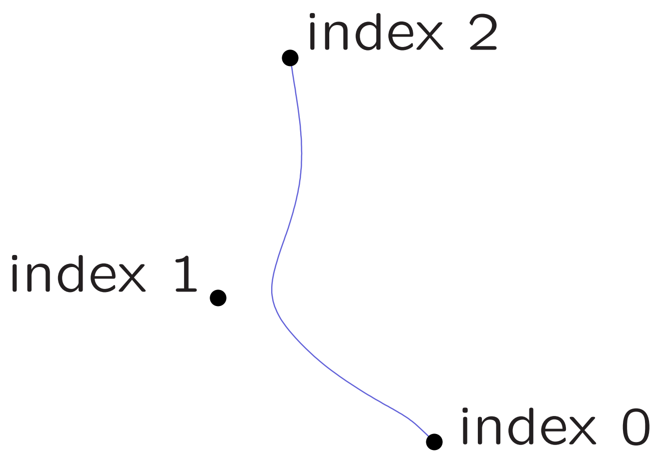

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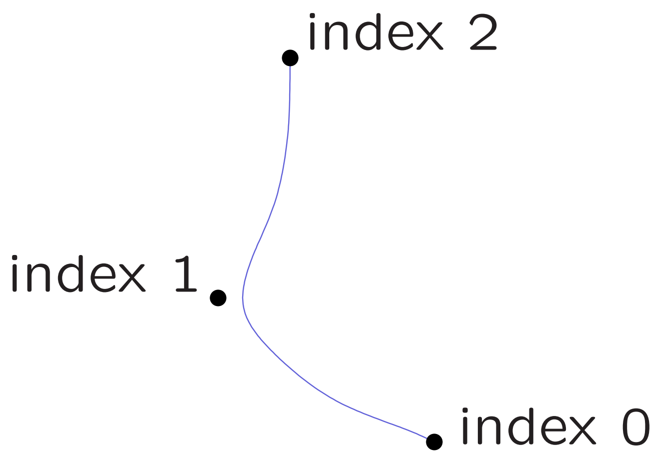

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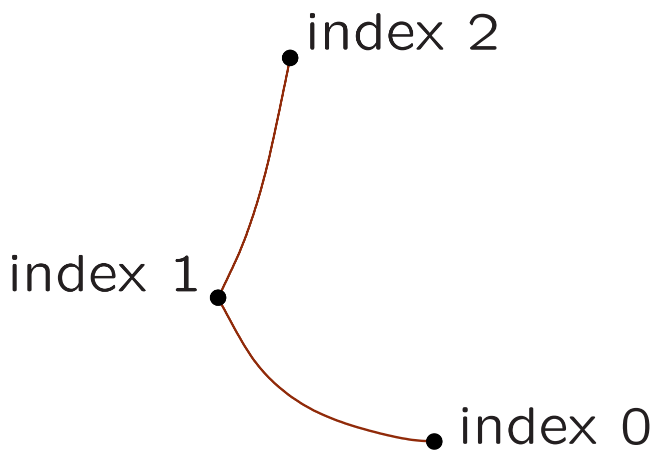

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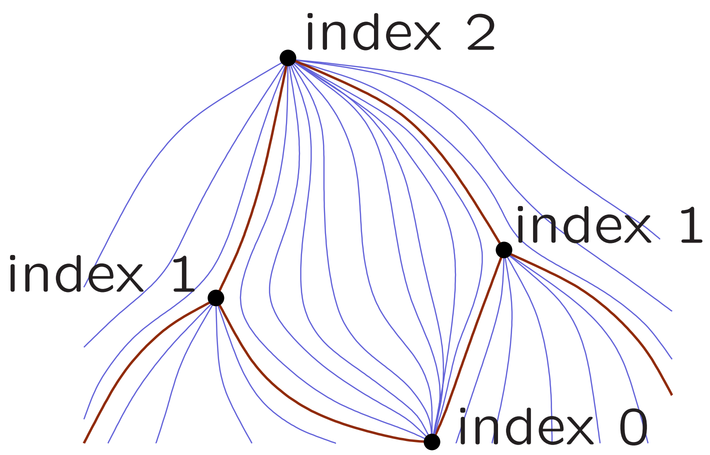

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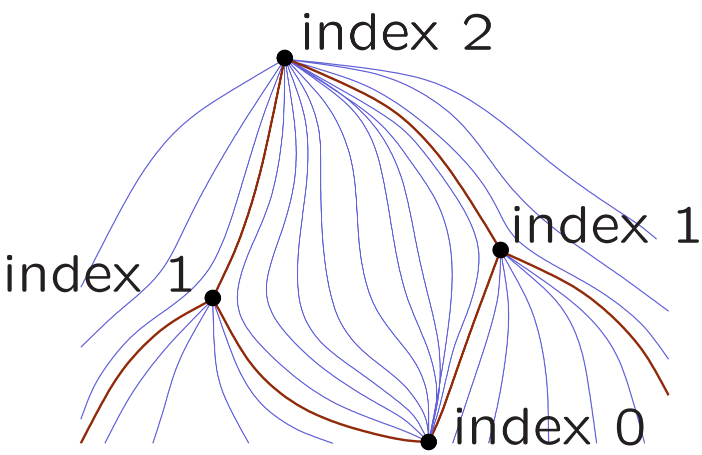

SFT of $(M, \xi = \ker \alpha)$:
"∞-dimensional Morse theory" for the *contact action functional*

$$\Phi : C^\infty(S^1, M) \to R : x \mapsto \int_{S^1} x^* \alpha,$$

with $\mathrm{Crit}(\Phi) = \{$periodic Reeb orbits$\}$.

Gradient flow:
Consider 1-parameter families of loops $\{u_s \in C^\infty(S^1, M)\}_{s\in\mathbb{R}}$ with

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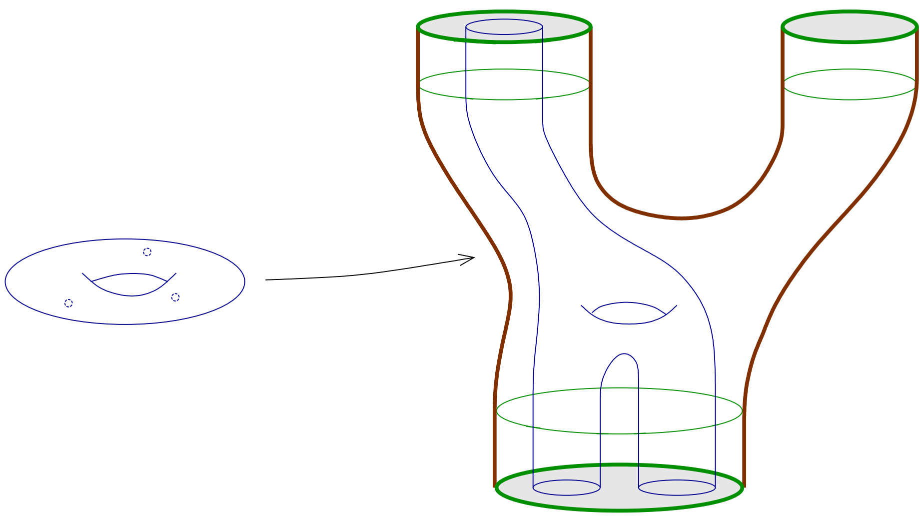
For a symplectic cobordism W and Riemann surface Σ, consider J-holomorphic curves

$$u : \Sigma \setminus \{z_1, \dots, z_n\} \to W$$

approaching Reeb orbits at the punctures.



The Cauchy-Riemann equation is **elliptic**:

$$\|u\|_{W^{1,p}} \leq \|u\|_{L^p} + \|\partial_s u + i\,\partial_t u\|_{L^p}$$

$\Rightarrow$ Spaces of holomorphic curves are (often)

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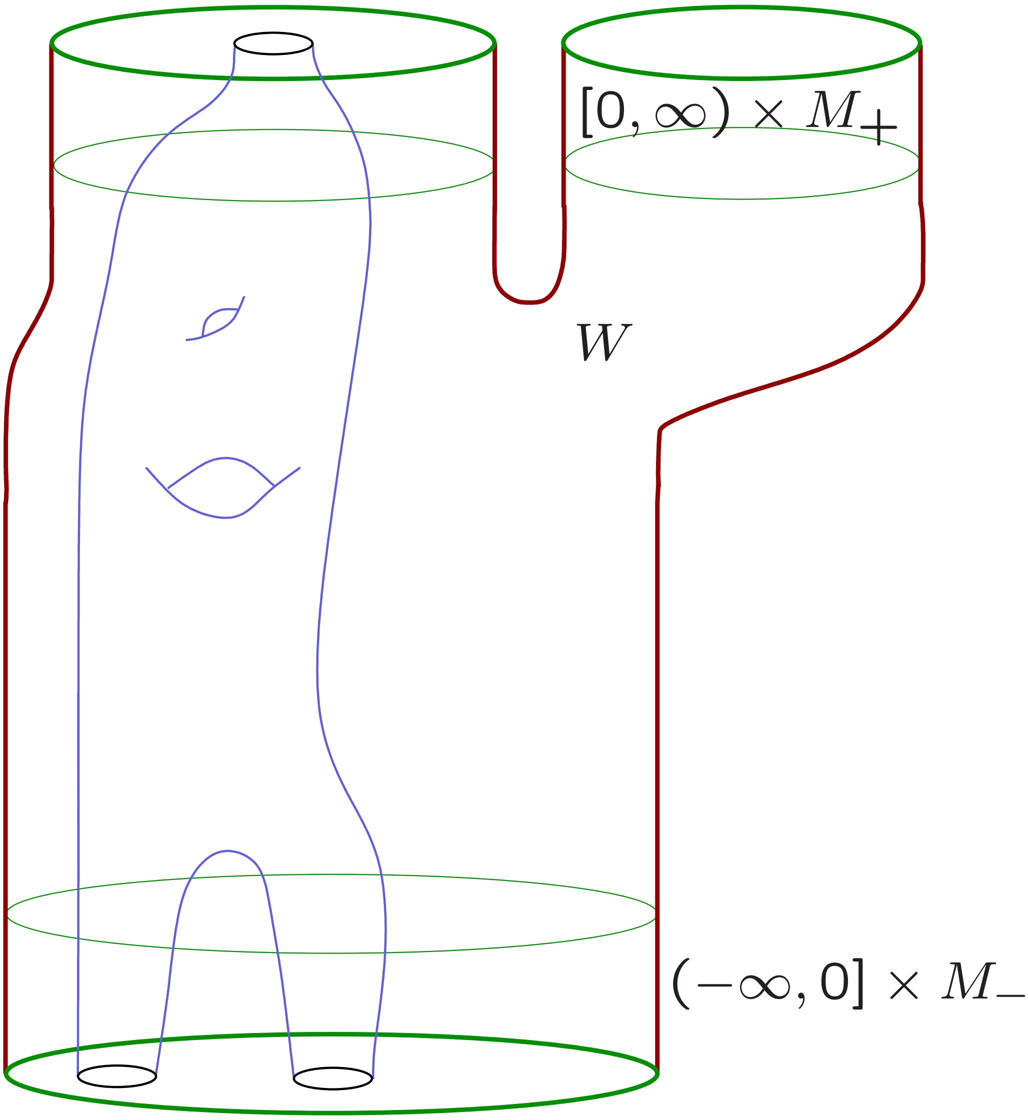

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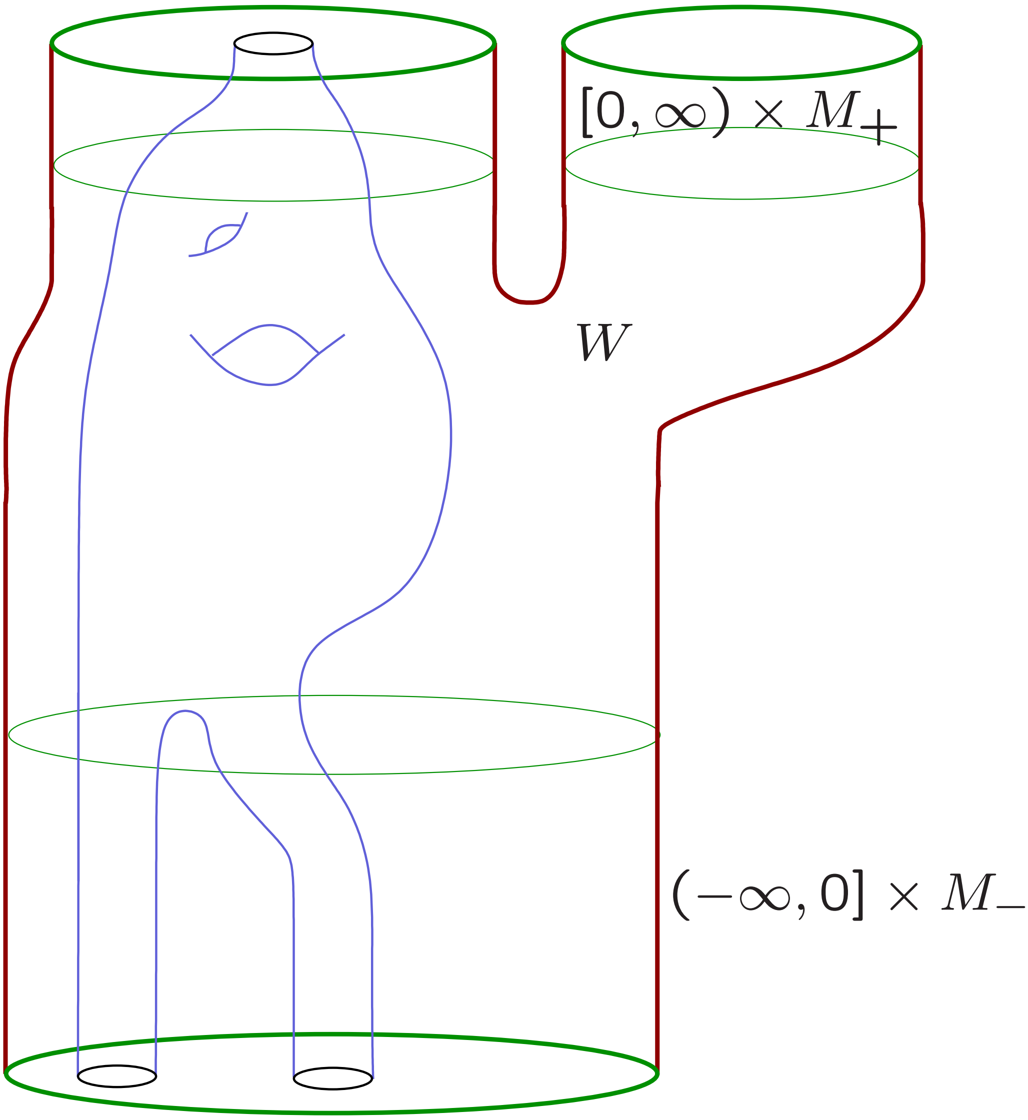

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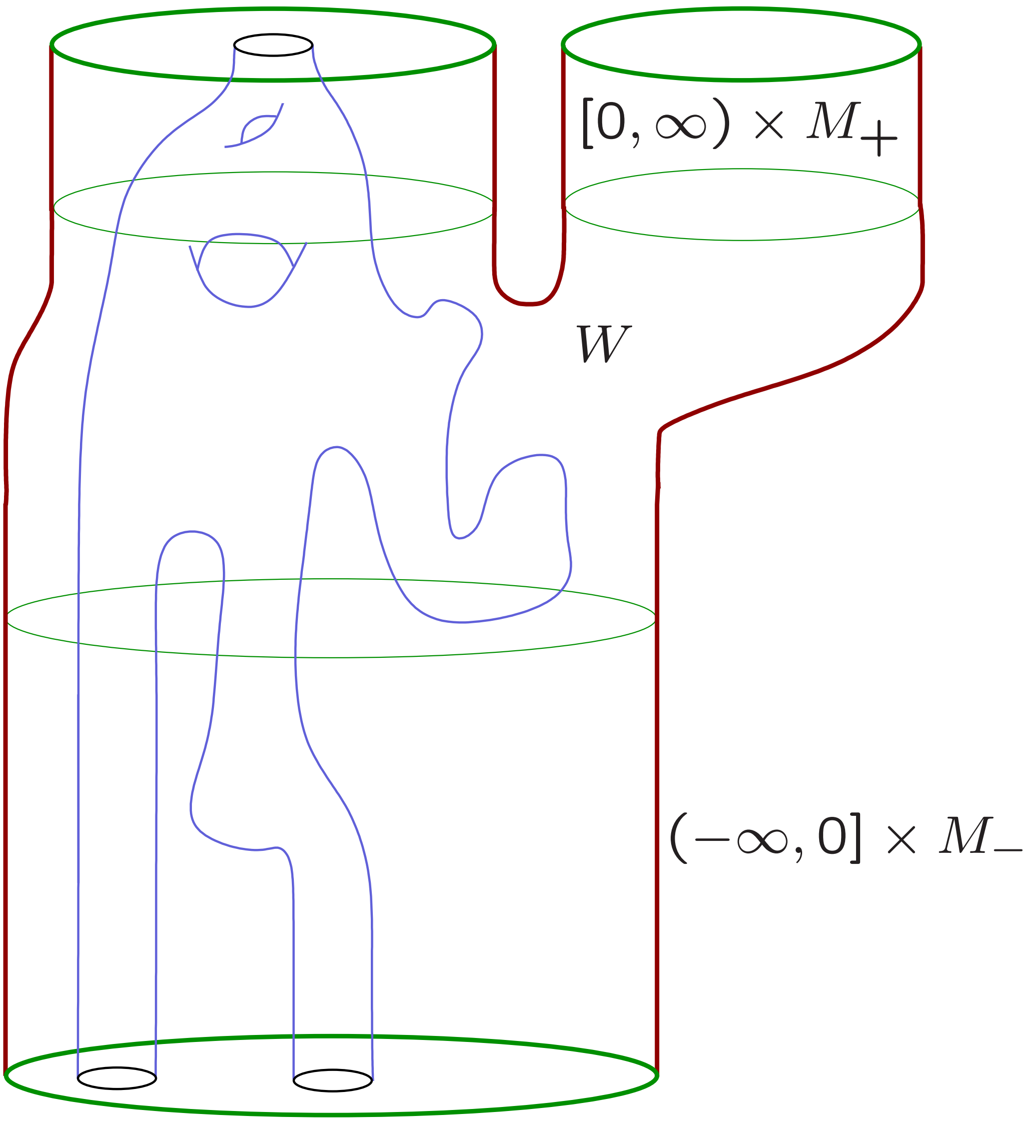

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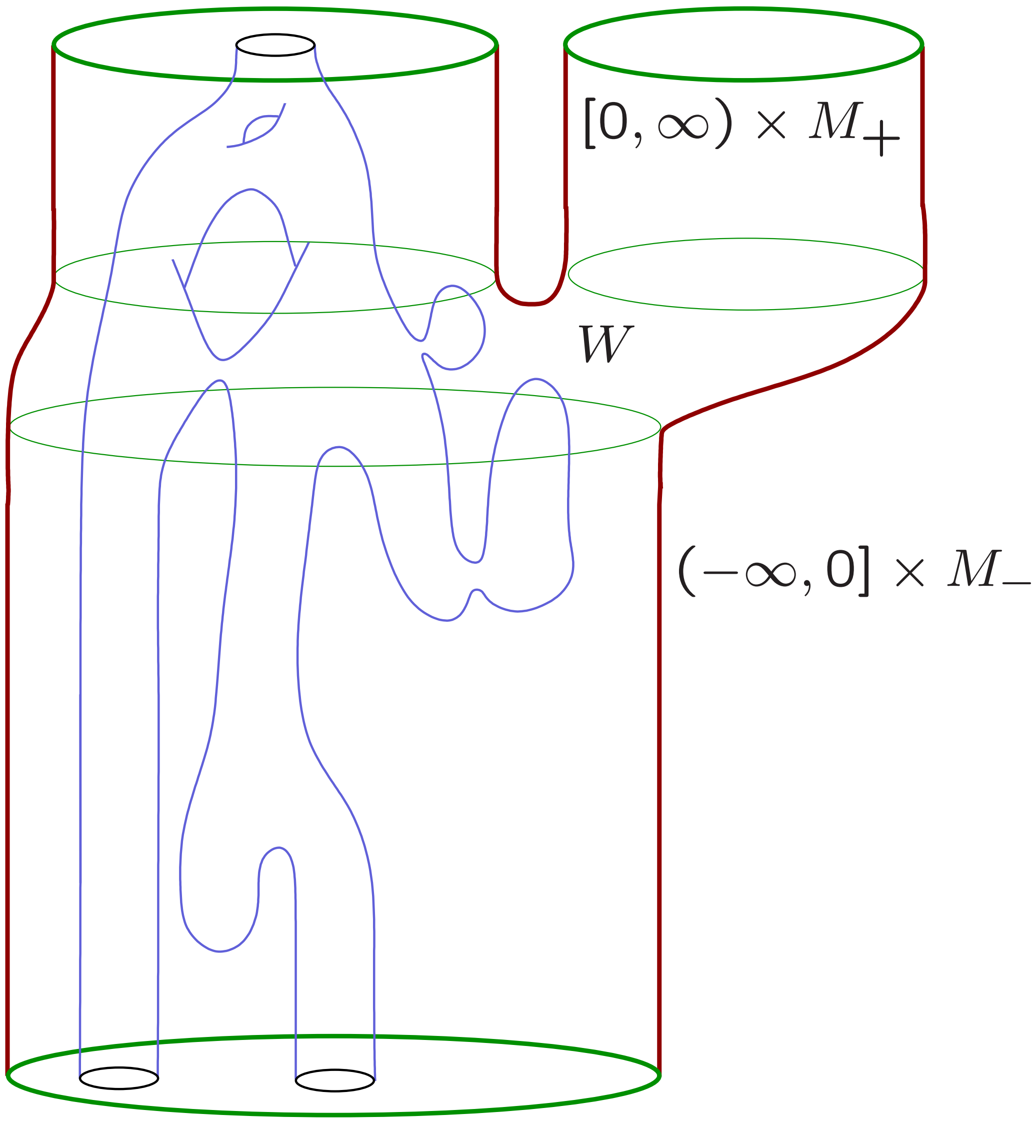

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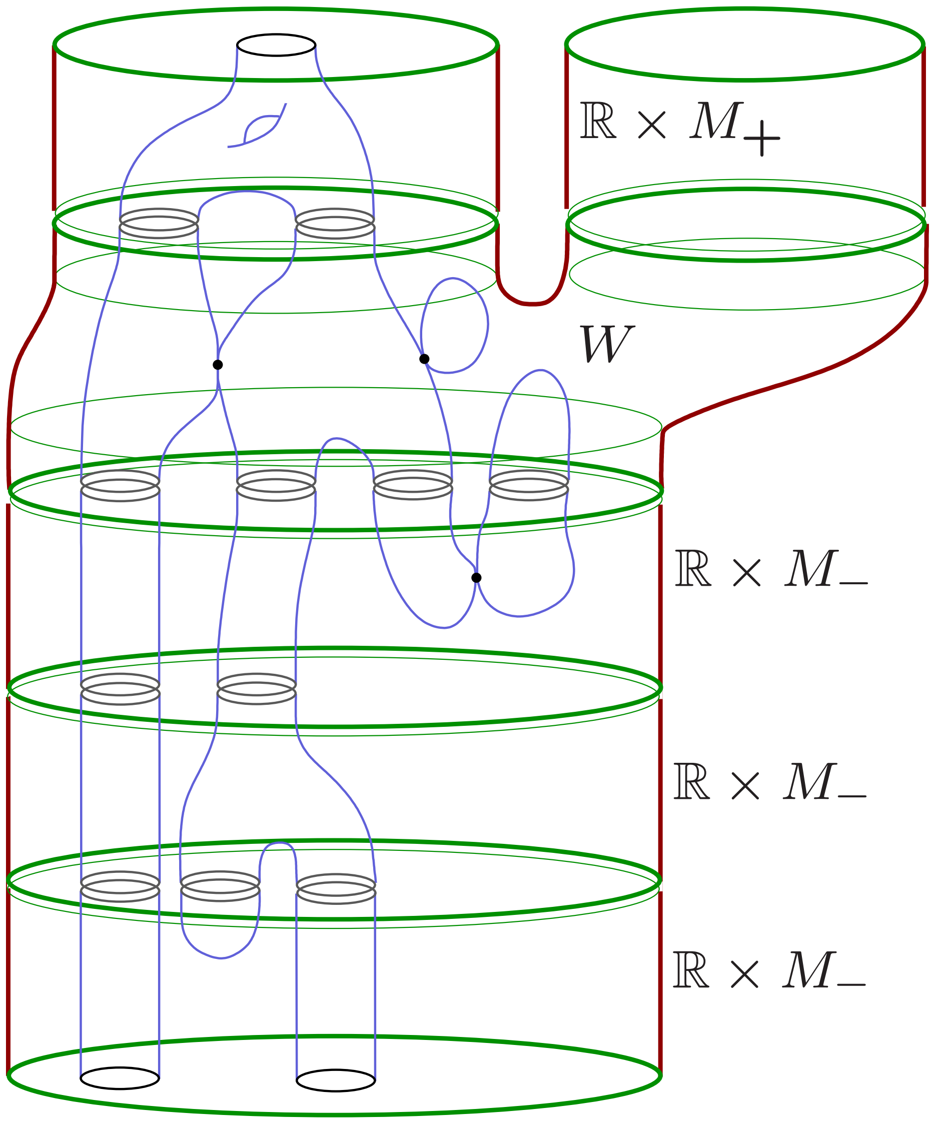

Definition of $\mathcal{H}$

$\Gamma^{\pm} := (\gamma_1^{\pm}, \ldots, \gamma_{k_{\pm}}^{\pm})$ lists of Reeb orbits

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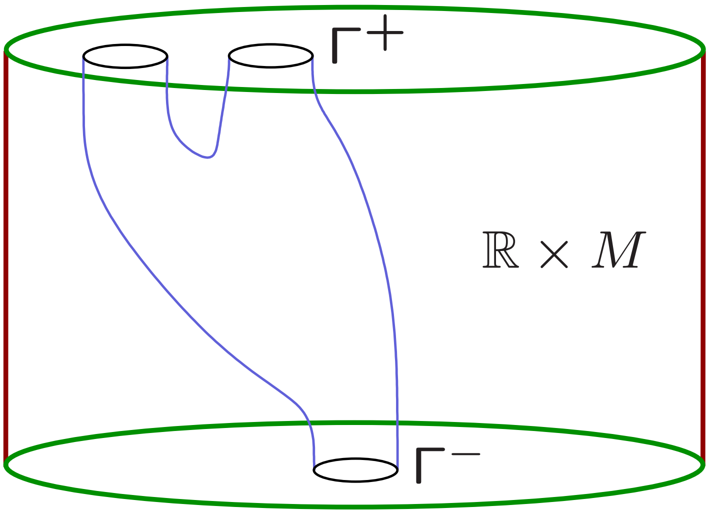

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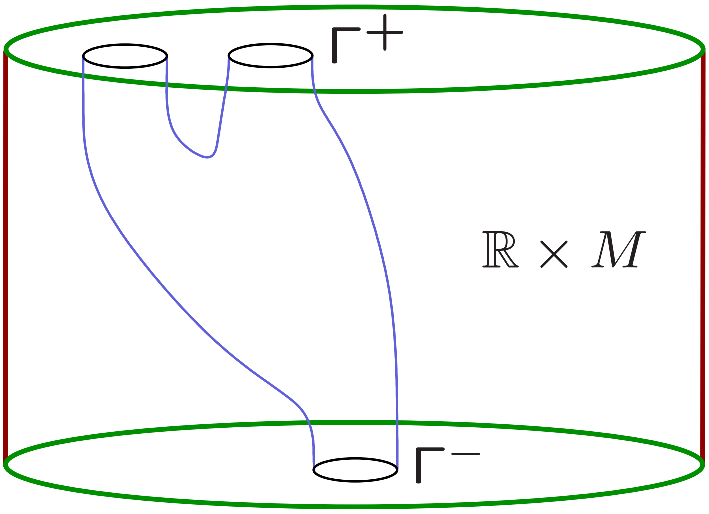

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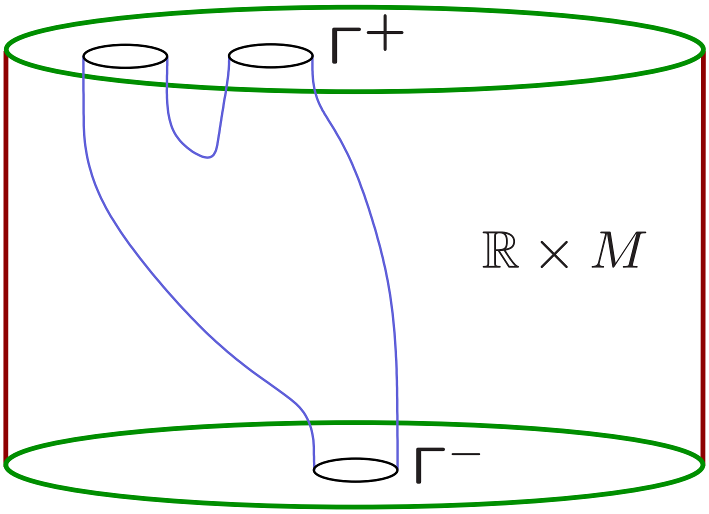

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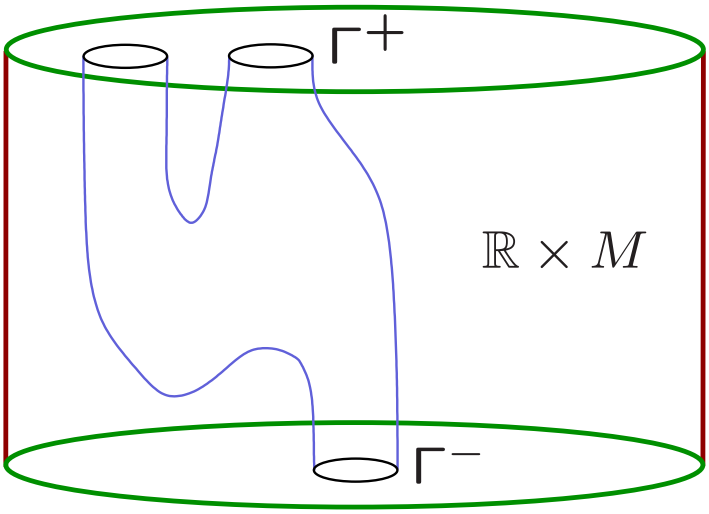

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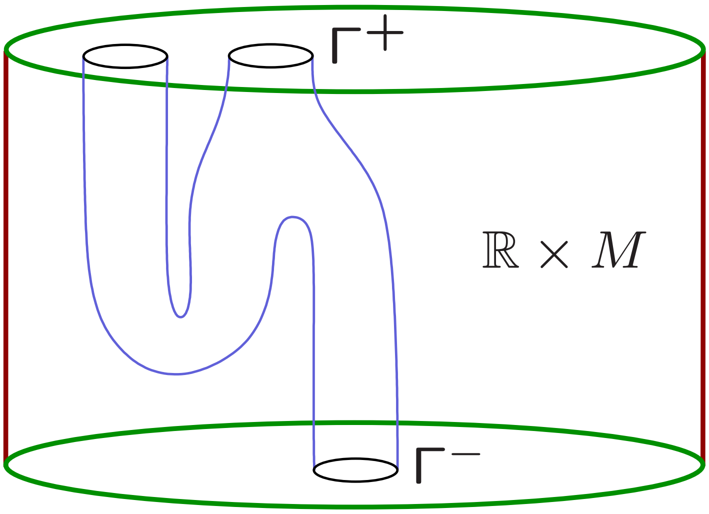

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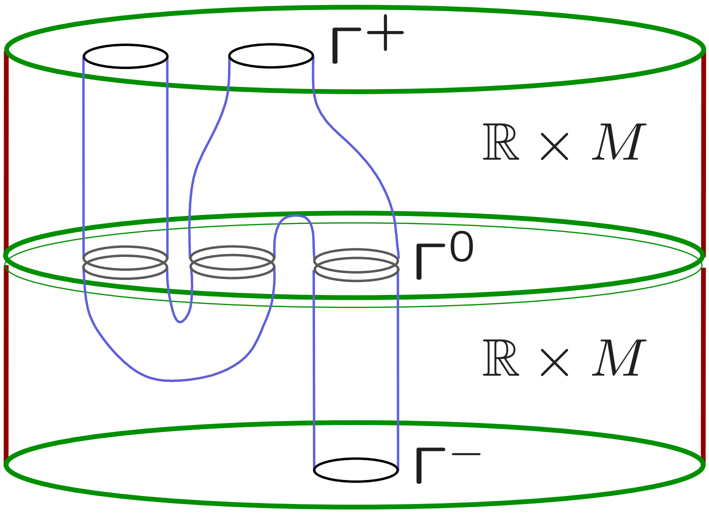

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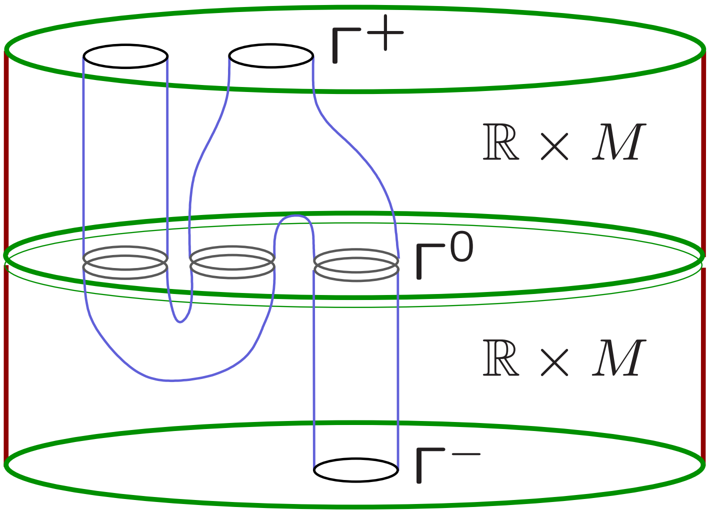

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$\mathcal{H}^2$ counts the boundary of a 1-dimensional space $\Rightarrow \mathcal{H}^2 = 0$.

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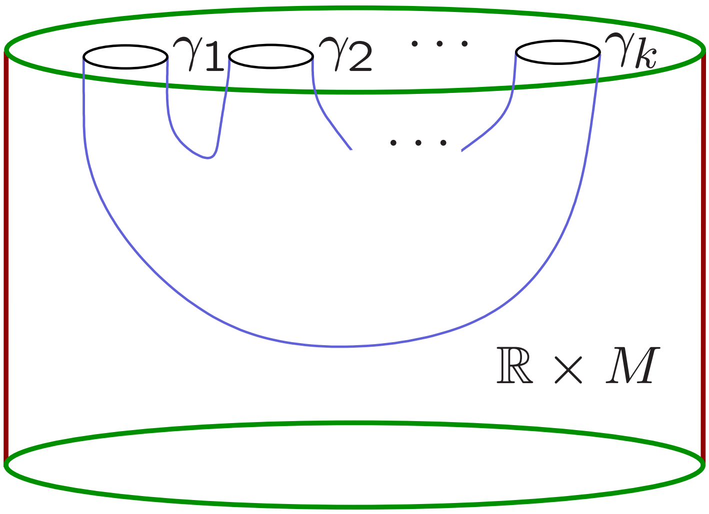
Suppose $\mathbb{R}\times M$ has exactly one rigid J-holomorphic curve, with genus 0, no negative ends, and positive ends at orbits $\gamma_1, \ldots, \gamma_k$.



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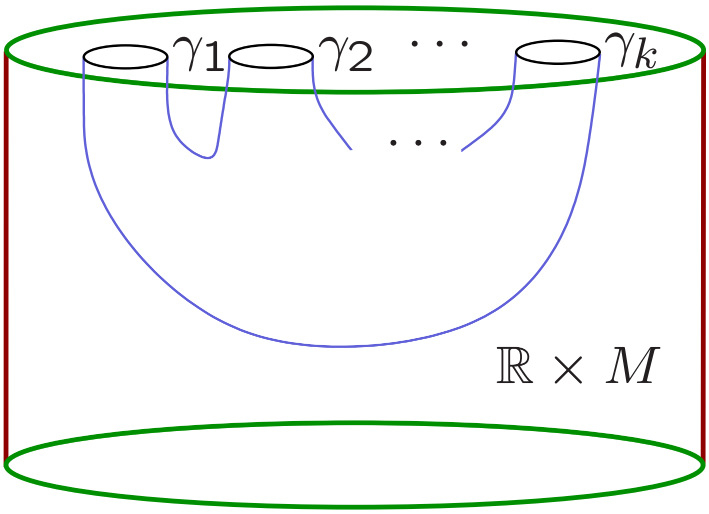

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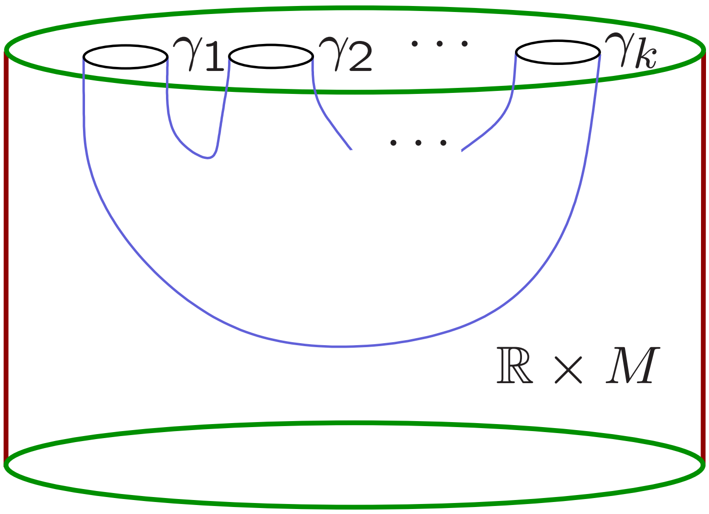

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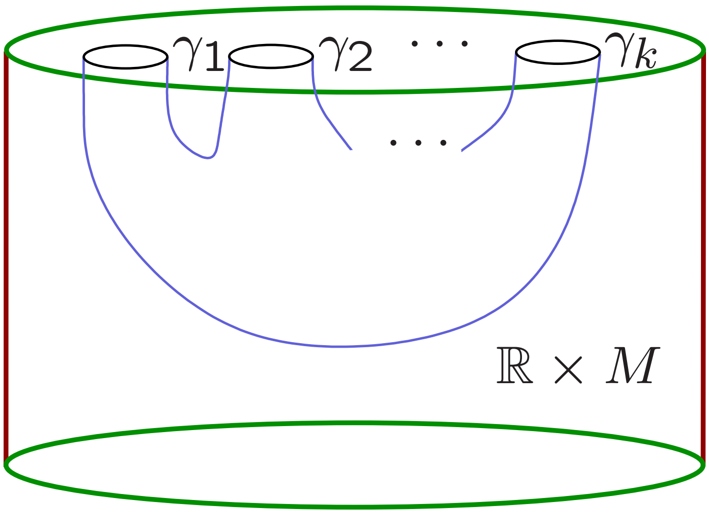

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$\Rightarrow [\hbar^{k-1}] = 0 \in H_*^{\mathsf{SFT}}(M,\xi)$

$\Rightarrow \mathsf{AT}(M,\xi) \leq k-1.$

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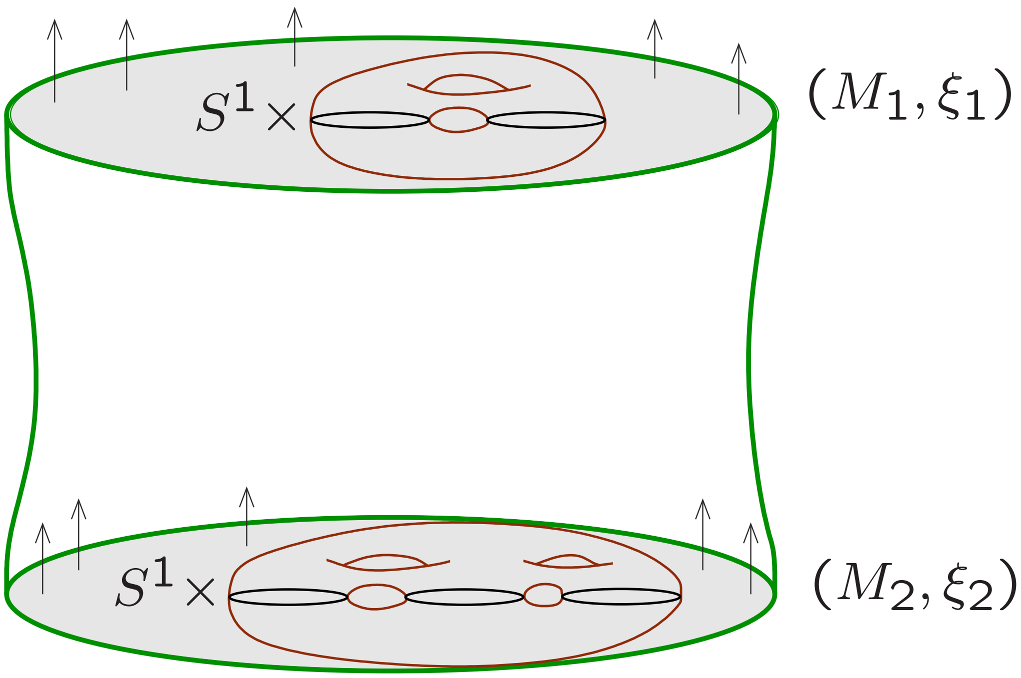

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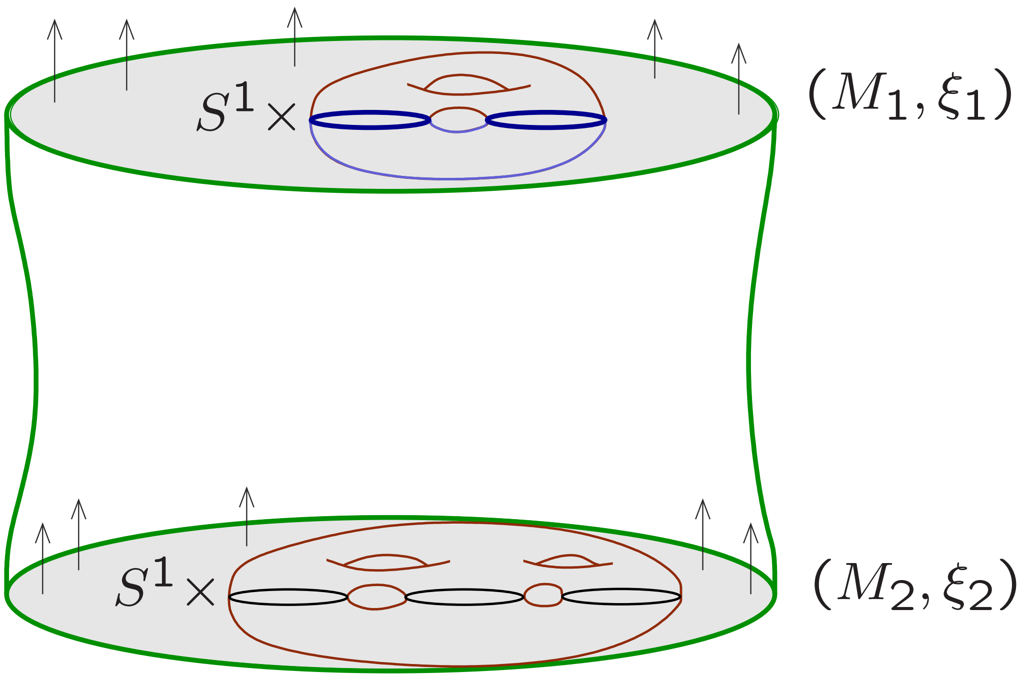

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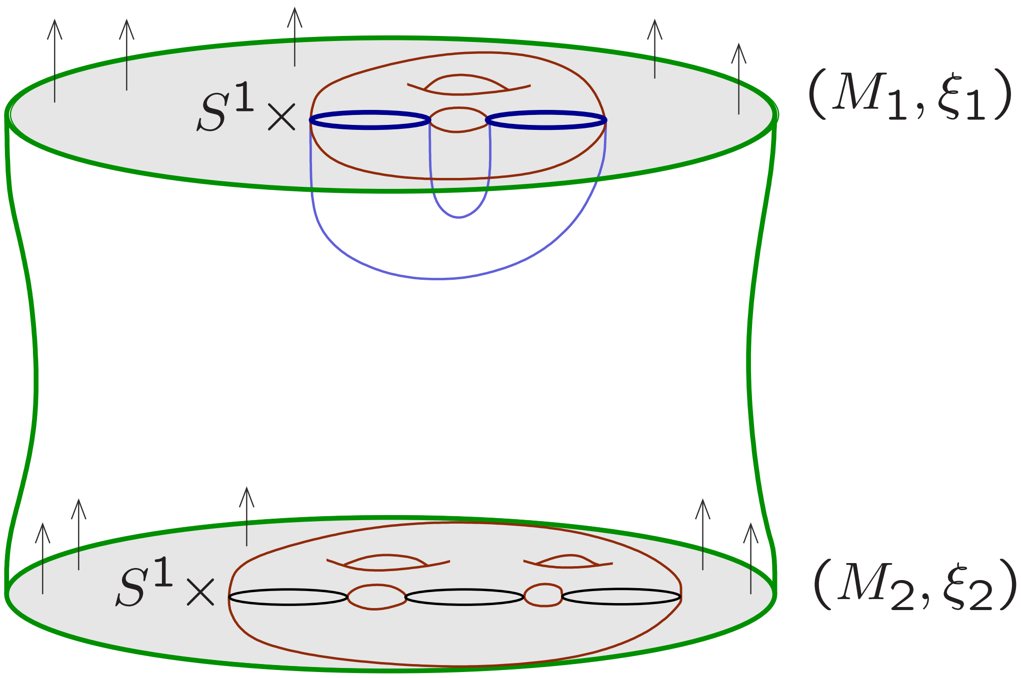

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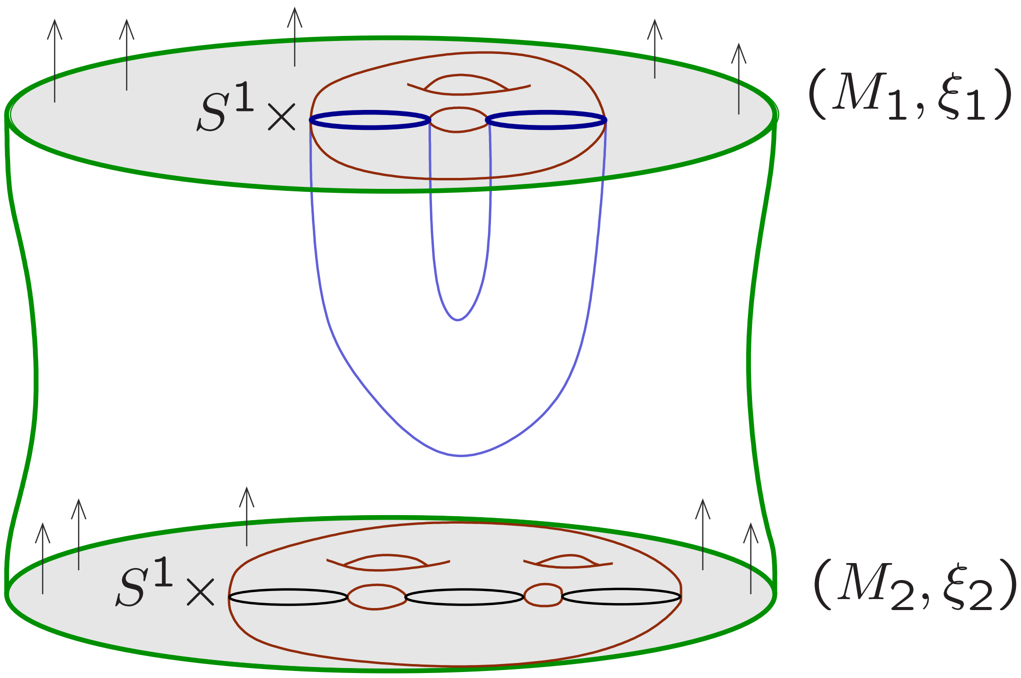

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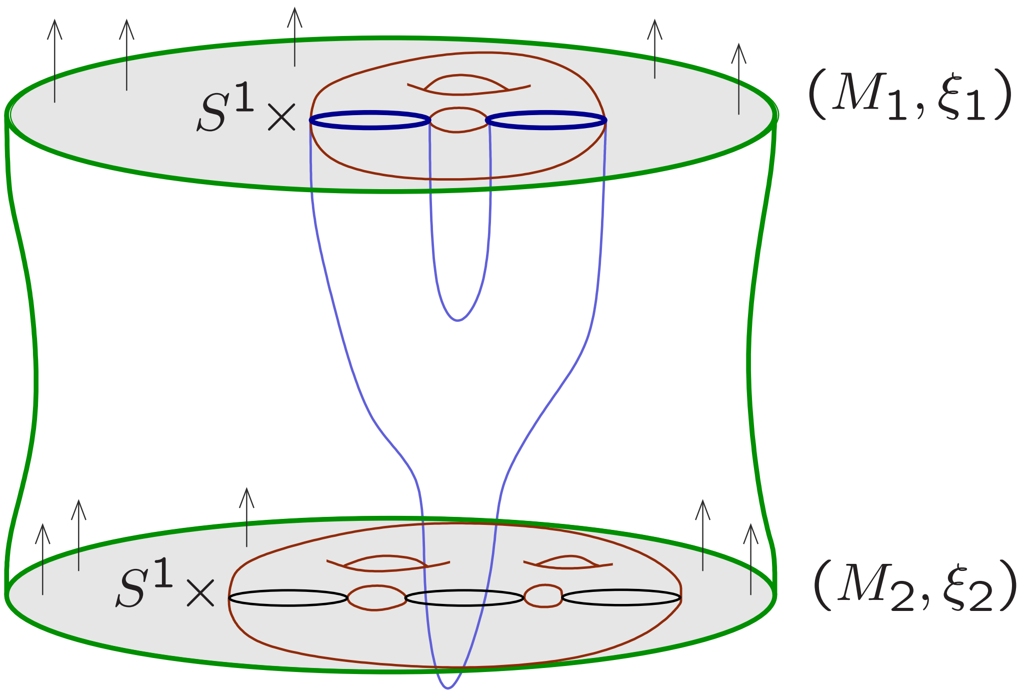

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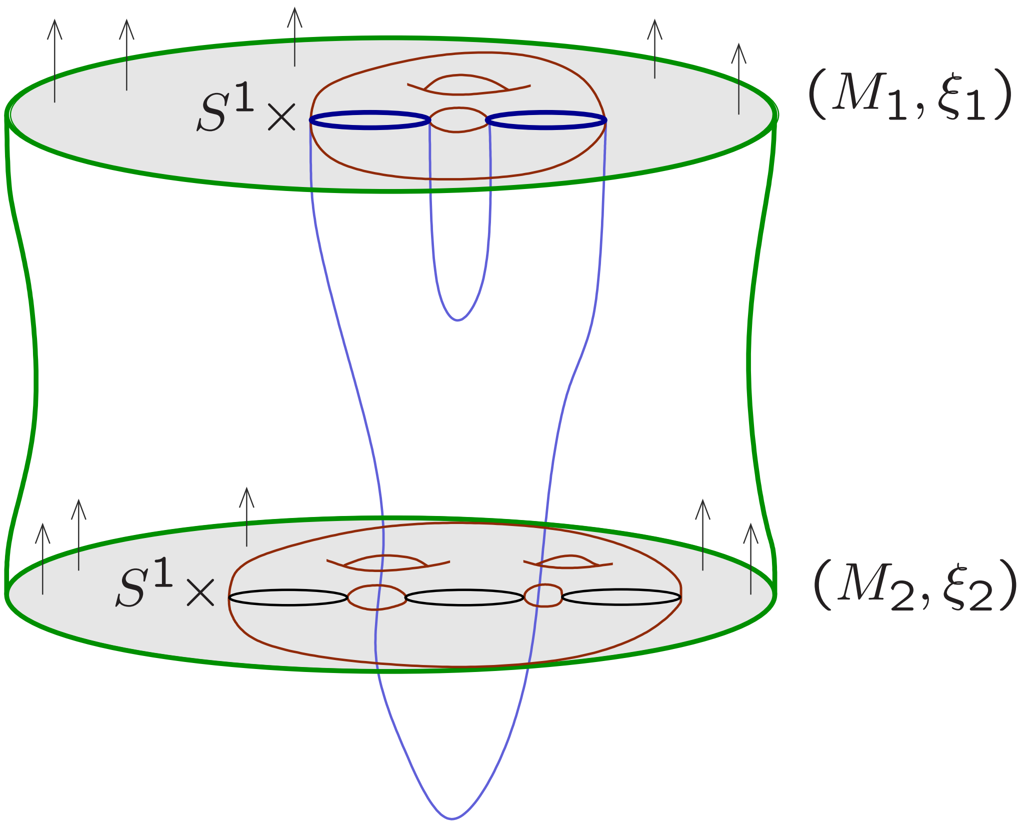

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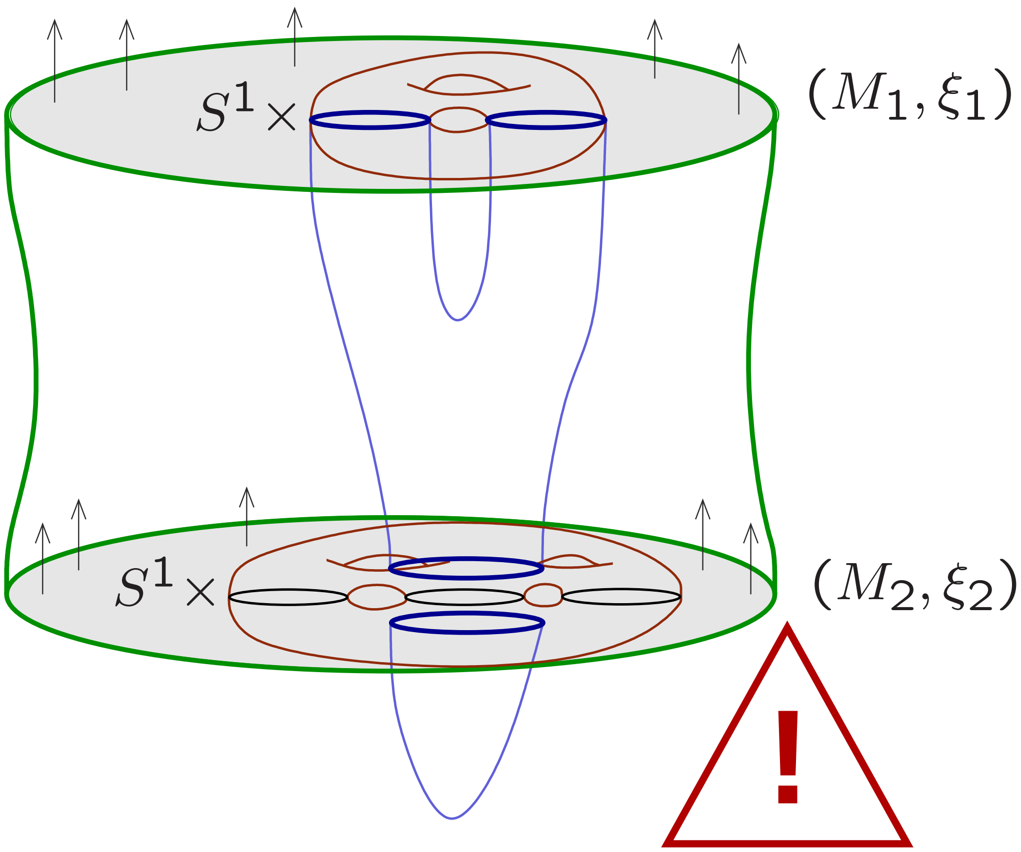

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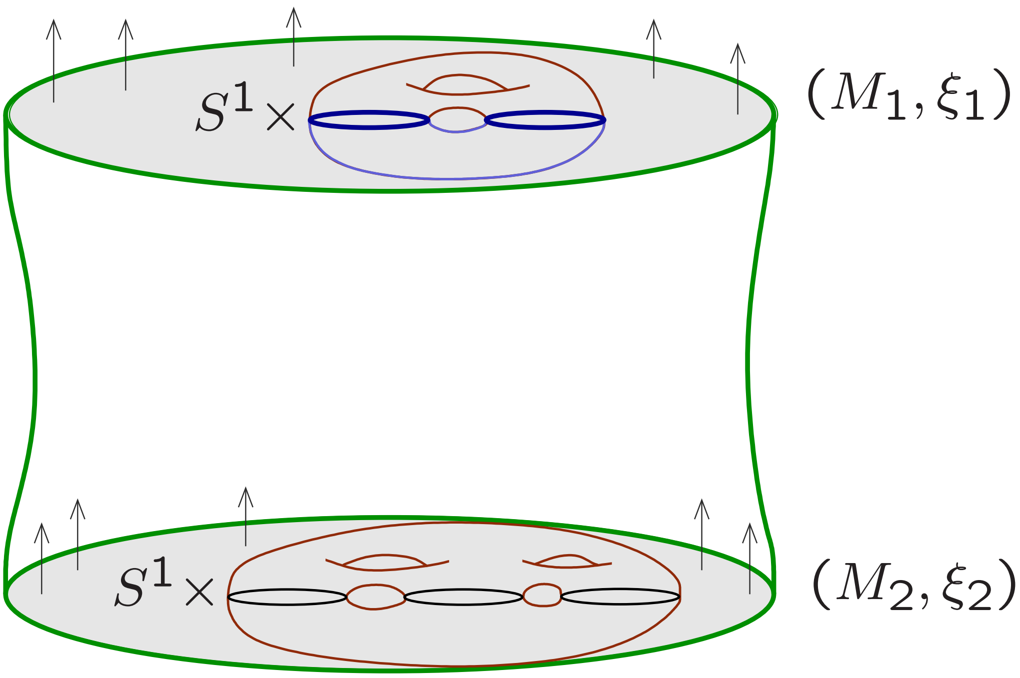

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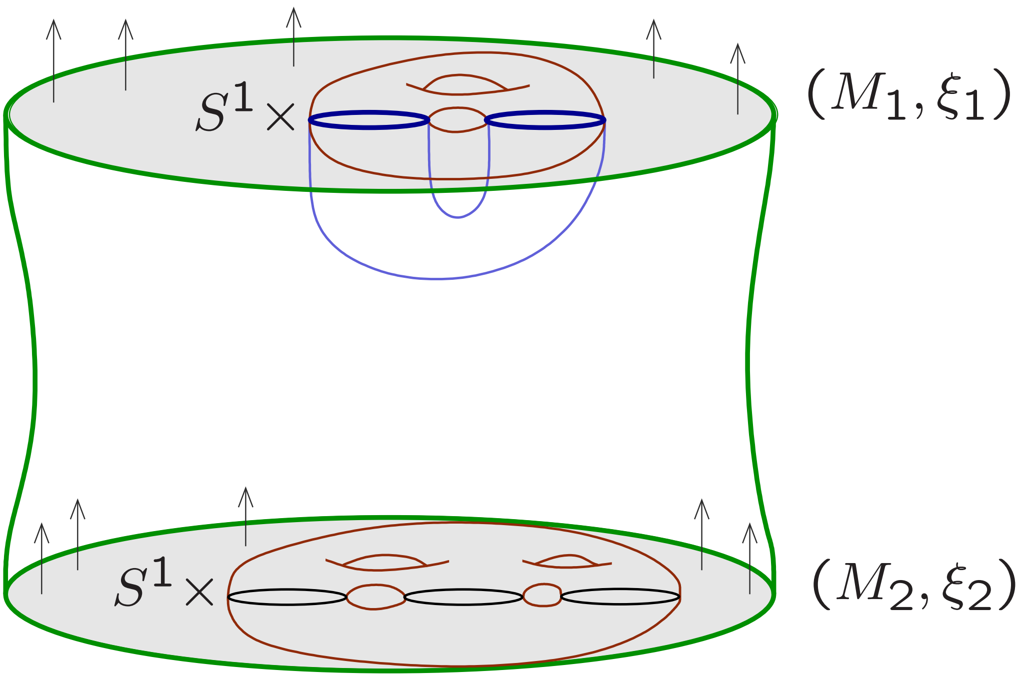

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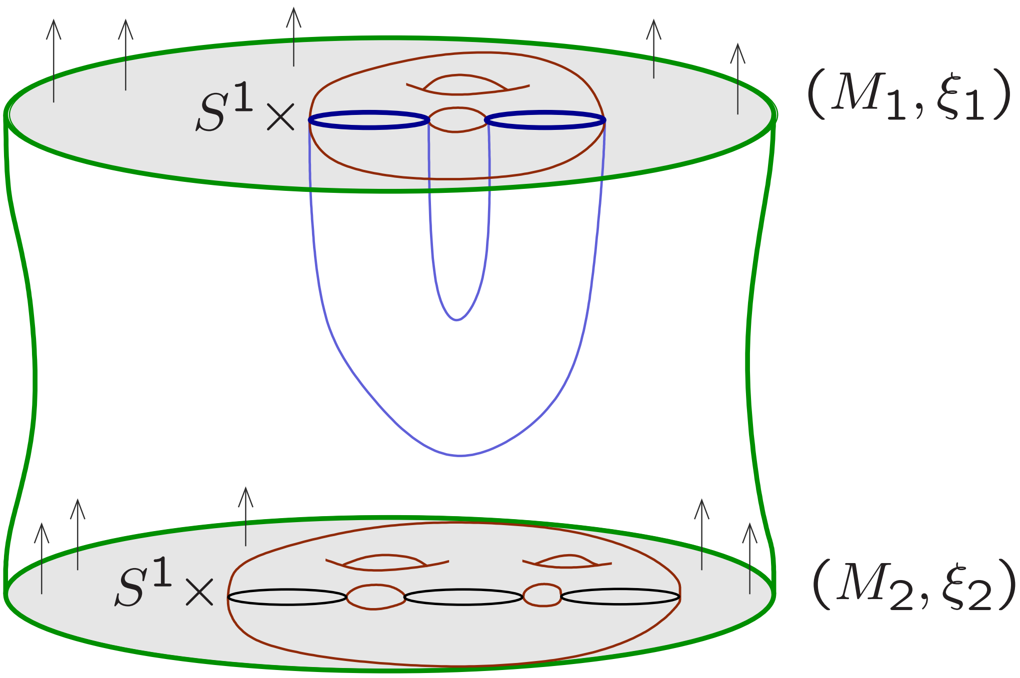

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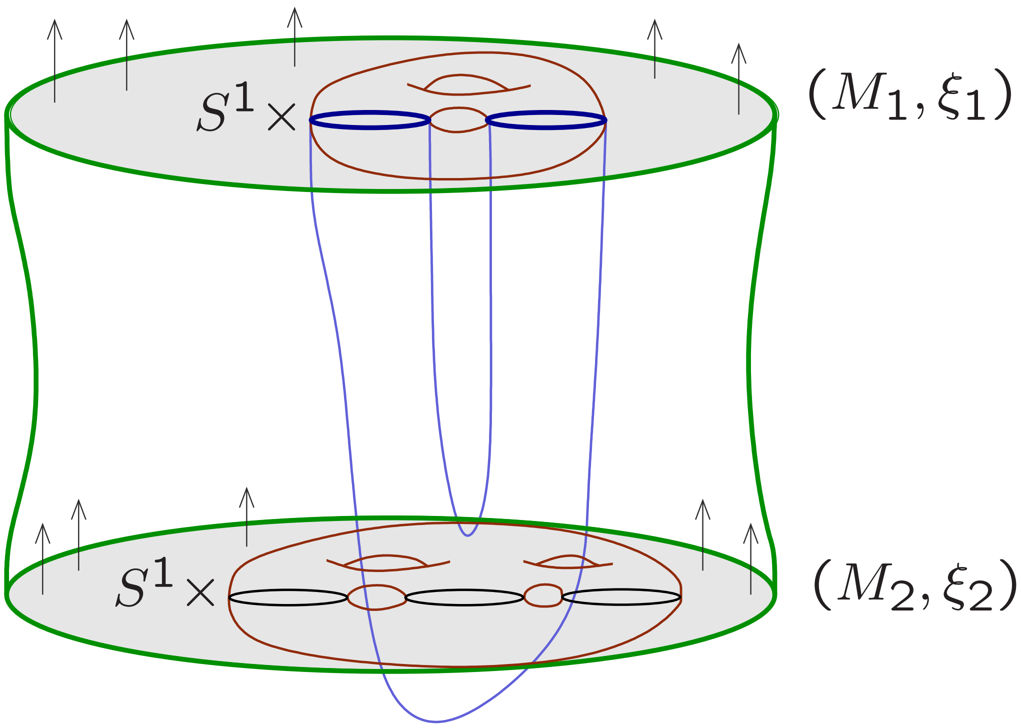

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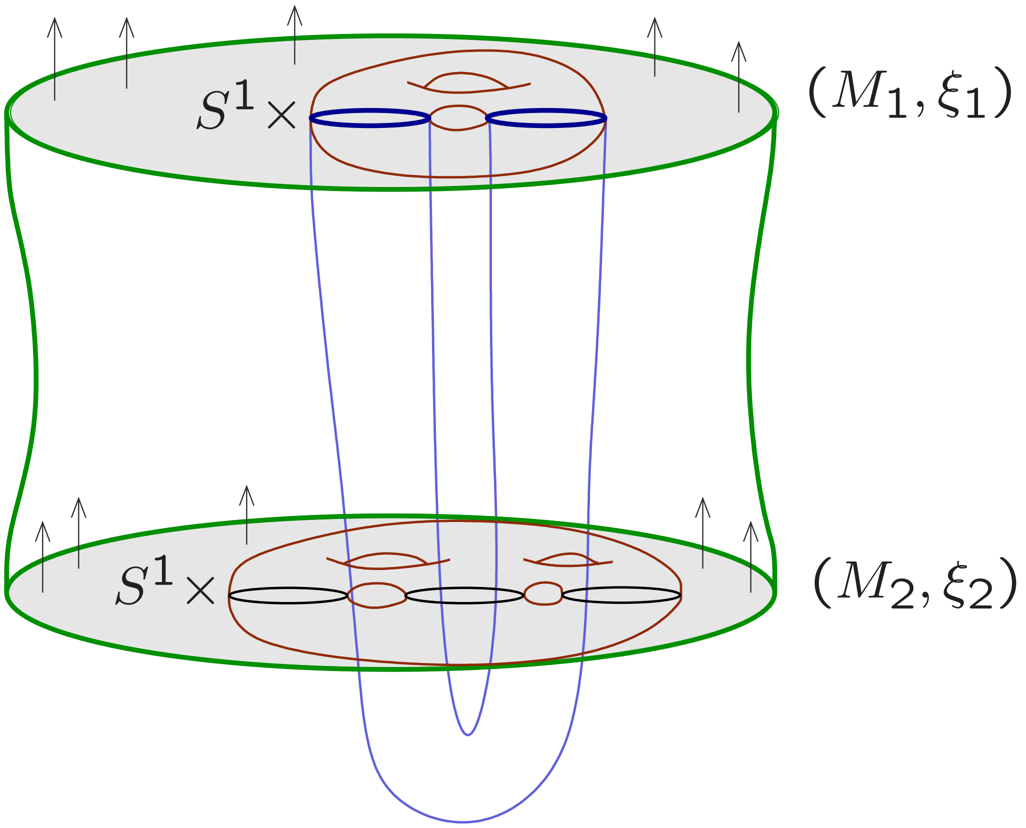

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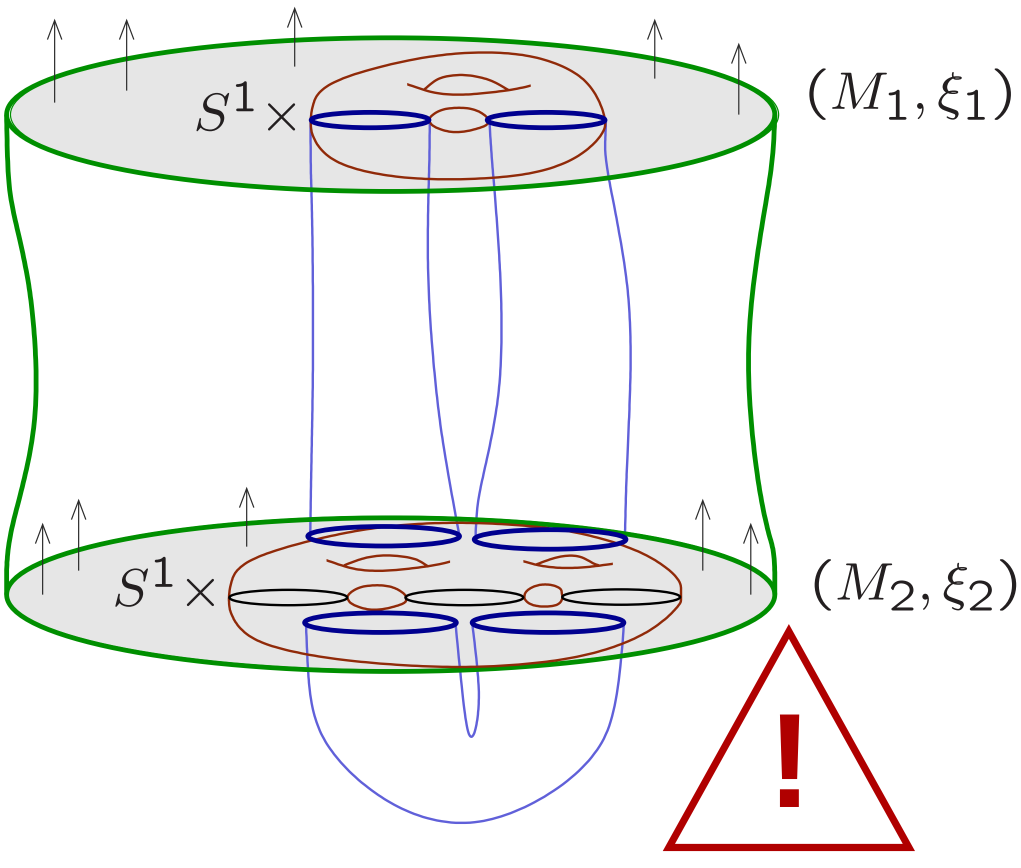

Some open questions and partial answers

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   - Overtwistedness, Giroux torsion, planar torsion:
     C. Wendl, *A hierarchy of local filling obstructions for contact* 3*-manifolds*, Preprint 2010, arXiv:1009.2746.

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     V. Colin, E. Giroux and K. Honda, *Finitude homotopique et isotopique des structures de contact tendues*, Publ. Math. Inst. Hautes Études Sci. **109** (2009), no. 1, 245-293.

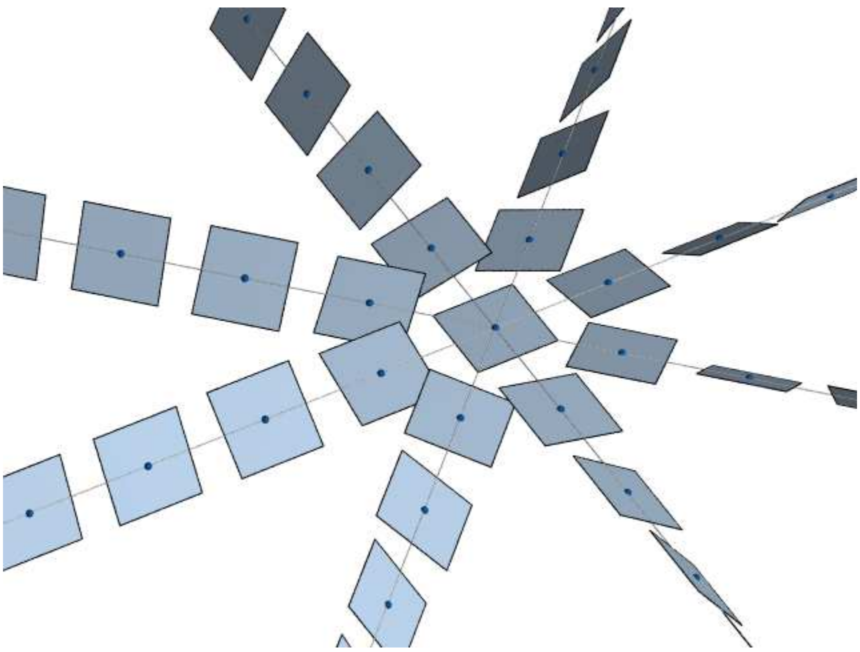
Main reference

- Janko Latschev and Chris Wendl, *Algebraic torsion in contact manifolds*, Geom. Funct. Anal. **21** (2011), no. 5, 1144-1195, with an appendix by Michael Hutchings.

Acknowledgment

Contact structure illustrations by Patrick Massot:

`http://www.math.u-psud.fr/~pmassot/`



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