A Functional Analytic Approach to Morse Homology

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Abstract

The aim of this project is to define the cup and cap product operations on the Morse cohomology of closed manifolds. We first construct the Morse homology by strict analogy to the methods of Floer homology. We explore the following topics: the geometry of manifolds of maps, Fredholm operators, genericness of transversality, complementarity of compactness and gluing and orientation of these Banach spaces, which form the foundation of this functional analytic Morse theory. By adapting the techniques of this construction, we define the cup and cap products by counting Y-shaped trajectories that arise by consideration of a triple of Morse functions.

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1 Overview of Functional Analytic Morse Homology

In this section, we give a summary of the main ideas that we will explore in this article.

We start by endowing the space of sections of the pullback bundles of TM by smooth compact curves with a $H^{1,2}$-topology. Then by applying the exponential map to these sections, we obtain a space of trajectories in M which is a Banach manifold. We construct Banach bundles over this manifold by using the fibre derivatives of the exponential map. It turns out that the smooth curves adapted to the gradient flow lines of a Morse function, f are the zeros of a section map, $F(\gamma) = \dot{\gamma} + \nabla f \circ \gamma$. These solutions are called time independent trajectories.

We then move on to show that this map is in fact a Fredholm map i.e. its differential is a Fredholm operator and its index is equal to the difference in Morse index of a pair of critical points. This is first done in the trivial case then we show that the results can be transferred to the manifold by means of appropriate trivialisation.

Although zero is not always a regular value of this map, there exists a generic set of Riemannian metrics for which this holds. So the Implicit function theorem allows us to conclude that the zeros of F is a Banach submanifold (without boundary) and has as tangent space the kernel of DF and hence due the regularity its dimension is equal to the Fredholm index.

We may compactify this submanifold by adding broken trajectories. This is essentially a consequence of the Arzelà-Ascoli theorem and the fact that M is compact. Since compactness does not guarantee that each broken trajectory arises as a boundary, we are led to construct a gluing operation that glues two trajectories into one trajectory in a higher dimensional space such that we may construct a sequence converging to that broken trajectory. In this sense, compactification and gluing can be viewed as complementary operations.

In order to define a homology theory over $\mathbb{Z}$ we need to define an orientation concept for this Banach manifold. This is achieved by constructing a determinant bundle using the Fredholm operators associated the trajectories. The main obstacle here is that we need a concept of orientation compatible with the gluing of trajectories. Along with the time independent trajectories, we consider time dependent trajectories and λ-parametrised trajectories which join critical points of a pair of different Morse functions.

These trajectories allow us to define chain maps between chain complexes generated by two Morse functions such that they induce an isomorphism in homology. Hence we conclude that the Morse homology is in fact independent of the choice of Morse function. We deduce the Poincaré Duality theorem as a direct consequence of this independence. In the final chapter, we define using a triple of Morse functions a space of trajectories which only go halfway between their critical points. We then show that the space of such trajectories that meet at the ends is a Banach submanifold whose dimension can be described by the index of an appropriate section map. By a corresponding compactness-gluing result, we deduce that for appropriate Fredholm indices, the connected components of this manifold are closed one dimensional manifold which can be compactified by adding similar broken trajectories. The cup and cap products can then be defined by counting these Y-shaped trajectories.

2 Space of Trajectories

Throughout this article we denote by (M, g) a smooth and closed Riemannian manifold.

Definition 2.1. *The Hessian of a smooth function, f on M at p is a bilinear map on T_pM defined by:*

$$H_p(f)(X,Y) := g(\nabla_X \nabla f(p), Y(p)) = X(Yf)(p) + (\nabla_X Y)f(p).$$

Definition 2.2 (Morse function)**.** *A smooth function, f on a manifold, M is a Morse function if its Hessian (matrix) is invertible at each of its critical point, p.*

Observe that if p is a critical point of f, then $(\nabla_X Y)f(p) = 0$ and since $X(Yf)(p) - Y(Xf)(p) = df(p)[X,Y] = 0$, where $[\cdot,\cdot]$ denotes the Lie brackets, the Hessian is in fact a symmetric bilinear form. Note that this is equivalent to $df \pitchfork 0$ in T^*M, since the zero section can be naturally identified with M so that the horizontal bundle at $(p, 0)$ is T_pM and the vertical bundle is T_p^*M. The invertibility of the Hessian implies it defines an isomorphism between T_pM and T_p^*M.
Recall that in classical Morse theory, the gradient flow lines between two critical points, x and y satisfy the differential equation:

$$\dot{\gamma} = \nabla f \circ \gamma(t),$$

with the boundary conditions $\lim_{t\to-\infty} \gamma(t) = x$ and $\lim_{t\to+\infty} \gamma(t) = y$. We want to define a Banach manifold structure on this space of trajectories connecting the critical points of f and so we need to construct an appropriate coordinate chart. In order to do so, we need to set up some structures. We refer to [3] and [11] for the details.

Definition 2.3. *We define a compactification of the real line and endow it with a smooth manifold structure as follows: Let $\overline{\mathbb{R}} := \mathbb{R} \bigcup \{\pm\infty\}$ with chart map,*

$$h : \overline{\mathbb{R}} \to [-1, 1]$$

$$h(t) = \begin{cases} -1, & t = -\infty \\ \frac{t}{\sqrt{1+t^2}}, & t \in \mathbb{R} \\ 1, & t = +\infty. \end{cases}$$

We can define the set of compact smooth curves in M joining x,$y \in M$ by:

$$C^{\infty}_{x,y}(\overline{\mathbb{R}}, M) := \{\gamma \in C^{\infty}(\overline{\mathbb{R}}, M) : \gamma(-\infty) = x\ , \gamma(+\infty) = y\}.$$

Using this definition of $\overline{\mathbb{R}}$, we see that if $f \in C^1(\overline{\mathbb{R}}, \mathbb{R})$ then,

$$|f'(t)| = |(f \circ h^{-1})' \circ h'(t))| \leqslant \sup_{s \in [-1,1]} (f \circ h^{-1})'(s) \cdot \frac{1}{(1+t^2)^{3/2}}.$$

Since $(f \circ h^{-1})$ is a C^1-function on a compact interval, $(f \circ h^{-1})'(s)$ attains in fact a maximum value. Hence in particular we see that $f' \in L^2$. The foundation of this Banach manifold will be based on the Sobolev spaces.

Definition 2.4. *We define the Sobolev Space, $W^{k,p}(\mathbb{R}, \mathbb{R}^n)$ where $p \in [1, +\infty)$ and $k \in \mathbb{N}_0$ by*

$$W^{k,p}(\mathbb{R}, \mathbb{R}^n) := \{f \in L^p(\mathbb{R}, \mathbb{R}^n) : f^{(i)} \in L^p(\mathbb{R}, \mathbb{R}^n) \text{ for } 0 \leqslant i \leqslant k\},$$

with norm $||f||_{k,p} = \big(\sum_{i=0}^{k} ||f^{(i)}||^p_{L^p}\big)^{\frac{1}{p}}$, where $f^{(i)}$ are to be understood as the i^{th} derivative of f in the sense of distributions.

We shall only deal with the case $k = 1$ and $p = 2$, so that $H^{1,2} = W^{1,2}$ is also a Hilbert space with the natural inner product given by:

$$\langle f, g\rangle_{1,2} := \langle f, g\rangle_{L^2} + \langle f', g'\rangle_{L^2}.$$

Note that the space of smooth functions is not Banach, its completion w.r.t the Sobolev norm equivalently defines the Sobolev space. We shall also need the following fundamental theorems associated with Sobolev spaces:

Theorem 2.1 (Sobolev Embedding Theorem)**.** *Let $\mathcal{U} \subset \mathbb{R}$ be open, if $kp > n$ and $k - np < 1$, then the following continuous embedding into the Hölder space holds:*

$$W^{k,p}(\mathcal{U}) \hookrightarrow C^{0,\alpha}(\overline{\mathcal{U}}) \text{ for each } \alpha \in (0, k - n/p]$$

hence, in particular, $H^{1,2}(\mathcal{U}, \mathbb{R}^n) \hookrightarrow C^0(\overline{\mathcal{U}}, \mathbb{R}^n)$.

Theorem 2.2 (Rellich-Kondrachov Theorem)**.** *Let $\mathcal{U} \subset \mathbb{R}$ be bounded, if $d \in \mathbb{Z}_0^+$, $kp > n$ and $k - np < 1$, then the following embedding holds:*

$$W^{k+d,p}(\mathcal{U}) \hookrightarrow W^{k,p}(\overline{\mathcal{U}}) \text{ for each } q \in [p, \infty)$$

and moreover it is compact.

From the above estimate, we get that if $A \in C^1(\overline{\mathbb{R}}, GL(n,\mathbb{R}))$ and $s \in H^{1,2}(\mathbb{R},\mathbb{R}^n)$ then,

$$||As||_{1,2} \leqslant c\ ||s||_{1,2}\ , \text{ where } c = 2\max\{||A||_\infty, ||A'||_\infty\}.$$

If we know further that $f \in C^1(\overline{\mathbb{R}}, \mathbb{R})$ satisfies $f(\pm\infty) = 0$, then from fundamental theorem of calculus and the above estimate, we obtain

$$\begin{aligned} |f(t) - f(t_0)| = \left|\int_{t_0}^{t} f'(s)ds\right| &\leqslant \left|\int_{t_0}^{t} \frac{c}{(1+s^2)^{3/2}} ds\right| \\ &\leqslant c \left|\frac{1}{t^2} - \frac{1}{t_0^2}\right| \end{aligned}$$

$\Longrightarrow |f(t)| \leqslant \frac{1}{t^2}$ by setting $t_0 = \pm\infty \Rightarrow\ f \in L^2(\mathbb{R},\mathbb{R}) \Rightarrow\ f \in H^{1,2}(\mathbb{R},\mathbb{R})$.

The reason for introducing the Sobolev space is that we want our space of trajectories to be a Banach manifold together with a notion of differentiability and since the Sobolev space is the ideal candidate that satisfies these properties, we can model our space of trajectories using $H^{1,2}(\mathbb{R},\mathbb{R}^n)$.

Definition 2.5. *Let ξ be a $C^\infty(\overline{\mathbb{R}})$ finite dimensional vector bundle on $\overline{\mathbb{R}}$ with smooth trivialisation ϕ given by $\phi : \xi \xrightarrow{\simeq} \overline{\mathbb{R}} \times \mathbb{R}^n$. We are able to define a Sobolev structure on the associated vector space of sections of ξ as follows:*

$$H^{1,2}(\xi) = \{\phi_*^{-1}(s)(t) = \phi^{-1}(t, s(t)) : s \in H^{1,2}(\mathbb{R},\mathbb{R}^n)\}$$

so that $||\phi_*^{-1}(s)||_{1,2} := ||s||_{1,2}$.

Remarks:

1. The Banach Space topology induced on $H^{1,2}(\xi)$ is independent of our choice of trivialisation. Indeed, suppose ψ is another trivialisation then the change in trivialisation is given by $\psi \circ \phi^{-1} \in C^\infty(\overline{\mathbb{R}}, GL(n,\mathbb{R}))$ hence, $||\phi_*^{-1}s||_{1,2}^{\psi} =: ||\psi \circ \phi_*^{-1}(s)||_{1,2} \leqslant c_1\ ||\phi_*^{-1}s||_{1,2}^{\phi}$. Similarly we obtain $||\psi_*^{-1}s||_{1,2}^{\phi} \leqslant c_2\ ||\psi_*^{-1}s||_{1,2}^{\psi}$, which show that the norms induced by each trivialisation are equivalent.

2. The contractibility of $\overline{\mathbb{R}}$ implies the existence of a global trivialisation. If two maps, $f, g : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ are homotopic then their pullback bundles are isomorphic. Since $\overline{\mathbb{R}}$ is contractible, this means that the constant map and the identity map are homotopic. As the pullback bundle of a constant map is trivial, the vector bundle is trivial as well.

We also notice that the map, $H^{1,2}$ is a (covariant) functor from the category of smooth vector bundles into the category of Banach spaces. In fact, this functor is continuous.

Definition 2.6. *A functor*

$$\Omega : \mathrm{Vec}_{C^\infty}(\overline{\mathbb{R}}) \to \mathrm{Ban}$$

is called a section functor if it associates to each smooth vector bundle, ξ on $\overline{\mathbb{R}}$, a vector space $\Omega(\xi)$ of sections in ξ such that Ω maps each smooth bundle homomorphism $\phi : \xi \to \eta$ to a linear map $\Omega_* \phi \in \mathcal{L}(\Omega(\xi), \Omega(\eta))$ defined in a natural way by $(\Omega_* \phi) \cdot s = \phi \cdot s$, and

$$\Omega_* : C^\infty(\mathrm{Hom}(\xi, \eta)) \to \mathcal{L}(\Omega(\xi); \Omega(\eta))$$

is continuous (w.r.t the associated norms).

By analogy to the definition of the $H^{1,2}$ functor, we can also define the L^2 functor and furthermore we have:

Lemma 2.3. *The functors $H^{1,2}, L^2 : \mathrm{Vec}_{C^\infty}(\overline{\mathbb{R}}) \to \mathrm{Ban}$ are section functors.*

Proof. In the above remark, we have checked that the induced norm defines a Banach space topology and that the norms induced by any trivialisations are equivalent. In view of definition 2.5, w.l.o.g we may assume we are dealing with the trivial case, i.e. $\xi = \overline{\mathbb{R}} \times \mathbb{R}^n$, $\eta = \overline{\mathbb{R}} \times \mathbb{R}^m$, $H^{1,2}(\xi) = H^{1,2}(\mathbb{R}, \mathbb{R}^n)$, $L^2(\xi) = L^2(\mathbb{R}, \mathbb{R}^n)$ and $C^\infty(\mathrm{Hom}(\xi, \eta)) = C^\infty(\overline{\mathbb{R}}, M(m \times n, \mathbb{R}))$. Now it suffices to verify the continuity of $H^{1,2}_*$ and L^2_*. Let $A \in C^\infty(\overline{\mathbb{R}}, M(m \times n, \mathbb{R}))$, then

$$||As||_{0,2} \leqslant ||A||_\infty \cdot ||s||_{0,2} \xRightarrow{\sup_{||s||_{0,2}=1}} ||A||_{\mathcal{L}(L^2, L^2)} \leqslant ||A||_\infty,$$

which proves continuity of L^2_*.
By direct calculation and using that $2\langle u, v\rangle \leqslant ||u||^2_{0,2} + ||v||^2_{0,2}$, we obtain

$$||(As)'||^2_{0,2} \leqslant 2(\ ||A's||^2_{0,2} + ||As'||^2_{0,2} \ ) \leqslant 2(\ ||A'||^2_{0,2} \cdot ||s||^2_\infty + ||A||^2_\infty \cdot ||s'||^2_{0,2} \ ).$$

Using the same estimate as above and the fundamental theorem of calculus, we obtain

$$||s(t_1)|^2 - |s(t_0)|^2| = \left| \int_{t_0}^{t_1} \frac{d}{dt} \langle |s(t)|^2, |s(t)|^2 \rangle \, \mathrm{dt} \ \right| \leqslant ||s||^2_{1,2} \Rightarrow ||s||_\infty \leqslant ||s||_{1,2}.$$

Combining these two estimates yields,

$$||As||_{1,2}^2 \leqslant c^2(\ ||A'||_{0,2}^2 + ||A||_\infty^2\ ) \cdot ||s||_{1,2}^2$$
$$\xRightarrow{\sup_{||s||_{1,2}=1}} ||A||_{\mathcal{L}(H^{1,2},H^{1,2})} \leqslant c\sqrt{\ ||A'||_{0,2}^2 + ||A||_\infty^2\ }\ . \qquad (1)$$

Hence if $A_n \to A$ in $C^\infty(\mathrm{Hom}(\xi,\eta))$ then $||A_n' - A'||_{0,2}, ||A_n - A||_\infty \to 0$ implying $||A_n - A||_{\mathcal{L}(H^{1,2},H^{1,2})} \to 0$ which proves continuity of $H_*^{1,2}$. □

Remarks:

1. Our definition of $\overline{\mathbb{R}}$ ensures that all the norms appearing in the above proof are finite.
2. For $A \in H^{1,2}(\mathrm{Hom}(\xi,\eta))$, the estimate $||A||_\infty \leqslant ||A||_{1,2}$ and (1) imply
$$H_*^{1,2} : H^{1,2}(\mathrm{Hom}(\xi,\eta)) \to \mathcal{L}(H^{1,2}(\xi); H^{1,2}(\eta))$$
is continuous which is a stronger condition as it does not require smoothness but just the square integrability of the homomorphism and that of its weak first derivative.
3. The Sobolev Embedding theorem gives the continuous embedding of sections, $H^{1,2}(\xi) \hookrightarrow C^0(\xi)$.

Definition 2.7. *We define the set,*

$$\mathcal{P}_{x,y}^{1,2} = \{\exp \circ s \in C^0(\overline{\mathbb{R}}, M) : s \in H^{1,2}(h^*\mathcal{O}),\ \ h \in C_{x,y}^\infty(\overline{\mathbb{R}}, M)\}$$

where $(\exp \circ s)(t) = \exp_{h(t)} s(t)$*. Due to the Sobolev Embedding theorem,* $s \in H^{1,2}(h^*\mathcal{O}) \hookrightarrow C^0(h^*\mathcal{O})$ *and as such* $s(\pm\infty) = 0$*, so* $\mathcal{P}_{x,y}^{1,2}$ *contains continuous curves joining critical points* x *and* y*.*

$\mathcal{P}_{x,y}^{1,2}$ can be described as the set of continuous curves on M that can be homotoped via the exponential map to smooth curves between critical points. In order to show that this set is indeed a smooth Banach manifold, we need to construct a smooth atlas.

Lemma 2.4. *Given* $\xi, \eta \in \mathrm{Vec}_{C^\infty}(\overline{\mathbb{R}})$ *and an open and convex neighbourhood of the zero section,* $\mathcal{O} \subset \xi$*, then any smooth map* $f \in C^\infty(\mathcal{O}, \eta)$ *satisfying* $f(\pm\infty, 0) = (\pm\infty, 0)$ *(w.r.t any trivialisation) induces a continuous map on* $H^{1,2}(\mathcal{O}) = \{s \in H^{1,2}(\xi) : s(\overline{\mathbb{R}}) \subset \mathcal{O}\}$ *given by,*

$$\begin{aligned} f_* : H^{1,2}(\mathcal{O}) &\to H^{1,2}(\eta) \\ s &\mapsto f_* s = f \circ s. \end{aligned}$$

Although the above lemma holds for any smooth vector bundle on $\overline{\mathbb{R}}$, in this article we only deal with the pullbacks of the tangent bundle by smooth and compact curves and in this case we notice that there is an obvious choice for such an f, namely the exponential map (acting between pullback bundles). So let $h \in C^{\infty}_{x,y}(\overline{\mathbb{R}}, M)$, then we define the pullback by

$$\begin{array}{ccc} h^*TM & \xrightarrow{pr_2} & TM \\ {\scriptstyle pr_1}\big\downarrow & & \big\downarrow{\scriptstyle \tau} \\ \overline{\mathbb{R}} & \xrightarrow{\quad h \quad} & M \end{array}$$

where $\tau \circ pr_2(t, \xi) = h \circ pr_1(t, \xi)$. Also, $h^*\mathcal{O} \subset h^*TM$ is open since $\mathcal{O}$ is.
For a closed (compact and boundaryless) Riemannian manifold, (M, g), we know that $\exp|_p$ is a local diffeomorphism in an open neighbourhood of zero in T_pM for any $p \in M$. So there exists an open neighbourhood, $\mathcal{O}$ of the zero section in the tangent bundle $\tau : TM \to M$ such that

$$\begin{array}{rcl} (\tau, \exp) : \mathcal{O} & \xrightarrow{\simeq} & \Delta \subset M \times M \\ \xi & \mapsto & (\tau(\xi), \exp(\xi)) \end{array}$$

is a diffeomorphism.

Proof of lemma 2.4. Since f is a smooth bundle map on $\overline{\mathcal{O}}$, so in particular the differential of its fibre restriction defines a continuous function on $\overline{\mathbb{R}}$ and can be uniformly bounded due to the compactness of $\overline{\mathbb{R}}$ and as such f_t are Lipschitz continuous. So w.r.t any trivialisation of ξ, we have:

$$|f_t(x_t) - f_t(y_t)| \leqslant c\ |x_t - y_t|, \text{where } x_t, y_t \text{ are fibre elements at } t \in \overline{\mathbb{R}}$$

and given a smooth section, s, letting $x_t = s(t)$ and y_t be the zero section we obtain from the triangle inequality:

$$|f \circ (s(t))| = |f_t(s(t))| \leqslant c\ |s(t)| + |f_t(0)|.$$

Since $f_t(0) \in C^{\infty}(\overline{\mathbb{R}}, \eta)$ hence is in $H^{1,2}(\eta)$, again w.r.t any trivialisation implying $f \circ s \in L^2(\eta)$. Considering the second order differential of f_t, similarly as above we obtain the estimate:

$$|df_t(\dot{x}_t) - df_t(\dot{y}_t)| \leqslant c'\ |\dot{x}_t - \dot{y}_t|$$

and analogously this shows that $(f \circ s)' \in L^2(\eta)$ hence f_* is well-defined and is continuous w.r.t the $H^{1,2}$ norm since $||s_n||_{1,2} \to 0$ implies $||(f \circ s)_n||_{1,2} \to 0$. $\square$

We can now give the fundamental theorem that will allow us to define smooth charts for $\mathcal{P}^{1,2}_{x,y}$.

Theorem 2.5. *The map, f_* defined in lemma 2.4 is smooth and its k^{th} derivative is given by $D^k f_*(s) = H^{1,2}_*(F^k f \circ s)$.*
Here $Ff : \mathcal{O} \to \mathrm{Hom}(\xi;\eta)$ denotes the fibre derivative of f, i.e. the derivative of f_p w.r.t the fibre elements in $\mathcal{O}_p$, more explicitly
$Ff(v)(w) := \frac{d}{dt}\Big|_{t=0}(f(v+tw))$ *where v, w are in the same fibre.*
$F^k f : \mathcal{O} \to \mathrm{Hom}\,(\underbrace{\xi \oplus ... \oplus \xi}_{k\ times};\eta) \cong \mathrm{Hom}\,(\underbrace{\xi \oplus ... \oplus \xi}_{k\text{-}1\ times};\mathrm{Hom}(\xi;\eta))$

Sketch of proof. The proof is by induction. The case $k = 0$ has been proven in lemma 2.3. For $k = 1$, given $x, y \in \mathcal{O}$ in the same fibre, we define:

$$\Theta : \mathcal{O} \oplus \mathcal{O} \to \mathrm{Hom}(\xi,\eta)$$
$$\Theta(x,y)\cdot z = \left[\int_0^1 Ff(x+t(y-x))\,\mathrm{dt} - Ff(x)\right]\cdot z.$$

By a change of variable we have that

$$\Theta(x,y)(y-x) = \int_x^y Ff(\alpha)d\alpha - Ff(x)(y-x) = f(y) - f(x) - Ff(x)(y-x),$$

which implies together with the second remark on page 9,

$$f_*(s) = f_*(s_0) + H^{1,2}_*(Ff\circ s_0)\cdot(s-s_0) + H^{1,2}(\Theta)(s_0,s)\cdot(s-s_0).$$

Using that $H^{1,2}(\Theta)(s_0,s_0) = 0$, $\lim_{s\to s_0} H^{1,2}(\Theta)(s,s_0) = 0$ and the above equation, this gives

$$Df_*(s_0) = H^{1,2}_*(Ff\circ s_0)$$

which is moreover continuous on $H^{1,2}(\mathcal{O})$. For $k = 2$, we consider the map $g(t,\xi) = Ff(t,\xi) - Ff(t,s_0(t))$ which satisfies the properties of f and so from case $k = 1$ this gives: $D^2 f_*(s_0) = Dg_* s_0 = H^{1,2}_*(F^2 f\circ s_0)$ and repeating the same argument (formally by induction) the proof is accomplished. □

Note that in the above proof, the convexity of $\mathcal{O}$ ensures that Θ is well-defined.

Theorem 2.6. *$\mathcal{P}^{1,2}_{x,y}$ is a smooth Banach manifold with atlas (strictly speaking this is a parametrisation, i.e. the chart is actually given by the inverse)*

$$\{H^{1,2}(h^*\mathcal{O}), H^{1,2}(\exp_h)\}_{h\in C^\infty_{x,y}(\mathbb{R},M)\}},\ \textit{where}\ H^{1,2}(\exp_h)(s) = \exp\circ s.$$

Proof. Define

$$\begin{aligned}\phi_h : h^*\mathcal{O} &\to \overline{\mathbb{R}} \times M \\ (t,\xi) &\mapsto (t, \exp_{h(t)} \xi)\end{aligned}$$

As seen earlier this is a local diffeomorphism, i.e. ϕ_h is an embedding in an open neighbourhood of the graph of h. Let $U_h := \phi_h(h^*\mathcal{O}) \subset \overline{\mathbb{R}} \times M$, $H^{1,2}(U_h) := \{g \in C^0(\overline{\mathbb{R}}, M) : (t, g(t)) \subset U_h,\ \phi_h^{-1} \circ (id, g) \in H^{1,2}(h^*\mathcal{O})\} \subset C^0_{x,y}(\overline{\mathbb{R}}, M)$ and

$$\begin{aligned}H^{1,2}(\phi_h^{-1}) : H^{1,2}(U_h) &\to H^{1,2}(h^*\mathcal{O}) \\ g &\mapsto \phi_h^{-1} \circ (id, g)\end{aligned}$$

then it follows that from these definitions that

$$\bigcup_{h \in C^\infty_{x,y}(\overline{\mathbb{R}}, M)} H^{1,2}(U_h)$$

defines an open cover for $\mathcal{P}^{1,2}_{x,y}(\mathbb{R}, M)$ and $\{H^{1,2}(\phi_h^{-1})\}_{h \in C^\infty_{x,y}(\overline{\mathbb{R}}, M)\}}$ is a family of bijective maps. To conclude that $\mathcal{P}^{1,2}_{x,y}$ is a smooth Banach manifold we need to check that the transition maps are indeed smooth. So suppose $U_h \bigcap U_f \neq \varnothing$, we need to show $H^{1,2}(\phi_f^{-1}) \circ H^{1,2}(\phi_h^{-1})^{-1}$ is a diffeomorphism. We now appeal to our fundamental theorem 2.5 above. Let $\mathcal{O}_h = \phi_h^{-1}(U_h \bigcap U_f) \subset h^*\mathcal{O}$ and analogously define $\mathcal{O}_f$ so that these sets satisfy the conditions of lemma 2.4. Considering the maps:

$$\begin{aligned}\Phi_{fh} &= \phi_f^{-1} \circ \phi_h : \mathcal{O}_h \to \mathcal{O}_f \\ \Phi_{fh}(t,\xi) &= (t, \exp^{-1}_{f(t)}(\exp_{h(t)} \xi))\end{aligned}$$

which satisfy $\Phi_{fh}(\pm\infty, 0) = (\pm\infty, 0)$ since our curves converge to critical points x and y, thus applying theorem 2.5 gives that $H^{1,2}(\phi_f^{-1}) \circ H^{1,2}(\phi_h^{-1})^{-1}$ and $(H^{1,2}(\phi_f^{-1}) \circ H^{1,2}(\phi_h^{-1})^{-1})^{-1} = H^{1,2}(\phi_h^{-1}) \circ H^{1,2}(\phi_f^{-1})^{-1}$ are smooth as required. □

Given this Banach manifold, we now want to construct a vector bundle on it which will allow us to define the section map, F upon which this whole theory is founded. To do so we need to look at the Levi-Civita connection, K on (M, g). We first recall some facts about the vertical and horizontal bundle of a fibre bundle, $\tau : E \to B$. The vertical bundle, V is a subbundle of TE defined as:

$$V = \{\xi \in TE : d\tau(\xi) = 0\}$$

i.e. V_e is the tangent space to the fibre at each point, $e \in E$ and the horizontal bundle, H is a (non-canonical) complementary space of V such that $TE = V \oplus H$. A connection is a projection, K onto V and hence defines H. In the case of a vector bundle, the fibres are simply vector spaces and as such there is a natural isomorphism between the fibre and V_e, thus we can write $K : TE \to E$. Returning to our case, where $E = TM$, $B = M$, we simply have the decomposition $T(TM) = V \oplus H = \ker(d\tau) \oplus \ker(K)$. Hence we also see that $K(\xi)_{|V_\xi}$ and $d\tau(\xi)_{|H_\xi}$ are isomorphisms. Since $\exp : \mathcal{O} \to M$ is a local diffeomorphism, $d\exp(\xi)$ is an isomorphism for each $\xi \in \mathcal{O}$, so the following mappings are well-defined isomorphisms:

$$\nabla_1 \exp(\xi) = d\exp(\xi) \circ (d\tau(\xi)_{|H_\xi})^{-1} : T_{\tau(\xi)}M \xrightarrow{\simeq} T_{\exp(\xi)}M,$$
$$\nabla_2 \exp(\xi) = d\exp(\xi) \circ (K(\xi)_{|V_\xi})^{-1} : T_{\tau(\xi)}M \xrightarrow{\simeq} T_{\exp(\xi)}M.$$

$\nabla_2 \exp(\xi)$ is thus the fibre derivative of the exponential map at ξ. In the zero section, the horizontal bundle, H can be canonically identified with the tangent bundle, TM giving that $\nabla_1 \exp(0)(\gamma'(0)) = \frac{d}{dt}(\exp(\gamma(t), 0))|_{t=0} = \gamma'(0)$, i.e. $\nabla_1 \exp(0) = \mathrm{Id}$. Similarly, $\nabla_2 \exp(0) = \mathrm{Id}$. Consequently,

$$\Theta(\xi) = (\nabla_2 \exp(\xi))^{-1} \circ \nabla_2 \exp(\xi) = (K(\xi)_{|V_\xi}) \circ (d\tau(\xi)_{|H_\xi})^{-1}$$

is a smooth map with $\Theta(0) = \mathrm{Id}$ and its fibre derivative at 0 is 0. These definitions allows us to write the fibre derivatives of the transition map, Φ_{fh} by

$$F\Phi_{fh}(t, \xi) = \nabla_2 \exp(\exp_{f(t)}^{-1}(\exp_{h(t)} \xi))^{-1} \circ \nabla_2 \exp(\xi). \qquad (*)$$

Also, using that $\nabla_t \xi := K(\dot{\xi})$ and $d\tau(\dot{\xi}) = \dot{h}$, for section ξ based at h we have

$$\frac{\partial}{\partial t}(\exp \xi) = \nabla_1 \exp(\xi)(\dot{h}) + \nabla_2 \exp(\xi)(\nabla_t \xi). \qquad (**)$$

The following theorem allows us to define Banach bundles, including the tangent bundle, on our space of trajectories, $\mathcal{P}^{1,2}_{x,y}$.

Theorem 2.7. *Let* $\Pi : \mathrm{Vec}_{C^\infty} \to \mathrm{Ban}$ *be a section functor such that* $\Pi_* : H^{1,2}(\mathrm{Hom}(\xi, \eta)) \to \mathcal{L}(\Pi(\xi); \Pi(\eta))$ *is continuous. Then the domain of* Π *can be extended uniquely to continuous vector bundles which are of the form* g^*TM *for* $g \in \mathcal{P}^{1,2}_{x,y}$ *and moreover*

$$\Pi(\mathcal{P}^{1,2*}_{x,y}TM) = \bigcup_{g \in \mathcal{P}^{1,2}_{x,y}(\mathbb{R}, M))} \Pi(g^*TM)$$

is a Banach bundle on $\mathcal{P}^{1,2}_{x,y}$.

The proof is essentially defining a fibre at each $g = \exp_h s$ by

$$\Pi(g^*TM) = \{J_{gh} \cdot v : v \in \Pi(h^*TM)\},$$

where $J_{gh}(t) = \nabla_2 \exp\,(\xi(t)) \in C^0(\mathrm{Hom}(h^*TM, g^*TM))$ and $\xi \in C^0(h^*\mathcal{O})$ and verifying that J_{gh} provides a smooth trivialisation which is again achieved by theorem 2.5. We note that L^2 and from the second remark on page 9, $H^{1,2}$ both satisfy the condition on Π. Moreover from $(*)$ we have the identity,

$$\nabla_2 \exp(\Phi_{fh} \circ s)) \circ (F\Phi_{fh} \circ s) \cdot v = \nabla_2 \exp(s) \cdot v,$$

from which we deduce that:

$$(J_{gf})^{-1} \circ J_{gh} = H^{1,2}_*(F\Phi_{fh} \circ s) = D\Phi_{fh*}(s).$$

Since the tangent space of a manifold isomorphic to the modelled space by the differential of the chart map, we have that the image of J_{gh} is in fact the tangent space at $g \in \mathcal{P}^{1,2}_{x,y}$. Hence

$$H^{1,2}(\mathcal{P}^{1,2*}_{x,y}TM) = \bigcup_{g \in \mathcal{P}^{1,2}_{x,y}(\mathbb{R},M))} H^{1,2}(g^*_{x,y}TM)$$

is the tangent bundle of $\mathcal{P}^{1,2}_{x,y}$, i.e. $T\mathcal{P}^{1,2}_{x,y} = H^{1,2}(\mathcal{P}^{1,2*}_{x,y}TM)$.

Theorem 2.8. *Given a smooth vector field, X on M s.t $X(x) = X(y) = 0$, then the mapping $C^\infty_{x,y}(\overline{\mathbb{R}}, M) \ni \gamma \mapsto \dot{\gamma} + X \circ \gamma \in C^\infty(\gamma^*TM)$ can be extended to a smooth section, Γ from $\mathcal{P}^{1,2*}_{x,y}(\mathbb{R}, M)$ in the Banach bundle $L^2(\mathcal{P}^{1,2*}_{x,y}TM)$.*

We omit the proof of this theorem which is again just a verification of smoothness. We point out here that using our chart coordinates, the above trivialisation given by J_{gh}, and $(**)$ we can trivialised Γ by:

$$\begin{aligned} &\Gamma : H^{1,2}(h^*\mathcal{O}) \to L^2(h^*\mathcal{O}) \\ &\xi \mapsto \nabla_t \xi + \Theta(\xi)\ \dot{h} + \nabla_2 \exp(\xi)^{-1}(X \circ \exp_h)(\xi) \end{aligned}$$

We shall refer to this crucial trivialisation throughout this article.

Corollary 2.8.1. *Given a smooth Morse function f on M, the mapping*

$$\begin{aligned} F : \mathcal{P}^{1,2}_{x,y} &\to L^2(\mathcal{P}^{1,2*}_{x,y}TM) \\ s &\mapsto \dot{s} + \nabla f \circ s \end{aligned}$$

describes a smooth section in the L^2-bundle and locally can be represented at $\gamma \in C^\infty_{x,y}$ by

$$\begin{aligned} &F_{loc,\gamma} : H^{1,2}(\gamma^*\mathcal{O}) \to L^2(\gamma^*TM) \\ &F_{loc,\gamma}(\xi)(t) = \nabla_t \xi(t) + g(t, \xi(t)), \end{aligned}$$

where $g : \overline{\mathbb{R}} \times \gamma^*\mathcal{O} \to \gamma^*TM$ is smooth and maps to the same fibre, satisfies $g(\pm\infty, 0) = 0$ and is endowed with the asymptotic fibre derivatives $D_2 g(\pm\infty, 0) = 0$, which are the conjugated linear operators of the Hessians of f at x and y, respectively.

The fact that $D_2 g(\pm\infty, 0) = 0$ is conjugated to Hessian can easily be seen from the above trivialisation together with fact that $F\Theta(0) = 0$ and the fibre linearisation of ∇f corresponds to the Hessian of f. In general, we can replace the vector field, ∇f by

$$X_{h_t} = \frac{\nabla h_t}{\sqrt{1 + |\dot{h}_t|^2 |\nabla h_t|^2}}, \text{ where } \frac{\partial h_t}{\partial t} = 0, \text{ for } |t| > R.$$

This asymptotically constant time dependent vector field, which we will consider in more detail in the compactness section, will be crucial in proving that the Morse homology is independent of the choice of Morse function. The space of trajectories, $\mathcal{P}^{1,2}_{x,y}$ consists of continuous curves connecting x and y and in particular, contains the curves lying in the intersection of the stable manifold of y and unstable manifold of x. The reason for considering the above map, F lies in the fact that its zeroes correspond to these curves.

Proposition 2.1. *The zeroes of $F : \mathcal{P}^{1,2}_{x,y} \to L^2(\mathcal{P}^{1,2*}_{x,y} TM)$ are exactly the smooth curves which solve the ODE $\dot{s} = \nabla f \circ s$ and satisfy the conditions $\lim_{t\to-\infty} s(t) = x$ and $\lim_{t\to+\infty} s(t) = y$.*

This proof is accomplished by the following straightforward, yet crucial, lemma which will also be useful to prove the compactness of this submanifold of zeroes.

Lemma 2.9. *Let $X : U(0) \to \mathbb{R}$ be a smooth vector field defined on a neighbourhood of $0 \in \mathbb{R}^n$ such that $X(0) = 0$ and $DX(0)$ is non-singular and symmetric. Then, there exists $\epsilon > 0$ s.t for each solution $\dot{s} = X(s)$ with $\lim_{t\to\infty} s(t) = 0$ we can find constants $c > 0$ and $t_0(s) \in \mathbb{R}$ s.t $|s(t)| \leqslant c\, e^{-\epsilon t}$ for $t > t_0(s)$.*

proof of proposition 2.1. ($\Rightarrow$) Since each solution, γ is weakly differentiable (due to the definition of $\mathcal{P}^{1,2}_{x,y}$ using the Sobolev space) so it suffices to check for smoothness. We argue by bootstrapping and using the fact that f is smooth, i.e. $\gamma' = -\nabla f \circ \gamma \Rightarrow \gamma'' = -\frac{\partial}{\partial t}(\nabla f \circ \gamma) \Rightarrow \gamma^{(k)} = -\frac{\partial^k}{\partial t^k}(\nabla f \circ \gamma)$, so γ is smooth (in the weak sense) and the Sobolev Embedding theorem implies it is indeed smooth.

($\Leftarrow$) We now want to show that a smooth solution, s is indeed in the zero section of F. In view of the manifold structure we defined on $\overline{\mathbb{R}}$, it suffices to show that $s \circ h^{-1} : [\,-1, 1] \to M$ is C^1 and $(s \circ h^{-1})' \circ h(\pm\infty) = 0$ then it holds that is in the Sobolev space and as such is in $\mathcal{P}^{1,2}_{x,y}$. We already have from above, $(s \circ h^{-1})' \circ h = \dot{s}(t) \cdot (1 + t^2)^{\frac{3}{2}}$. For a Morse function, f, its Hessian (in local coordinates) satisfies the above lemma, with the vector field satisfying $\nabla f(x) = \nabla f(y) = 0$. The fact that M is compact and $X = \nabla f$ is smooth imply it is Lipschitz and the relation $X(s) = \dot{s}$ gives $|\dot{s}| \leqslant d|s|$ hence $\dot{s}$ decreases exponentially as well. This yields the desired limit on $(s \circ h^{-1})'(t)$. $\square$

In classical Morse theory, we needed the transversality condition, i.e. $W_u^f(x) \pitchfork W_s^f(y)$, such that the sum of their tangent space span the tangent space of the manifold itself. Then from the theorem about transversality (which is a consequence of the Implicit function theorem), $W_u^f(x) \bigcap W_s^f(y)$ is submanifold of M. In order, to achieve a similar result in this setup, we notice that if we show that $DF(s)$ is surjective for each $s \in F^{-1}(0)$ then the (Banach space version) Implicit function theorem will give us a submanifold structure on $F^{-1}(0)$. However in the infinite dimensional setting it is not that straightforward, this leads us to the notion of Fredholm operators.

3 Analysis of Fredholm Operators

Before analysing the section map F, we start with the trivial case and then transfer the results to the non-trivial case in a way that is independent of the choice of coordinate charts. So we look at linear operators on the trivial bundle, $\overline{\mathbb{R}} \times \mathbb{R}^n$ of the form,

$$\begin{aligned} F_A : H^{1,2}(\mathbb{R}, \mathbb{R}^n) &\to L^2(\mathbb{R}, \mathbb{R}^n) \\ s &\mapsto \dot{s} + \ As, \end{aligned}$$

where $A \in C^0_b(\mathbb{R}, \mathrm{End}(\mathbb{R}^n))$, i.e. $A(t)$ is an $n \times n$ matrix for each t and $||A||_\infty < +\infty$.

The fundamental theorem of this section is that Spectral flow is equal to the Fredholm Index, i.e. the net change in the number of negative eigenvalues of A is equal to the Fredholm index of F_A.

Definition 3.1. *A linear operator, $K : X \to Y$ is compact if $\overline{K(B(0,1))}$ is compact, i.e. the image of unit ball is precompact.*

Definition 3.2. *A Fredholm operator is a bounded linear operator, $F : X \to Y$, where X, Y are Banach spaces such that $\dim(\ker(F))$, $\dim(\mathrm{coker}(F)) < +\infty$ and $R(F)$ is closed. We define the Fredholm index of F by*

$$\mathrm{ind}(F) = \dim(\ker(F)) - \dim(\mathrm{coker}(F)).$$

If we drop the assumption that $\mathrm{coker}(F)$ is finite dimensional, F is then called a semi-Fredholm operator.

An equivalent definition:

Definition 3.3. *$F : X \to Y$ is a Fredholm operator if there exists a compact operator, $K : Y \to X$ such that $\mathrm{Id} - F \circ K$ and $K \circ F - \mathrm{Id}$ are compact operators.*

We shall denote the space of Fredholm operators by $\mathcal{F}(X, Y)$ and the space of compact operators by $Com(X, Y)$.
Remarks:

1. In the above definition, $R(F)$ is closed is a redundant condition: let C be a complementary space to $R(F)$ in Y, then since $C \cong \mathrm{coker}(F)$, it is finite dimensional. F induces an injective map, $\tilde{F} : X/\ker(F) \to Y$. So let $S : X/\ker(F) \oplus C \to Y$ by $S(x, y) = \tilde{F}(x) + y$, then $R(F) \cong S(X/\ker(F) \oplus \{0\})$. Since S is a bounded linear bijective map, the Inverse Bounded theorem implies it is a toplinear isomorphism and as such $S(X/\ker(F) \oplus \{0\})$ is closed since $X/\ker(F) \oplus \{0\}$ is.

2. A linear isomorphism is a Fredholm operator of index 0. So we may interpret Fredholm operators as nearly invertible operators when working in infinite dimensional spaces.

Properties of Fredholm Operators

1. If $F_1 \in \mathcal{F}(Y, Z)$ and $F_1 \circ F_2 \in \mathcal{F}(X, Z)$ then $F_2 \in \mathcal{F}(X, Y)$ and $\mathrm{ind}(F_1 \circ F_2) = \mathrm{ind}(F_1) + \mathrm{ind}(F_2)$

2. $\mathcal{F}(X, Y)$ is open in $\mathcal{L}(X, Y)$ and moreover, $\mathrm{ind} : \mathcal{F} \to \mathbb{Z}$ is a locally constant function.

3. If K is compact and F is Fredholm then $F + K$ is Fredholm and $\mathrm{ind}(F + K) = \mathrm{ind}(F)$

Sketch of proof.
1) We simply consider the short exact sequence $0 \to \ker(F_2) \hookrightarrow \ker(F_1 \circ F_2) \to \ker(F_1) \bigcap R(F_2) \to 0$ and by dimension counting it follows that $\ker(F_2)$ is finite dimensional. A similar argument can be carried out for the cokernel.
2) Denote by K, C and R the kernel, cokernel and range of F respectively. Then we can write $X = X_1 \bigoplus K$ and $Y = Y_1 \bigoplus R$. F induces an isomorphism, $F_0 : X_1 \to R$. We can choose $\epsilon > 0$ such that if $||F - L|| < \epsilon$ then the induced map, L_0 is an isomorphism as well. Let p be the projection on R and $i : X_1 \hookrightarrow X$, then $L_0 = p \circ L \circ i$ and p, L_0, i are Fredholm of indices $-\dim(K),\ 0,\ \dim(C)$ hence from property 1) $0 = \mathrm{ind}(S) - \mathrm{ind}(F)$.
3) is trivial from the second definition of Fredholm operator. $\square$

Definition 3.4. *Given a Hilbert space, H a bounded linear operator, A is self-adjoint (s.a) if $\langle Ax, y\rangle = \langle x, Ay\rangle$ for each $x, y \in H$.*

We will use the following notations;

$$\mathcal{S} = \{A \in GL(n, \mathbb{R}) : A \text{ is } s.a\}$$

$$\mathcal{A} = \{A \in C_b^0(\overline{\mathbb{R}}, \mathrm{End}(\mathbb{R}^n)) : A^{\pm} = A(\pm\infty) \in \mathcal{S}\}$$

In classical Morse theory, we saw that the Hessian matrix is self-adjoint and we defined the Morse index of a critical point of f as the number of negative entries of the Hessian matrix in Morse coordinates, or equivalently the number of negative eigenvalues. We define by analogy the Morse index of a self-adjoint operator, A by:

$$\mu(A) = \#(\sigma(A) \bigcap \mathbb{R}^-),$$

where $\sigma(A) = \{\lambda : \det(A - \lambda I) = 0\}$ denotes the finite spectrum of A, i.e. the set of eigenvalues counting multiplicity. The map, F can also be viewed as,

$$F : C_b^0(\mathbb{R}, \mathrm{End}(\mathbb{R}^n)) \to \mathcal{L}(H^{1,2}; L^2)$$
$$F(A(t)) = F_{A(t)}.$$

This is map is continuous since,

$$||(F_A - F_B)(s)||_{0,2} = \Big(\int_{-\infty}^{+\infty} |(A-B)s|^2 \, \mathrm{d}t \Big)^{1/2} \leqslant ||A - B||_\infty ||s||_{1,2}$$
$$\xRightarrow{\sup_{||s||_{1,2}=1}} ||F_A - F_B||_{\mathcal{L}(H^{1,2};L^2)} \leqslant ||A - B||_\infty,$$

so taking $A \to B$ in C_b^0 gives $F_A \to F_B$ in $\mathcal{L}$.
We aim to prove the following theorem:

Theorem 3.1. *If $A \in \mathcal{A}$, then F_A is a Fredholm operator.*

Lemma 3.2. *Given Banach spaces, X, Y and Z and $F \in \mathcal{L}(X;Y)$, $K \in Com(X;Z)$ and $c > 0$ such that*

$$||x||_X \leqslant c \; (||Fx||_Y + ||Kx||_Z), \;\; for \; all \; x \in X$$

then F is a semi-Fredholm operator.

Proof. We first show that $\ker(F)$ is finite dimensional then $R(F)$ is closed. Pick a sequence $\{x_k\}_{k=0}^\infty \subset \ker(F)$ with $||x_k|| = 1$, then we get $||x_n - x_m|| \leqslant c(||K(x_n - x_m))||)$. Since K is compact, we can extract a convergent subsequence of $\{Kx_k\}_{k=0}^\infty$ and hence of $\{x_k\}_{k=0}^\infty$. This implies $S = \{x \in \ker(F) : ||x||_X = 1\}$ is sequentially compact hence compact. By Riesz's lemma, the unit ball in an infinite dimensional space is not compact; hence $\ker(F)$ must be finite dimensional.
Suppose now that $Fx_k \to y \in Y$, then we need to show there exists $x \in X$ such that $Fx = y$. Let's assume $\{x\}_{k=1}^\infty$ is bounded then we can extract a convergent subsequence of $\{Kx_k\}_{k=1}^\infty$, say $\{Ky_k\}_{k=1}^\infty$. Hence we have

$$||y_n - y_m||_X \leqslant c \; (||F(y_n - y_m)||_Y + ||K(y_n - y_m)||_Z).$$

Since $\{y_k\}_{k=1}^\infty$ is Cauchy and X is Banach,

$$y_k \to z \Rightarrow Fy_k \to Fz$$

and by uniqueness of limit $y = Fz$. If $\{x\}_{k=1}^\infty$ was unbounded, then w.l.o.g we may assume $\{x\}_{k=1}^\infty \notin \ker(F)$ and furthermore, Hahn-Banach theorem implies that $\ker(F)$ has a closed complementary space. By normalising we may set $||x_k||_X = 1$, so we get that $Fx_k \to 0$. On the other hand, the compactness of K and our assumed inequality imply that we may extract a subsequence,$\{y_k\}_{k=0}^{\infty}$ converging to z with $||z||_X = 1$ and $Fz = 0$, which contradicts that $z \notin \ker(F)$. □

We shall also need some simple properties of the Fourier transform to prove theorem 3.1.

Definition 3.5. *The Fourier transform of f is defined by;*

$$[\mathcal{F}(f)](y) = \int_{\mathbb{R}} e^{ixy} f(x) dx$$

when it exists. (Note that it always exists for L^1 functions).

Theorem 3.3. *$\mathcal{F}$ is an isometry on $L^2(\mathbb{R}, \mathbb{R}^n)$, i.e. $||\mathcal{F}(f)||_{0,2} = ||f||_{0,2}$.*

Proof. $\mathcal{F}$ is an isometry on the Schwartz space and the latter is dense in L^2, so we may extend to L^2 by continuity. □

Properties of Fourier Transform

1. $\mathcal{F}$ is linear.
2. $\mathcal{F}(\dot{f}) = iy \cdot \mathcal{F}(f)$

Proof of theorem 3.1.
The proof is carried out in 4 steps:
S1: Let $A \in \mathcal{A}$ be a non-zero constant map. Then we define a map $\omega : L^2 \to L^2$ by $\omega \circ s(t) = t \cdot s(t)$. From the properties of $\mathcal{F}$, we can write

$$F_A = \mathcal{F}^{-1} \circ (i\omega + A) \circ \mathcal{F}.$$

Since A is invertible so 0 is not an eigenvalue. Denote by $\lambda_0 = \min |\sigma(A)| > 0$. If we define an operator,

$$\begin{aligned} B : \mathbb{R}^n &\to \mathrm{End}(\mathbb{C}^n) \\ \alpha &\mapsto i\alpha I + A \end{aligned}$$

For $A \in \mathcal{A}$, the spectral theorem implies that A has real eigenvalues and its eigenvectors provide an orthonormal basis, hence $0 \notin \sigma(B) = i\alpha + \sigma(A) \Rightarrow B^{-1}$ exists and has eigenvalues $(\sigma(B))^{-1} \Rightarrow ||B(\alpha)^{-1}|| \leqslant \sup_{\lambda \in \sigma(B)} |\lambda|^{-1} = 1/\sqrt{\lambda_0^2 + \alpha^2}$. Let

$$\begin{aligned} C : \mathbb{R}^n &\to \mathrm{End}(\mathbb{C}^n) \\ \alpha &\to \sqrt{1 + \alpha^2} \end{aligned}$$

so

$$||C(\alpha)B(\alpha)^{-1}|| \leqslant \sqrt{1+\alpha^2}\cdot\frac{1}{\sqrt{\lambda_0^2+\alpha^2}} \leqslant \max(\frac{1}{\lambda_0},1) =: d$$

By the isometry of $\mathcal{F}$ and the Polarisation Identity, we also have $\langle \mathcal{F}x, \mathcal{F}y\rangle_{L^2} = \langle x, y\rangle_{L^2}$. Hence

$$|\langle \mathcal{F}^{-1}CB^{-1}\mathcal{F}\xi, \eta\rangle_{L^2}| = |\langle CB^{-1}\mathcal{F}\xi, \mathcal{F}\eta\rangle_{L^2}| \leqslant ||CB^{-1}||_\infty \; ||\xi||_{0,2} \; ||\eta||_{0,2}$$

$$\begin{aligned} ||\mathcal{F}^{-1}CB^{-1}\mathcal{F}\mathcal{F}^{-1}B\mathcal{F}s||_{0,2}^2 &= ||\sqrt{1+\alpha^2}\mathcal{F}s||_{0,2}^2 \\ &= \int_{\mathbb{R}}(1+\alpha^2)|\mathcal{F}s(\alpha)|^2 d\alpha \\ &= ||\mathcal{F}(s)||_{0,2}^2 + ||\mathcal{F}(\dot{s})||_{0,2}^2 = ||s||_{1,2}^2 \end{aligned}$$

this yields $||s||_{1,2} \leqslant d \; ||\mathcal{F}^{-1}B\mathcal{F}||_{0,2} = d \; ||F_A s||$

S2: We can generalise the above inequality for any $A \in \mathcal{A}$ as follows: There exists $T > 0$ such that

$$||s||_{1,2} \leqslant c_T ||F_A s||_{0,2} \quad \text{if } s|_{[-T,T]} = 0.$$

The proof is essentially using the continuity of A, i.e. $||A^\pm - A(t)|| \to 0$ as $t \to \pm\infty$. Then we can apply the estimate from S1 to $A^\pm$.

S3: We now show that F_A is semi-Fredholm.

$$\frac{1}{2}|\dot{s}|^2 + 2\langle \dot{s}, As\rangle + 2|As|^2 = \frac{1}{2}\,\langle \dot{s} + 2As, \dot{s} + 2As\rangle \geqslant 0$$

$$\Rightarrow \int_{-T}^{T} |\dot{s} + As| \,\mathrm{dt} \geqslant \int_{-T}^{T} (|\dot{s}|^2/2 - |As|^2)\,\mathrm{dt} \geqslant \frac{1}{2}\int_{-T}^{T} |\dot{s}|^2\,\mathrm{dt} - ||A||_\infty \int_{-T}^{T} |s|^2\,\mathrm{dt}$$

So choosing $c > 0$ large enough gives

$$\Rightarrow \int_{-T}^{T} (|s|^2 + |\dot{s}|^2)\,\mathrm{dt} \leqslant c\int_{-T}^{T} (|s|^2 + |F_A s|^2)\,\mathrm{dt}\,.$$

Considering a smooth cut-off function,

$$\beta(t) = \begin{cases} 0, & |t| \geqslant T+1 \\ 1, & |t| \leqslant T \end{cases}$$

with $\dot{\beta}(t) \neq 0$ for $|t| \in (T, T+1)$, we have $s(t) = \beta(t)s(t) + (1-\beta(t))s(t)$. Hence βs satisfies the above estimate and $(1-\beta)s$ is as in S2, using the triangle inequality we obtain:

$$||s||_{1,2} \leqslant c(||s||_{L^2[-T-1,T+1]} + ||F_A s||_{0,2})$$

We can rewrite the above as:

$$||s||_{1,2} \leqslant c(||Ks||_{0,2} + ||F_A||_{0,2})$$

where,

$$K : H^{1,2}(\mathbb{R}, \mathbb{R}^n) \to H^{1,2}([-T-1, T+1], \mathbb{R}^n) \hookrightarrow L^2([-T-1, T+1], \mathbb{R}^n).$$

Here the first map is simply by restriction as $(1-\beta(t))s(t)$ has compact support, $[-T-1, T+1]$ and the second map is from Rellich-Kondrachov theorem which is a compact embedding. So lemma 3.2 allows us to conclude that F_A is semi-Fredholm for any $A \in \mathcal{A}$ and hence from self-adjointness in particular, F_{-A^T} is also semi-Fredholm.

S4: To conclude that F_A is Fredholm, we need to show $\text{coker}(F_A)$ is finite dimensional. Since L^2 is Hilbert, we have an orthogonal decomposition

$$L^2 = R(F_A) \oplus R(F_A)^\perp \text{ and } \text{coker}(F_A) \cong R(F_A)^\perp$$

$\Rightarrow$ for $r \in R(F_A)^\perp$, $\langle r, \dot{s} + As \rangle = 0 \ \forall s \in H^{1,2}$. $C_0^\infty(\mathbb{R}, \mathbb{R}^n)$ is dense in $H^{1,2}$ $\Rightarrow \langle r, \dot{\phi} \rangle_{L^2} = -\langle r, A\phi \rangle_{L^2} \ \forall \phi \in C_0^\infty$

$$\Rightarrow \langle r, \dot{\phi} \rangle_{L^2} = -\langle A^T r, \phi \rangle_{L^2}$$

i.e. r is weakly differentiable and $\dot{r} = A^T r \in L^2 \to r \in H^{1,2}$ and $0 = \dot{r} - A^T r = F_{-A^T}(r)$

$$\Rightarrow \text{coker}(F_A) \cong \ker(F_{A^T})$$

so from S3, $\dim(\text{coker}(F_A)) < +\infty$ □

In view of the above theorem, we denote by $\Sigma \subset \mathcal{F}(H^{1,2}(\mathbb{R}, \mathbb{R}^n), L^2(\mathbb{R}, \mathbb{R}^n))$, the set of operators, F_A. We can define an equivalence relation as follows: $F_A \sim F_B$ if $A^\pm = B^\pm$ i.e. the Fredholm operators are equivalent if they are equal at $\pm\infty$ and denote the equivalence classes by Θ_{F_A}. The following lemma will be crucial in orientating the space of trajectories.

Lemma 3.4. *Θ_{F_A} is contractible in Σ.*

Sketch of proof. Let A be a class representative. Define a homotopy by:

$$H : [0,1] \times \Theta_{F_A} \to \Theta_{F_A}$$
$$H(\alpha, F_B) = F_{(1-\alpha)\cdot B + \alpha \cdot A}$$

so that $H(0, \cdot)$ is the identity and $H(1, \cdot)$ is the constant map. The continuity of H follows by a simple proof by contradiction argument. □

From property 2) of Fredholm operators and the above lemma, it follows that the index is in fact constant on each equivalence class, i.e. the index of F_A depends only on $A^{\pm}$. Remember that our objective is to prove that the Spectral flow equals the Fredholm Index. In order to simplify the proof we make the following observation:
If $A \in \mathcal{S}$ then it is diagonalisable since the spectral theorem implies the existence of an invertible matrix, C such that

$$CAC^{-1} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

and w.l.o.g we may order the eigenvalues by sign, i.e. $\mathrm{sgn}(\lambda_{k+1}) \leqslant \mathrm{sgn}(\lambda_k)$ and furthermore, that by swapping columns of C if necessary, assume that $\det(C) > 0$. So applying the above to $A(\pm\infty) = A^{\pm}$, we can find $C^{\pm} \in C^{\infty}(\overline{\mathbb{R}}, GL(n, \mathbb{R}))$ such that

$$C^{\pm}A^{\pm}(C^{\pm})^{-1} = \mathrm{diag}(\lambda_1^{\pm}, ..., \lambda_n^{\pm})$$

and is asymptotically constant such that $C(t) = C^{\pm}$ for $|t| > 1$. Note that the fact that we chose $\det(C^{\pm}) > 0$ ensures that such a smooth function $C(t)$ exists since $C^{\pm}$ belong to the same pathwise connected component of $GL(n, \mathbb{R})$. Then a simple computation gives

$$CF_AC^{-1} = F_{\dot{C}^{-1}C + CAC^{-1}}.$$

By the asymptotic assumption, for large t, $\dot{C}^{-1} = 0$ and hence $CF_AC^{-1} \in \Theta_{\mathrm{diag}(\lambda_1^{\pm}, ..., \lambda_n^{\pm})} \Rightarrow \mathrm{ind}(CF_AC^{-1}) = 0\ + \mathrm{ind}(F_A)\ + 0 = \mathrm{ind}(F_A)$.
The following theorem will allow us to make the connection between the relative Morse index (or Spectral Flow) and the Fredholm index. We point out here that our definition of the relative Morse index using the spectral flow allows us to work in infinite dimensional settings since the calculation of the Spectral flow avoids us having to consider $\infty - \infty$ which does not makes sense for critical points of infinite indices.

Theorem 3.5.

$$\mathrm{ind}(F_A) = \mu(A^-) - \mu(A^+)$$

Proof. With regards to the above discussion and lemma 3.4 , we can simply consider $A(t) = \mathrm{diag}(\lambda_1(t), ..., \lambda_n(t))$ satisfying the properties in our discussion. Under these circumstances, we see that

$$\ker(F_A) = \{s \in H^{1,2} : \dot{s} = -As\} = \{s \in H^{1,2} : \dot{s}_i = -\lambda_i s_i,\ i = 1, .., n\}$$

The differential equation $\frac{dy}{dt} = -\lambda y \Leftrightarrow y = be^{-\lambda t}$ can have an asymptotically zero solution at $+\infty$ $(-\infty)$ only if $\lambda > 0 (\lambda < 0)$. We see immediately that $s \in \ker(F_A)$ if and only if $\lambda_i^- < 0$ and $\lambda_i^+ > 0$ giving that:

$$\begin{aligned}\dim(\ker(F_A)) &= \text{number of corresponding } \lambda \text{ s changing sign from -ve to +ve} \\ &= \max(\mu(A^-) - \mu(A^+), 0).\end{aligned}$$

Note that the second equality follows due to our ordering of the eigenvalues by sign. Similarly from the result in S4 of theorem 3.1,

$$\begin{aligned}\dim(\operatorname{coker}(F_A)) &= \dim(\ker(F_{-A^T})) \\ &= \dim(\ker(F_{-A})) \\ &= \max(-(\mu(A^-) - \mu(A^+)), 0).\end{aligned}$$

□

Having now proved that the change in Morse index is equal to Fredholm index, we need to transfer these results to the non-trivial case, i.e. when our bundles are $H^{1,2}(\xi)$ and $L^2(\xi)$. Referring back to our operator, $F : \mathcal{P}^{1,2}_{x,y} \to L^2(\mathcal{P}^{1,2*}_{x,y}TM)$ which has local representation, $F(\xi)(t) = \nabla_t \xi(t) + g(t, \xi(t))$, we wish to transfer the result of the above theorem to a linearisation of F (which we shall see is Fredholm). Since the covariant derivative, ∇_t is $\mathbb{R}$-linear, so the linearisation of F at s is of the form $\nabla_t + D_2 g(s)$. This should be enough motivation for the following definition.

Definition 3.6.

$$\Sigma_{\xi,\nabla} := \left\{ F_A : H^{1,2}(\xi) \to L^2(\xi) : \begin{array}{c} F_A s = \nabla_t s + As \ \textit{s.t}\ A \in C^0(\overline{\mathbb{R}}, \operatorname{End}(\xi)) \\ \textit{with } A^\pm = (A^\pm)^T \textit{ and are invertible} \end{array} \right\}$$

In the above definition, ∇ denotes a covariant derivative associated with the vector bundle, ξ on $\overline{\mathbb{R}}$. Note that since ξ is a trivial bundle, we may choose a frame field (dependent on a trivialisation), i.e. a set $\{s_1, ..., s_n\}$ of smooth sections which form a basis for the fibre at each $p \in \overline{\mathbb{R}}$. Then we may write:

$$\nabla_t s = \nabla_t (a_i s_i) = \dot{a}_i s_i \ + a_i \nabla_t s_i \ = \dot{a}_i s_i \ + a_i \Gamma_{ik} s_k$$

Here we have used the Einstein notation, i.e. $b_i s_i = \Sigma_{i=1}^n b_i s_i$. So we may write in coordinates given by the frame field together with a trivialisation,

$$\nabla_t^{triv} s = \dot{s} + \Gamma s, \text{ where } \Gamma \in C^\infty(\overline{\mathbb{R}}, \operatorname{End}(\mathbb{R}^n)) \text{ and } s \in H^{1,2}(\mathbb{R}, \mathbb{R}^n).$$

Remark that the trivialised covariant derivative is obtained by $\nabla_t^{triv} s = \phi_* \nabla_t \phi_*^{-1} s$ with ϕ as in definition 2.5. Before using the Fredholm property, we need to ensure that our calculations does not depend on our choice of trivialisation. By the product rule, we see that the change of trivialisation acts on the trivialised covariant derivative as follows:

$$\Psi^{-1}\nabla_t^{triv}(\Psi s) = (\frac{\partial}{\partial t} + \Psi^{-1}\dot{\Psi} + \Psi^{-1}\Gamma\Psi)s,$$

where $\Psi = \phi \cdot \psi^{-1} \in C^\infty(\overline{\mathbb{R}}, GL(n,\mathbb{R}))$. So the trivialised operator, $F_{A,\nabla}^{triv} = \phi \circ F_A \circ \phi^{-1}$ changes as

$$\Psi^{-1}F_{A,\nabla}^{triv}(\Psi s) = (\frac{\partial}{\partial t} + \Psi^{-1}\dot{\Psi} + \Psi^{-1}\Gamma\Psi + \Psi^{-1}A\Psi)s.$$

So by our analysis in section 1, $\dot{\Psi}(\pm\infty) = 0$ and we see that if $\Gamma(\pm\infty) = 0$, then

$$\Psi^{-1}F_{A,\nabla}^{triv}\Psi = \frac{\partial}{\partial t} + \Psi^{-1}A\Psi.$$

So if the Fredholm property (finite dimensional kernel and cokernel) holds for trivialisation, ϕ then it holds for any trivialisation, ψ with same index.

Lemma 3.6. *Given the tangent bundle, τ on Riemannian manifold, (M,g) with a covariant derivative, ∇, by pulling back by a curve, $u : \overline{\mathbb{R}} \to M$, we obtain a covariant derivative, $u^*\nabla$ on the pullback bundle u^*TM and for this induced covariant derivative $\Gamma(\pm\infty) = 0$.*

Proof. Choosing a local frame field for TM, we have the representation:

$$\nabla_v w(p) = Dw(p)\cdot v + v\cdot\tilde{\Gamma}(p)\cdot w,$$

where $\tilde{\Gamma}(p)$ is an $n \times n$ matrix. Since $\dot{u}(\pm\infty) = 0$ and in the pullback bundle, u^*TM, $[u^*\nabla]_t(\xi(u(t))) := u^*(\nabla_{\dot{u}(t)}\xi(t))$ (see [9]). So $\Gamma(t) = \dot{u}(t)\cdot\tilde{\Gamma}(u(t))$ and hence $\Gamma(\pm\infty) = 0$. ☐

Hence if $F_B \in \Sigma_{\xi,\nabla}$, then by trivialising we obtain

$$\phi\cdot F_B\cdot\phi^{-1} \in \Theta_{F_A}, \text{ with } A^\pm = \phi(\pm\infty)B(\pm\infty)\phi^{-1}(\pm\infty),$$

i.e. we have reduced our problem to the trivial case in a manner which is independent of ϕ, so we have a well-defined notion of Fredholm operator on the pullback bundle. Thus, we can say that F_B is a Fredholm operator and from theorem 3.5, $\mathrm{ind}(F_B) = \mu(A^-) - \mu(A^+)$. Returning back to the map $F : \mathcal{P}_{x,y}^{1,2} \to L^2(\mathcal{P}_{x,y}^{1,2*}TM)$, we have:

Corollary 3.6.1. *$F : \mathcal{P}_{x,y}^{1,2} \to L^2(\mathcal{P}_{x,y}^{1,2*}TM)$ is a Fredholm map, i.e. its differential, $DF(s)$ is a Fredholm operator, and*

$$\begin{aligned}\mathrm{ind}(DF(s)) &= \mu(D_2g(-\infty,0)) - \mu(D_2g(+\infty,0))\\ &= \mu(x) - \mu(y).\end{aligned}$$

4 Transversality

The aim of this section is to show that transversality is a generic condition, in other words, we can find perturbations such that $F \pitchfork 0$ and as a consequence, $F^{-1}(0)$ is a manifold. To give a motivation for transversality, we consider the surfaces given by $x^2 + y^2 - z^2 = 1$ and $x = 1$. The intersection of these two manifolds is given by $\{(1, y, \pm y) : y \in \mathbb{R}\}$ which is not a manifold. However notice that if we allow a small perturbation of $x = 1$ to $x = 1.01$, then the intersection is indeed a manifold. A more precise statement of the above observation is given by the following theorem.

Theorem 4.1. *Let $f : M \to N$ be a smooth map between finite dimensional manifolds and f is transverse to a submanifold, $P \subset N$, i.e. $df(p)(T_pM) + T_{f(p)}P = T_{f(p)}N \ \forall p \in f^{-1}(P)$, then $f^{-1}(P)$ is a submanifold of M and* $\mathrm{codim}(f^{-1}(P)) = \mathrm{codim}(P)$*. Moreover, if f is not transverse then it can be perturbed so that it becomes transverse.*

The first part of this theorem is proved by a direct application of the Implicit function theorem and the second part of the above theorem is proved using Sard's lemma.(See [10]) In the above example, f was the inclusion map, i. The trouble is that this theorem does not generalise directly to infinite dimensional manifolds due to the fact that Sard's lemma does not have an infinite dimensional analogue. However, Smale generalised the Sard's lemma for the class of Fredholm operators:

Theorem 4.2 (Sard-Smale's Theorem)**.** *If $F : M \to V$ is a smooth Fredholm map between Banach spaces, then the regular values of F is a generic set.*

The key idea of the proof is to use the Fredholm properties to reduce the theorem to the finite dimensional case and apply Sard's lemma. [12]

Definition 4.1. *A Baire space, X is a topological space such that if $\{U_n\}_{n=1}^{\infty}$ is a collection of open dense sets, then $\bigcap_{n=1}^{\infty} U_n$ is also dense.*

Theorem 4.3 (Baire Category theorem)**.** *Every complete metric space is a Baire space. Hence in particular, all Banach spaces are Baire.*

Definition 4.2. *A G_δ set is a subset of a topological space such that $G_\delta = \bigcap_{n=1}^{\infty} U_n$, where each U_n is open.*

Hence if $\Sigma \subset X$ such that Σ is a G_δ set with each U_n dense, then Σ is also dense. We say that $G \subset X$ is generic w.r.t a condition on the points on X if the condition holds for some set $\Sigma \subset G$.

Note: Genericness is stronger than simply being dense. The set, I of irrational numbers are generic in $\mathbb{R}$ since $I = \bigcap_{q\in\mathbb{Q}}\{q\}^c$ but $\mathbb{Q}$ is not generic although it is dense in $\mathbb{R}$.
We now state the fundamental theorem of this chapter:

Theorem 4.4. *If G and M are Banach manifolds and $\tau : E \to M$ is a Banach bundle with fibre, $\mathbb{E}$ and $\Phi : G \times M \to E$ is a smooth map such that for each $g \in G$,*

$$\Phi_g : M \to E \quad \Phi_g(m) = \Phi(g, m),$$

i.e. G can be viewed as the parameter set defining sections of E and there is a countable trivialisation, $\{(U, \psi)\}_{n=1}^{\infty}$ such that 0 is a regular value of

$$\Psi := pr_2 \circ \psi \circ \Phi : G \times U \to E|_U \to U \times \mathbb{E} \to \mathbb{E}$$

i.e. $D\Psi(g, m)$ is onto $\forall (g, m) \in \Psi^{-1}(0)$ and $\Psi_g : U \to \mathbb{E}$ is a Fredholm map of index r $\forall g \in G$. Then there exists a set, Σ (as above) such that $\Phi_g^{-1}(0)$ is a closed submanifold for each $g \in \Sigma$. (Here 0 is understood to be the zero section.) i.e. $\Psi_g \pitchfork 0$.

Before proving the above theorem we need the Banach version of the Implicit function theorem [8] and the following lemma:

Lemma 4.5. *Suppose $\Psi : E \times F \to G$ is a surjective linear map between Banach spaces given by $\Psi(e, f) = \Psi_1(e) + \Psi_2(f)$ such that Ψ_1, Ψ_2 are linear and Ψ_2 is a Fredholm operator, then $E \times F$ can be decomposed as $E \times F = \ker(\Psi) \oplus H$ for some closed subspace, H.*

This lemma is proved using the fact that the Fredholm map gives a decomposition of the codomain into the its range and a finite dimensional space then using that Ψ is onto.

Theorem 4.6 (Implicit function theorem)**.** *Let U, V be open sets in Banach spaces, E, F and $f : U \times V \to G$ be smooth such that $D_2 f(a, b)$ is an isomorphism with $f(a, b) = 0$, then there is a smooth map, $g : U_0 \to V$ where U_0 is an open neighbourhood of $a \in U$ with $f(x, g(x)) = 0$ $\forall x \in U_0$.*

proof of theorem 4.4. The proof consists of 3 steps:
S1: We show that $Z := \Psi^{-1}(0)$ is a submanifold.
Since we have a countable trivialisation satisfying the above properties hence if we can prove it for one given trivialisation (U, ψ), then we can simply take the intersection of all such sets and to obtain Σ by the Baire category theorem. So w.l.o.g we may assume that we are in the trivial case. So let $\Psi = pr_2 \circ \psi \circ \Phi$, then by the first assumption:

$$\Psi : G \times M \to \mathbb{E}$$

has 0 as a regular value and

$$\begin{aligned} D\Psi(g,m) : T_gG \times T_mM \cong T_{(g,m)}(G \times M) &\to T_{\Psi(g,m)}\mathbb{E} \cong \mathbb{E} \\ D\Psi(g,m)(v,w) &= D_1\Psi(g,m)v + D_2\Psi(g,m)w \end{aligned}$$

and by second assumption $D_2\Psi$ is a Fredholm operator of index r. By the above lemma we get the splitting condition on the Banach space, $T_gG \times T_mM$ and by surjectivity we have $\widehat{D_2\Psi}$ (w.r.t the coordinates from the lemma, i.e. $D\Psi\big|_H$) is an isomorphism hence we may apply the Implicit function theorem

$$\Rightarrow Z \text{ is a submanifold of } G \times M \text{ and } T_{(g,m)}Z = \ker(D\Psi(g,m))$$

S2: pr_1 is Fredholm of index r

$$\begin{tikzcd} Z \arrow[r, "\Psi"] \arrow[d, "pr_1"'] & 0 \in \mathbb{E} \\ G & \end{tikzcd}$$

Let $z = (g,m) \in Z$, then $Dpr_1 : T_zZ \to T_gG$. Since $pr_1(g,m) = g$ so $\ker(Dpr_1) = T_mM \bigcap T_zZ$. Since $D_2\Psi(z) : T_mM \to T_{\Phi(z)}\mathbb{E} \cong \mathbb{E}$ so let $\gamma(0) = m$ and $\gamma'(0) = v$ then

$$D_2\Psi(z)v = 0 \Leftrightarrow \Psi(g,\gamma(t)) = constant$$

Since $(g,\gamma(0)) = z \in Z$, so $\Psi(g,\gamma(t)) = 0$ i.e. $\ker(D_2\Psi) = T_mM \bigcap T_zZ$ hence $\dim(\ker(Dpr_1)) = \dim(\ker(D_2\Psi)) < +\infty$
We now look at the cokernels:

$$D_1\Psi : T_gG \to T_{\Phi(g)}\mathbb{E} \cong \mathbb{E}$$

so $\operatorname{coker}(D_1\Psi) = \mathbb{E}/R(D_1\Psi)$. Similarly $\operatorname{coker}(Dpr_1(z)) = T_gG/R(Dpr_1)$.
$D_1\Psi$ induces a map:

$$\begin{aligned} \widetilde{D_1\Psi} : T_gG/R(Dpr_1) &\to \mathbb{E}/R(D_2\Psi) \\ [v] &\to [D_1\Psi(v)] \end{aligned}$$

$D\Psi(z)(v,w) = 0$ for $(v,w) \in T_zZ$ gives $D_1\Psi(z)(v) = D_2\Psi(z)(-w)$ so $[D_1\Phi(v)] = 0 \Rightarrow D_1\Psi(z)(v) = D_2\Psi(-w)$ for some $-w \Rightarrow (v,w) \in T_zZ \Rightarrow [v] = 0$. Since $D\Psi(z)$ is onto, so is $\widetilde{D_1\Psi}$ and hence is an isomorphism i.e. $\dim(\operatorname{coker} Dpr_1)) = \dim(\operatorname{coker}(D_2\Psi)) < +\infty$ so Dpr_1 is Fredholm with index r.
By Sard-Smale's theorem, the regular values of pr_1 are generic, i.e. we have

found a generic set, Σ.

S3: $\Phi_g^{-1}(0)$ is a submanifold $\forall g \in \Sigma$

Let $b \in \Sigma$ such that $\Psi(b,m) \neq 0$ $\forall m \in M$, then it does not intersect the 0-section and as such is trivially transversal. So suppose $\Psi(b,m) = 0$, i.e $(b,m) \in Z$, for some $m \in M$, so it suffices to show that $D\Psi_b(m) = D_2\Psi(b,m)$ is onto, then we can conclude by the Implicit function theorem. By surjectivity of $D\Psi(b,m)$ we know that for any $\gamma \in \mathbb{E}$ there exists $(\alpha,\beta) \in T_bG \times T_mM$ such that

$$\gamma = D\Psi(b,m)(\alpha,\beta) = D_1\Phi(b,m)(\alpha) + D_2\Phi(b,m)(\beta)$$

And from the above, since $b \in \Sigma$, $Dpr_1(b,m)$ is also surjective and so there exists $(\alpha',\beta') \in T_{(b,m)}Z$ such that $Dpr_1(b,m)(\alpha',\beta') = \alpha$ hence $\alpha = \alpha'$. Furthermore, $(\alpha,\beta') \in T_{(p,m)}Z = \ker(D\Phi(z))$ so $D_1\Psi(\alpha) + D_2\Psi(\beta') = 0$ $\Rightarrow \gamma = D_2\Psi(\beta - \beta')$, i.e. $D_2\Psi$ is onto so we conclude this proof. □

Note that map, F is dependent on g and f. We shall show that $F^{-1}(0)$ is a manifold for generic choice of the metric, g. (We point out here that instead we could have fixed g and show that $F^{-1}(0)$ is a manifold for generic choice of f.) Recall that ∇f is a vector field on (M,g) defined by $g(\nabla f, X) = df(X)$ for any smooth vector field, $X \in \Gamma(TM)$. So in order to ensure that $F \pitchfork 0$, we need to find an appropriate Riemannian metric, g and to show that this is a generic property. We also need to endow it with a Banach manifold structure so that we may apply the above theorem. To do so, we fix a metric, g_0 and find a generic set of variations. We define $g(X,Y) = g_0(AX,Y)$ where $A \in \mathrm{End}(TM)$ such that A is self-adjoint w.r.t g_0, positive definite and $||A - \mathrm{Id}\,|| \leqslant \epsilon$, i.e. A is close to the identity map. These restrictions on A ensure that g indeed defines a Riemannian metric. Hence w.r.t the new metric, g the vector field has the form $\nabla_g f = A \cdot \nabla_{g_0} f$. We still need a Banach manifold structure on the space of symmetric endomorphisms, hence we construct an appropriate norm which will endow it with a Banach topology.

Definition 4.3. *Let ξ be the smooth vector bundle of endomorphisms of the tangent bundle, i.e. $\xi = \mathrm{End}(TM)$, endowed with a norm $|\cdot|$ and a covariant derivative, ∇ (naturally induced by the Levi-Civita connection) and $\{e_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers. We define a norm on the space of smooth sections of ξ, by :*

$$||s||_e = \Sigma_{k=0}^{\infty} e_n \max_M |\nabla^k s|,$$

where $|\nabla^k s|(p) = \max\{|\nabla_{x_1}...\nabla_{x_k} s(p)| : ||x_i|| = 1,\ x_i \in T_pM\ i = 1,2,...,k\}$.

This norm defines a Banach space,

$$C^e(\xi) = \{s \in C^\infty(\xi) : ||s||_e < +\infty\}$$

provided the sequence, $\{e_k\}_{k=0}^\infty$ is chosen such that $C^\epsilon \neq \emptyset$. The fact that the space is Banach can easily be seen from the fact that any Cauchy sequence in $C^e(\xi)$ necessarily converge to a point in the completion of $C^e(\xi)$ and if $s_n \to s$ then the triangle inequality implies $s \in C^e(\xi)$.

Lemma 4.7. *There exists a sequence $\{e_n\}$ such that $\overline{C^e(\xi)} = L^2(\xi)$.*

See [5] p.808 for details of the proof. The key idea is to construct an appropriate sequence such that $C^e(\xi)$ contains functions which approximate characteristics functions which in turn are dense in L^2 hence so is $C^e(\xi)$. It is clear that if $0 \leqslant e_n \leqslant e'_n$ $\forall n$ then $||s||_e \leqslant ||s||_{e'}$ and obviously we have $||s||_{C^k} \leqslant C \cdot ||s||_e$ where $C = \max\{e_1, ..., e_k\}$.

Definition 4.4. *We denote by $E_{g_0} := \mathrm{End}_{s,g_0}(TM)$, the vector bundle on M whose fibres are linear maps which are self-adjoint w.r.t the Riemannian metric, g_0 (which clearly is a vector space). Notice that E_{g_0} is a subbundle of ξ. So we define a Banach space of sections in E_{g_0} by $C^e(E_{g_0})$. Let $T_{g_0} \subset E_{g_0}$ consists of those endomorphisms which are positive definite w.r.t g_0 hence T_{g_0} naturally defines a Riemannian metric on M via the formula, $g(X,Y) = g_0(AX,Y)$ for $A \in T_{g_0}$.*

So we obtain a Banach manifold of parameters by:

$$G_{g_0} = C^e(T_{g_0}) := \{s \in C^e(E_{g_0}) : s(p) \in (T_{g_0})_p \ for\ each\ p \in M\},$$

which from the above lemma, is continuously embedded in $C^0(T_{g_0})$ and so defines an open set in $C^e(E_{g_0})$ containing the identity map, Id. We note that $C^e(T_{g_0}) \subset C^e(E_{g_0})$ is open (since if A is positive definite then a direct computation shows that if $||B - A|| < \frac{1}{4}||A||$ then B is positive definite as well) and the latter is a Banach space hence $T_A C^e(T_{g_0}) \cong C^e(E_{g_0})$.
As we mentioned earlier in order to show that our homology theory is independent on our choice of Morse function, we will have to consider two more types of trajectories (in addition to the time independent trajectories). Below we define a homotopy between two Morse functions and we define the $h^{\alpha\beta}$-trajectories or time dependent trajectories by $\gamma(t) = X_{h_t} \circ \gamma(t)$. These are trajectories which transit smoothly over time from trajectories of f^α to trajectories of f^β, i.e. they connect critical points of f^α to those of f^β and it is exactly these trajectories that will enable us to define chain homomorphisms between Morse complexes.

Definition 4.5. *A homotopy, $h^{\alpha\beta}: \mathbb{R} \times M \to \mathbb{R}$ between f^α and f^β is finite if there exists $R > 0$ large enough such that*

$$h_t = \begin{cases} f^\alpha, & \text{if } t \leqslant -R \\ f^\beta, & \text{if } t \geqslant R. \end{cases}$$

If moreover, for each $x_0 \in M$ such that $\nabla h_t(x_0) = 0$ for each t, i.e. x_0 is a critical point for all functions, h_t, (x_0 is said to be an $\mathbb{R}$-critical point) we have that

$$\frac{\partial}{\partial t} + H_{x_0}(h_t) : H^{1,2}(x_0^* TM) \to L^2(x_0^* TM)$$

is onto then the homotopy is said to be regular.
Here x_0^*TM is the pullback bundle given by the constant map, $\gamma : \overline{\mathbb{R}} \to \{x_0\}$ and $H_{x_0}(h_t)$ is the Hessian. Since the Hessian at x_0 defines a bilinear form, it also defines a unique endomorphism of $H^{1,2}$ (by Riesz Representation theorem).

Note that for any given pair of Morse functions, f^α and f^β, we can always find a finite regular homotopy. Suppose we have a critical point, x_0 such that $\frac{\partial}{\partial t} + H^2(h_t)(x_0)$ is not surjective then we can replace $h^{\alpha\beta}$ by a new homotopy $h^{\alpha\beta} + k$, where k is a perturbation in a small neighbourhood of x_0, i.e. $k_t \in C_0^\infty(M, \mathbb{R})$ such that $dk_t \neq 0$ for some t, hence x_0 is no longer a critical point an $\mathbb{R}$-critical point. Since our manifold is compact, this process can be repeated finitely many times for each such critical point.
We now need to consider that the map,

$$\begin{gathered} \Phi : G_{g_0} \times \mathcal{P}^{1,2}_{x,y} \to L^2(\mathcal{P}^{1,2}_{x,y}) \\ \Phi(A, \gamma) \mapsto \dot{\gamma} + \frac{A \cdot \nabla_{g_0} h_t}{\sqrt{1 + |\dot{h}_t|^2 \cdot |A \cdot \nabla_{g_0} h_t|^2}} \circ \gamma. \end{gathered}$$

In order to apply theorem 4.4, we need to check that $D(pr_2 \circ \psi \circ \Phi)(A, \gamma)$ is surjective for each pair $(A, \gamma) \in \Phi^{-1}(0)$ and is a Fredholm map of index, $\mu(x) - \mu(y)$. Since the fibre at γ is simply $L^2(\gamma^* TM) = \mathbb{E}$, so $(pr_2 \circ \psi \circ \Phi)(A, \gamma)$ can simply be represented in local coordinates by the map:

$$\begin{aligned} \Phi : &G_{g_0} \times H^{1,2}(\gamma^* \mathcal{O}) \to L^2(\gamma^* TM) \\ &(A, \xi) \mapsto \nabla_t \xi + \Theta(\xi)\dot{h} + (\nabla_2 \exp(\xi))^{-1} \circ X_{h_t} \circ (\exp_\gamma \xi). \end{aligned}$$

From corollary 3.6.1, we know that for any $A \in G_{g_0}$, $\Phi(A, \cdot)$ is a Fredholm map i.e. $D\Phi_A$ is a Fredholm operator and has index $\mu(x) - \mu(y)$, so we just need to check that the zero section is a regular value of Φ.

Theorem 4.8.

$$D\Phi(A,\xi) : C^e(E_{g_0}) \times H^{1,2}(\gamma^*\mathcal{O}) \to L^2(\gamma^* TM)$$

is surjective for each (A,ξ) such that $\Phi(A,\xi) = 0$ for any finite regular homotopy, h_t.

Proof. We already know that $D\Phi_A(\xi) = D_2\Phi(A,\xi)$ is Fredholm and also

$$D_1\Phi(A,\xi)B = \nabla_2 \exp(\xi))^{-1} \cdot \frac{d}{ds}\Big|_{s=0} \left(\frac{\nabla_2 \exp(\xi)^{-1}(A+sB)\nabla h_t}{\sqrt{1+|\dot{h}_t|^2 \cdot |A \cdot \nabla_{g_0} h_t|^2}} \circ \exp_\gamma(\xi) \right)$$
$$= \frac{(1+|\dot{h}_t|^2|A\nabla h_t|^2)(B\nabla h_t) - A\nabla h_t(\langle A\nabla h_t, B\nabla h_t\rangle)}{(1+|\dot{h}_t|^2|A\nabla h_t|)^{3/2}} \circ \exp_\gamma(\xi).$$

For simplicity of notation we rewrite $d(t) = D_1\Phi(A,\xi)$, and

$$d(t) = \frac{B(t)x(t) + \alpha(t)(B(t)x(t)|y(t)|^2 - y(t)\langle y(t), B(t)x(t)\rangle)}{(1+|\alpha(t)|^2|y(t)|^2)^{3/2}},$$

so pick $(A,\xi) \in \Phi^{-1}(0)$, then by the Fredholm property, $R(D_2\Phi(A,\xi))$ is a closed subspace of L^2 with finite codimension. Since $R(D\Phi) \supset R(D_2\Phi) \Rightarrow \operatorname{codim}(R(D\Phi)) \leqslant \operatorname{codim}(R(D_2\Phi)) < +\infty$ and hence closed as well. So let $c(t)$ be in the (orthogonal) complement of the range such that

$$\langle D\Phi(A,\xi)(B,\eta), c\rangle_{L^2} = 0.$$

To complete this proof, we need to show that $c = 0$, hence $R(D\Phi) = L^2$, i.e. $D\Phi$ is onto. From the above inclusion, we get $\langle D_2\Phi\eta, c\rangle = 0$ Since $c \in C \cong \operatorname{coker}(D_2\Phi) = \ker(F_A)$ for some $A \in \mathcal{A}$, so $c(t)$ is a smooth solution to $\dot{c}(t) = X(t)c(t)$. The theory of PDE implies that there exists a unique solution (for $c(t) \neq$ *constant*). Moreover if $c(t_0) = 0$ for some $t_0 \in \mathbb{R}$, then $c^{(n)}(t_0) = 0$ for all n so $c(t) = 0$. So we may choose $B(t)$ with arbitrarily small compact support and hence we may reduce the above to a pointwise problem, i.e. replace $c(t)$ by $\tilde{c} = c(t_0)$ for some fix $t_0 \in \mathbb{R}$.

$$\Rightarrow \langle Bx + \alpha(Bx|y|^2 - y\langle y, Bx\rangle), \tilde{c}\rangle = 0. \tag{2}$$

Here $B = B(t_0)$, similarly for x, α, y. Since $B \in \operatorname{End}_{s,g_0}(T_pM)$, where $p = \exp_{\gamma(t_0)} \xi(t_0)$, so

$$Pz := z\langle y, y\rangle - y\langle y, z\rangle$$

is symmetric w.r.t $\langle\cdot,\cdot\rangle$ as $\langle Pz,w\rangle = \langle z,Pw\rangle$. Moreover, $\langle Pz,z\rangle = |y|^2|z|^2 - \langle y,z\rangle \geqslant 0$ by Cauchy-Schwarz inequality hence P is a positive operator, hence $\sigma(P) \geqslant 0$ implying that $(P-\lambda I)^{-1}$ exists for any $\lambda < 0$. We rewrite (1) as

$$\begin{aligned}&\langle (I+\alpha P)Bx, \tilde{c}\rangle = 0\\ \Rightarrow &\langle Bx, (I+\alpha P)\tilde{c}\rangle = 0 \ \forall B \in \mathrm{End}_{s,g_0}(T_pM)\end{aligned}$$

In particular, we may choose symmetric B such that $\langle Bx, (I+\alpha P)\tilde{c}\rangle \neq 0$, so we get contradiction unless $x(t) = 0$. Since $\Phi(A,\xi) = 0$ hence $\exp_\gamma \xi$ is constant but since $h^{\alpha\beta}$ is a finite regular homotopy, $D_2\Phi$ must be onto, hence $c(t) = 0$ as required. $\square$

Hence theorem 4.4 applied to the above map, Φ gives that $F^{-1}(0) = \mathcal{M}^f_{x,y}$ and $\mathcal{M}^{h^{\alpha\beta}}_{x,y}$ are closed submanifold of $\mathcal{P}^{1,2}_{x,y}$ for generic Riemannian metric and has dimension $\mu(x) - \mu(y)$ since the regularity at 0 implies the Fredholm index is equal to the dimension of the kernel of DF and the Implicit function theorem also tells us that its tangent space at u is given by the kernel of DF_u, where DF_u is the differential of F in the local trivialisation at u, in other words, the dimension of the manifold is equal to the dimension of its tangent space which is equal to the Fredholm index.
As we mentioned earlier, we also need the above result for the more general λ-parametrised trajectories which will be crucial to show that the induced chain homomorphism, $\Phi^{\alpha\beta}$ by $h^{\alpha\beta}$ is in fact independent of the actual choice of homotopy.
Suppose we have two finite regular homotopies:

$$h_i^{\alpha\beta}(t,\cdot) = \begin{cases} f^\alpha, & t \leqslant -R\\ f^\beta, & t \geqslant R\end{cases}$$

for $i = 1,2$ which we homotope by:

$$\begin{aligned}H^{\alpha\beta} &: [0,1]\times\mathbb{R}\times M \to \mathbb{R}\\ H^{\alpha\beta}(\lambda,t,\cdot) &= \begin{cases} f^\alpha, & t \leqslant -R\\ f^\beta, & t \geqslant R\end{cases}\\ H^{\alpha\beta}(0,\cdot,\cdot) = h_0^{\alpha\beta}, &\quad H^{\alpha\beta}(1,\cdot,\cdot) = h_1^{\alpha\beta}\end{aligned}$$

The Fredholm map, $G^{\alpha\beta}$ for the λ-trajectories is obtained by replacing ∇h_t by $\nabla H^{\alpha\beta}(\lambda,t,\cdot)$ in X_{h_t},

$$\begin{aligned}G^{\alpha\beta} :&[0,1]\times\mathcal{P}^{1,2}_{x_\alpha,y_\beta} \to L^2(\mathcal{P}^{1,2}_{x_\alpha,y_\beta} * TM)\\ &(\lambda,\gamma) \mapsto \dot{\gamma}(t) + \frac{\nabla H^{\alpha\beta}(\lambda,t,\cdot)}{\sqrt{1+|\dot{H}^{\alpha\beta}|^2\cdot|\nabla H^{\alpha\beta}|^2}}\end{aligned}$$

Theorem 4.9. *Let $h_0^{\alpha\beta}, h_1^{\alpha\beta}$ be regular smooth finite homotopies with associated Morse-Smale metrics, then there exists a generic set of $\lambda-$homotopies, $H^{\alpha\beta}$ and a generic set of suitable homotopies of the Riemannian metric, such that $(G^{\alpha\beta})^{-1}(0)$ is a $(\mu(x_\alpha) - \mu(y_\beta) + 1)$ dimensional submanifold of $[0,1] \times \mathcal{P}^{1,2}_{x_\alpha, y_\beta}$.*

Proof. The proof that 0 is a regular value of $G^{\alpha\beta}$ is same as in theorem 4.8, hence we just need to satisfy the Fredholm condition. So let $G^{\alpha\beta}(\lambda, \eta) = 0$, where we allow λ to vary. We consider the linearisation of $G^{\alpha\beta}$ in local coordinates,

$$\begin{aligned} DG^{\alpha\beta}(\lambda, \eta) : \mathbb{R} \times H^{1,2}(\gamma^* TM) &\to L^2(\gamma^* TM) \\ (\tau, \xi) &\mapsto D_1 G^{\alpha\beta}(\lambda, \eta)\tau + D_2 G^{\alpha\beta}(\lambda, \eta)\xi \end{aligned}$$

Here $\mathbb{R}$ denotes $T_\lambda[0,1]$. Since $h_0^{\alpha\beta}$ *and* $h_1^{\alpha\beta}$ are regular, so $D_2 G^{\alpha\beta}(0, \eta)$ and $D_2 G^{\alpha\beta}(1, \eta)$ are surjective from theorem 4.8. Also $D_2 G^{\alpha\beta}(\lambda, \eta)$ are Fredholm operators (although we cannot say that they are surjective as well), so it follows that $DG^{\alpha\beta}$ is Fredholm as well since $\mathbb{R}$ is 1-dimensional.

$$\begin{aligned} \operatorname{ind}(D_2 G(0, \eta)) &= \dim(\ker(D_2 G(0, \eta))) - \dim(\operatorname{coker}(D_2 G(0, \eta))) \\ &= \mu(x_\alpha) - \mu(y_\beta) - 0 \end{aligned}$$

Since both $D_2 G(0, \eta)$ and $DG(0, \eta)$ are surjective so this increases the dimension of the kernel of $DG(0, \eta)$ by 1, more precisely for each $\tau \in \mathbb{R}$, we can find ξ such that $DG^{\alpha\beta}(0, \eta)(\tau, \xi) = 0$, and by continuity of $DG^{\alpha\beta}$ in λ and the fact that the Fredholm index is locally constant, we get

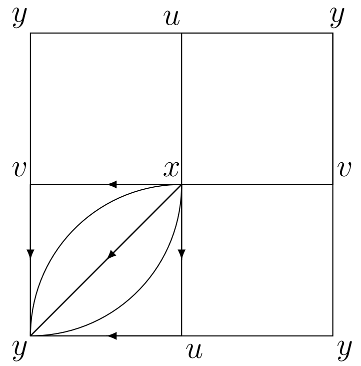
$$\operatorname{ind}(DG^{\alpha\beta}(\lambda, \eta)) = \operatorname{ind}(DG^{\alpha\beta}(0, \eta)) = 1 + \mu(x_\alpha) - \mu(y_\beta).$$

□

Remark here that $(G^{\alpha\beta})^{-1}(0)$ unlike $F^{-1}(0)$ is not a boundaryless manifold since it has boundaries corresponding to $\lambda = 0, 1$. Throughout the rest of this article, we shall always assume that a generic metric has been chosen so that the transversality property holds.

5 Compactness

Observe that the Implicit function theorem only tells us that $\mathcal{M}^f_{x,y}$ is a finite dimensional boundaryless manifold. In this section we aim to define what it means to compactify this manifold. To motivate this idea, we consider the 2-torus with critical points, u, v, x and y.



We observe that the space of trajectories on the lower left in $\mathcal{M}^f_{x,y}$ approach arbitrarily the edges in $\mathcal{M}^f_{x,v} \times \mathcal{M}^f_{v,y}$, this is the so-called broken trajectory. In this case we observe that there are 2 possible broken trajectories in each square, hence a total of 8 distinct broken trajectories. Before making this idea rigorous, we need to introduce the concept of unparametrised trajectories. More precisely, if $\gamma(t) \in \mathcal{M}^f_{x,y}$ then so is $\gamma(t + c)$ for any $c \in \mathbb{R}$. So we may identify these 2 curves under the equivalence relation, $\gamma \sim \hat{\gamma}$ if $\hat{\gamma}(t) = \gamma(t+c)$ for some real c.

Lemma 5.1. *The group,* $(\mathbb{R}, +)$ *act freely and properly on* $\mathcal{M}^f_{x,y}$ *by*

$$\begin{aligned} \mathbb{R} \times \mathcal{M}^f_{x,y} &\to \mathcal{M}^f_{x,y} \\ (\tau, \gamma) &\mapsto \gamma \bullet \tau = \gamma(\cdot + \tau) \end{aligned}$$

Sketch of proof. If we could identify $\mathcal{M}^f_{x,y}$ with $W^u(x) \bigcap W^s(y)$, then we know from classical Morse theory that the induced group action would be free and proper. (See for e.g [1]) Here $W^u(x)$ and $W^s(y)$ denote the unstable manifold of x and stable manifold of y, respectively. This identification is done by a simple evaluation map, E which we restrict to $\mathcal{M}^f_{x,y}$ i.e.

$$\begin{aligned} E : \mathcal{M}^f_{x,y} &\to W^u(x) \bigcap W^s(y) \in M \\ \gamma &\mapsto \gamma(0) \end{aligned}$$

It is straightforward to check in local coordinates that this map is indeed a diffeomorphism. Then our equivalence relation can be identified via:

$$\gamma \bullet \tau = E^{-1} \circ \psi_\tau \circ E(\gamma)$$

where ψ_τ denotes the flow along integral curves defined by $-\nabla f$ which as we mentioned above acts freely and properly.

□

We denote the space of group orbits by $\widehat{\mathcal{M}}^f_{x,y} = \mathcal{M}^f_{x,y}/\mathbb{R}$, i.e. this is the set of unparametrised trajectories. Also observe that if we define $\phi = f \circ E$, then $\phi^{-1}(a) = \mathcal{M}^{f,a}_{x,y}$ is a $(\mu(x) - \mu(y) - 1)$ dimensional manifold for any regular value $a = \gamma(0)$. Geometrically, $\mathcal{M}^{f,a}_{x,y}$ consists of those curves, $\gamma \in \mathcal{M}^f_{x,y}$ such that $f(\gamma(0)) = a$ which by the above lemma identifies with $f^{-1}(a) \bigcap W^u(x) \bigcap W^s(y)$. So $|\widehat{\mathcal{M}}^f_{x,y}|$ is simply the number of unparametrised curves joining x to y.
Recall that for metric spaces, compactness is equivalent to sequential compactness so we may formulate the notion of compactness w.r.t to our space of (unparametrised) trajectories as follows.

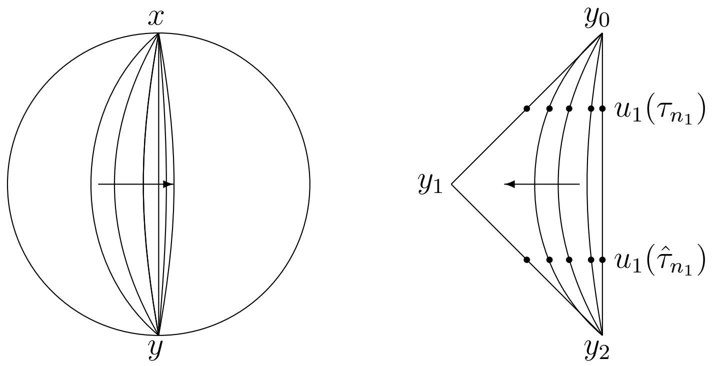
Definition 5.1. *A subset, $K \subset \widehat{\mathcal{M}}^f_{x,y}$ is compact up to broken trajectories of order k if for any sequence $\{\hat{u}_n\}_{n\in\mathbb{N}} \subset K$ either*

1. *$\hat{u}_n$ has a convergence subsequence in K, or*
2. *there exists critical points, $x = y_0, y_1, ..., y_i = y$ such that $2 \leqslant i \leqslant k$ with trajectories, $v_j \in \mathcal{M}^f_{y_j,y_{j+1}}$ and suitable reparametrisation times, $\tau_{n_{k,j}}$ such that*

$$u_{n_k} \bullet \tau_{n_{k,j}} \xrightarrow{C^\infty_{loc}} v_j$$

   for some subsequence $\{n_k\}_{k\in\mathbb{N}}$.

Note: Convergence in $\widehat{\mathcal{M}}^f_{x,y}$ is understood to be in the quotient topology induced by our equivalence relation so that $\hat{u}_n \to \hat{u}$ in $\widehat{\mathcal{M}}^f_{x,y}$ if and only if there exists a sequence t_n such that $u_n(\cdot + t_n) \to u(\cdot)$ in $\mathcal{M}^f_{x,y}$.



The picture on the left corresponds to case 1 and the picture on the left shows convergence to broken trajectories and we also see that the choice of reparametrisation times determines to which broken trajectory the curves converge to. It is due to this splitting that we cannot have $H^{1,2}$ convergence but just C^∞_{loc} convergence.

Lemma 5.2. *Every sequence of trajectories, $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{M}^f_{x,y}$ has a convergent subsequence in the C^∞_{loc} norm, i.e. there exist n_k s.t*

$$u_{n_k} \xrightarrow{C^l([-R,R])} v \in C^\infty(\mathbb{R}, M)$$

$\forall l \in \mathbb{N}$ *and* $\forall R > 0$

We shall appeal to the Arzelà-Ascoli theorem to extract such a subsequence:

Theorem 5.3. *Arzelà-Ascoli theorem*
Let $\{f_i\}_{i\in I}$ be a family of continuous functions from a compact metric space, X to a metric space, Y. Then $\{f_i\}_{i\in I}$ is precompact (in $C^0(X,Y)$) if and only if each $\{f_i\}_{i\in I}$ are pointwise precompact and equicontinuous.

proof of lemma 5.2. In order to apply Arzelà-Ascoli theorem, we need equicontinuity and pointwise precompactness. The latter follows directly from the fact that M is a compact manifold. The equicontinuity condition is obtain from the estimate:

$$\begin{aligned}
\int_s^t |\dot{u}_n(\tau)|^2 d\tau &= \int_s^t \langle -\dot{u}_n, \nabla f \circ u_n \rangle d\tau \\
&= -\int_s^t \frac{d}{d\tau}(f \circ u_n(\tau)) d\tau \\
&= f(u_n(s)) - f(u_n(t)) \\
&\leqslant f(x) - f(y), \quad \because u_n(\tau) \to x, y \text{ as } \tau \to -\infty, +\infty \text{ respectively}
\end{aligned}$$

Note that our compact Riemannian manifold, M has the distance metric given by:

$$d(x,y) = \inf\left\{ \int_0^T |\dot{\gamma}| \,\mathrm{dt} : \gamma(0) = x,\ \gamma(T) = y \right\}$$

Also by the Hopf-Rinow theorem this metric makes our manifold into a complete metric space. So

$$\begin{aligned} d(u_n(t), u_n(s)) &\leqslant \int_s^t |\dot{u}_n(\tau)| d\tau \\ &\overset{\text{Hölder}}{\leqslant} \sqrt{|t-s|} \cdot \sqrt{\int_s^t |\dot{u}_n(\tau)|^2 d\tau} \\ &\leqslant \sqrt{|t-s|} \cdot (f(x) - f(y)) \end{aligned}$$

so by Arzelá-Ascoli, on any compact interval $[-R, R]$ we can find a subsequence, n_k such that

$$u_{n_k}\Big|_{[-R,R]} \xrightarrow{C^0([-R,R])} v \in C^0([-R,R])$$

hence we have C^0_{loc} convergence. To conclude convergence in C^∞_{loc} we once again as in lemma 2.8 use a bootstrapping argument. Since ∇f is smooth, we have

$$\dot{u}_{n_k} = -\nabla f \circ u_{n_k} \xrightarrow{C^\infty_{loc}} -\nabla f \circ v = \dot{v}$$

The fact that $\dot{v}$ exists follows by uniqueness of convergence. Since $\ddot{u} = D(\nabla f)\nabla f \circ u$ so we may repeat this argument and so on (formally by induction), to conclude $u_{n_k} \xrightarrow{C^\infty_{loc}} v$ with $\dot{v} = \nabla f \circ v$. □

With consideration to convergence of type 1, we need to show:

Lemma 5.4. *If $v \in \mathcal{M}^f_{x,y}$ then $u_n \xrightarrow{\mathcal{P}^{1,2}_{x,y}} v$, i.e. convergence is in the $H^{1,2}$ norm (rather than simply C^∞_{loc}).*

Proof. We proceed in 2 steps. We first show that $u_n(t) \to y$ as $t \to +\infty$ uniformly in $n \in \mathbb{N}$, i.e. $\forall \epsilon > 0\ \exists T > 0\ s.t\ \forall t > T,\ d(u_n(t), y) < \epsilon\ \ \forall n$. Since we know that for any $R > 0$, $u_n \xrightarrow{C^\infty[-R,R]} v$ and $v(+\infty) = y$, so we need to look at the convergence near y. Observe that

$$N_\epsilon = \{p \in B(y) : |f(p) - f(y)| < \epsilon \text{ and } |\nabla f| < \epsilon\}$$

defines a fundamental system of neighbourhood of y, where $B(y)$ is a neighbourhood of y with y as the only critical point, since $\nabla f(y) = 0$ and hence

for any neighbourhood, $U(y) \ni y$ by choosing $\epsilon > 0$ small enough we can always ensure that $N_\epsilon \subset U(y)$.
Since M is compact, $|\nabla f|$ attains a maximum on M hence in particular is Lipschitz continuous so from the explicit calculation in the previous lemma we have:

$$||\nabla f(u_n(s))| - |\nabla f(u_n(t))|| \leqslant c \cdot \sqrt{|s-t|} \text{ , where } c = \sqrt{f(x) - f(y)} \cdot \max_M |\nabla f|.$$

Suppose by contradiction that convergence is not uniform, then this means that we can extract sequences $\{t_k\}, \{n_k\} \to +\infty$ such that $|\nabla f \circ (u_{n_k}(t_k))| > \epsilon$ for each k. So if $|t_k - s| < \delta = \epsilon^2/4c^2$ then $|\nabla f(u_{n_k}(s))| > \epsilon$. Hence

$$\begin{aligned} f(u_{n_k}(t_k)) - f(u_{n_k}(t_k + \delta)) = \int_{t_k}^{t_k+\delta} \frac{d}{d\tau} f(u_{n_k}(\tau)) d\tau &= \int_{t_k}^{t_k+\delta} df(\dot{u}_{n_k}(\tau)) d\tau \\ \geqslant \delta \cdot |\nabla f(u_{n_k})|^2 = \delta\epsilon^2/4 & \end{aligned}$$

here we once again used that $df(\dot{u}) = \langle \nabla f \circ u, \dot{u} \rangle$ and $-\nabla f \circ u = \dot{u}$. Since flow lines are decreasing, $f(u_{n_k}(t)) \geqslant f(y)$ for each t. So we get

$$f(u_{n_k}(t_k)) - f(y) \geqslant \delta\epsilon^2/4 \quad \forall k \in \mathbb{N}$$

Since $u_{n_k} \xrightarrow{C^\infty[-R,R]} v$ for any $R > 0$ and $t_k \to \infty$, we get $d(u_{n_k}(t_k), y) \leqslant d(u_{n_k}(t_k), v(t_k)) + d(v(t_k), y) \to 0$ which is a contradiction. The same argument can be carried out for x. This accomplishes the first step. From lemma 2.8, we get in local coordinates (taking x, y to be 0 under the chart),

$$|u_n(t)| \leqslant c\, e^{-\lambda|t|} \quad \forall\ |t| > T$$

It is by step 1 here that we may assume that this T is independent on u_n. For u_n close to v, we have

$$u_n = \exp_v \xi(t) \quad for \quad \xi(t) \in H^{1,2}(v^*TM)$$

so $||u_n - v||_{1,2} \to 0$ since we have a uniform exponential decay near the ends and C^∞ convergence on the interior, $[-T, T]$. □

We may now state the fundamental theorem of this section:

Theorem 5.5. *$\widehat{\mathcal{M}}^f_{x,y}$ is compact up to broken trajectories of order $\mu(x) - \mu(y)$, i.e. any trajectory can split into at most $\mu(x) - \mu(y)$ broken trajectories.*

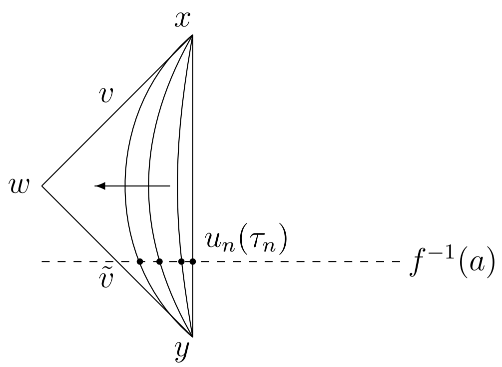
Proof. From lemma 5.3, we know that there exists a sequence $u_n \xrightarrow{C^\infty_{loc}} v \in C^\infty(M, \mathbb{R})$ and $\dot{v} = -\nabla f \circ v$. If $v \in \mathcal{M}^f_{x,y}$ then from lemma 5.4, this proves type 1 convergence. So we need to show if $v \notin \mathcal{M}^f_{x,y}$ then $v \in \mathcal{M}^f_{w,z}$ for some critical points w, z of f. Since $f(u_n) \in [f(y), f(x)]$ so by C^∞_{loc} convergence, $f(v) \in [f(y), f(x)]$ as well. We also have the estimate:

$$\int_{-T}^{T} |\dot{v}(s)|^2 ds \leqslant f(x) - f(y) \Rightarrow \int_{-\infty}^{+\infty} |\dot{v}(s)|^2 ds < \infty$$

From this inequality and by the C^∞_{loc}-convergence, we deduce that as $t \to \pm\infty$:

$$|\nabla f(v(t))| = |\dot{v}(t)| \to 0$$

which tells us from proposition 2.1 that indeed $v \in \mathcal{M}^f_{w,z}$. As we saw in the picture earlier, the reparametrisation times determine to which broken trajectory convergence occurs to. w.l.o.g we may assume $z \neq y$ and $w = x$, so let $f(y) < a < f(z)$ and we pick sequence τ_n such that $f(u_n \bullet \tau_n) = a$. Then by lemma 5.3, we get a subsequence such that $u_n \bullet \tau_n \xrightarrow{C^\infty_{loc}} \tilde{v}$ and now we have $f(\tilde{v}) \in [f(y), f(w)]$ (otherwise it would coincide with v by uniqueness of convergence) .



So we may repeat this process iteratively until we end up in $H^{1,2}$ convergence. Under the Smale transversality condition, there can only exists a non-constant trajectory if $\mu(v(-\infty)) - \mu(v(+\infty)) \geqslant 1$, so ignoring trivial trajectories we have that the order is indeed $\mu(x) - \mu(y)$ since we have the ordering $\mu(y) < ... < \mu(v(-\infty)) < \mu(v(+\infty)) < ... < \mu(x)$, i.e. any sequence can break into at most $\mu(x) - \mu(y)$ broken trajectories. $\square$

Note that we assumed that M is compact, in general if M is only complete then we need f to be a coercive function, i.e. $f^{-1}(-\infty, a]$ is compact for each $a \in \mathbb{R}$. (which is trivial in our situation). We now need to analyse the compactification for $h^{\alpha,\beta}$ and H_λ trajectories.

Theorem 5.6. *Let* $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{M}^{h^{\alpha\beta}}_{x^0_\alpha, x^l_\beta}$ *be a sequence of* $h^{\alpha\beta}$*-trajectories, then either*

1. *there is a subsequence* n_k *such that* $u_{n_k} \to v \in \mathcal{M}^{h^{\alpha\beta}}_{x^0_\alpha, x^l_\beta}$ *in the* $H^{1,2}$*-norm, or*

2. *there are critical points,* $x_\alpha = x^0_\alpha, ..., x^k_\alpha$ *of* f^α *and* $x^0_\beta, ..., x^l_\beta = x_\beta$ *of* f^β *with* $1 \geqslant k + l \geqslant \mu(x^0_\alpha) - \mu(x^l_\beta)$ *and* $h^{\alpha\beta} -$ *trajectories*

$$v^i_\alpha \in \mathcal{M}^{f^\alpha}_{x^i_\alpha, x^{i+1}_\alpha}, \quad v^j_\beta \in \mathcal{M}^{f^\beta}_{x^j_\beta, x^{j+1}_\beta}, \quad v_{\alpha\beta} \in \mathcal{M}^{h^{\alpha\beta}}_{x_\alpha, x_\beta}$$

with reparametrisation times $\tau^i_{\alpha,n}$, $\tau^j_{\beta,n}$ *such that*

$$u_n \bullet \tau^i_{\alpha,n} \xrightarrow{C^\infty_{loc}} v^i_\alpha, \quad u_n \bullet \tau^j_{\beta,n} \xrightarrow{C^\infty_{loc}} v^j_\beta, \quad u_n \xrightarrow{C^\infty_{loc}} v_{\alpha\beta}$$

and, as in the previous theorem, we have

$$\mu(x^0_\alpha) < ... < \mu(x^k_\alpha) \leqslant \mu(x^0_\beta) < ... < \mu(x^l_\beta).$$

Sketch of proof. The proof of this theorem is similar to that of the time-independent case. In order to apply Arzelà-Ascoli we need equicontinuity since precompactness is guaranteed by compactness of M, so let $\{\gamma_n\}_{n\in\mathbb{N}} \subset \mathcal{M}^{h^{\alpha\beta}}_{x_\alpha, y_\beta}$ then,

$$\begin{aligned}
d(\gamma_n(t), \gamma_n(s)) &\leqslant \int_s^t |\dot\gamma_n(\tau)| d\tau \\
&\leqslant \sqrt{|t-s|} \cdot \sqrt{\int_{-\infty}^{+\infty} |\dot\gamma(\tau)|^2 d\tau} \\
&= \sqrt{|t-s|} \cdot \sqrt{\int_{-\infty}^{+\infty} \frac{|\nabla h_\tau|^2}{1 + |\dot h_\tau|^2 |\nabla h_\tau|^2} d\tau} \\
&\leqslant \sqrt{|t-s|} \cdot \sqrt{\int_{-\infty}^{+\infty} \frac{d}{d\tau} h(\tau, \gamma_n(\tau)) d\tau - \int_{-\infty}^{+\infty} \dot h(t, \gamma_n(t))\, \mathrm{dt}} \\
&= \sqrt{|t-s|} \cdot \sqrt{f^\alpha(x_\alpha) - f^\beta(x_\beta) + 2CR}, \quad C = \max_M |\dot h_t|
\end{aligned}$$

Here we used the fact that:

$$\frac{d}{dt} h(t, \gamma(t)) = \left(\dot h_t - \frac{|\nabla h_t|^2}{\sqrt{1 + |\dot h_t|^2 |\nabla h_t|^2}} \right) \circ \gamma(t)$$

and $\dot h_t(\cdot) := \frac{\partial}{\partial t} h(t, \cdot)$ so that for $|t| \geqslant R$, $\dot h_t(\cdot) \equiv 0$. By Arzelà-Ascoli, we have proved the analog of lemma 5.3 for $h^{\alpha\beta}$-trajectories, i.e. there exists n_k such that

$$u_{n_k} \xrightarrow{C^\infty_{loc}} v \in C^\infty(\mathbb{R}, M)$$

so for sequence $\{\tau^i_{\alpha,n}\}_{n\in\mathbb{N}}$ we must have that $\tau^i_{\alpha,n} \leqslant -R$ and for $\{\tau^j_{\beta,n}\}_{n\in\mathbb{N}}$ we must have that $\tau^j_{\beta,n} \geqslant R$. Note that reparametrisation does not affect the above estimate. So again picking sequences such that $f(u_{n_k} \bullet \tau^i_{\alpha,n}) = a$ (similarly for $\tau^j_{\beta,n}$) and arguing as in the previous theorem we can conclude this proof. $\square$

We now state the corresponding result for λ-parametrised trajectories.

Theorem 5.7. *Let $\{(\lambda_n, u_n)\}_{n\in\mathbb{N}} \subset \mathcal{M}^{H^{\alpha\beta}_\lambda}_{x_\alpha,y_\beta}$ be a sequence of H_λ trajectories. Then either there exists a subsequence which converge in the $H^{1,2}$-norm or we can find reparametrisation times $\tau^i_{\alpha,n}$, $\tau^j_{\beta,n}$ and trajectories*

$$v^i_\alpha \in \mathcal{M}^{f^\alpha}_{x^i_\alpha,x^{i+1}_\alpha}, \quad v^j_\beta \in \mathcal{M}^{f^\beta}_{x^j_\beta,x^{j+1}_\beta}, \quad (\lambda, v^\lambda_{\alpha\beta}) \in \mathcal{M}^{H^{\alpha\beta}_\lambda}_{x_\alpha,y_\beta}$$

such that

$$u_n \bullet \tau^i_{\alpha,n} \xrightarrow{C^\infty_{loc}} v^i_\alpha, \quad u_n \bullet \tau^j_{\beta,n} \xrightarrow{C^\infty_{loc}} v^j_\beta, \quad (\lambda_n, u_n) \xrightarrow{C^\infty_{loc}} (\lambda, v^\lambda_{\alpha\beta})$$

The proof is along the same lines as the $h^{\alpha\beta}$ case, the equicontinuity and pointwise boundedness follows from the fact that both M and $[0,1]$ are compact. We also point out here that $\mathcal{M}^{H^{\alpha\beta}_\lambda}_{x_\alpha,y_\beta}$ is 0-dimensional if $\mu(x_\alpha) - \mu(y_\beta) + 1 = 0$ and so it is a finite set (again by compactness) of $\{(\lambda_i, u_i)\}^k_{i=1}$ trajectories.

6 Gluing

Compactness of the space of trajectories only asserts that trajectories in $\mathcal{M}^f_{x,y}$ for $\mu(x) - \mu(y) \geqslant 2$ can break into 2 or more trajectories in $\mathcal{M}^f_{x,y_1} \times \mathcal{M}^f_{y_1,y_2} \times \ldots \times \mathcal{M}^f_{y_{n-1},y}$ for $n \geqslant 2$, however this does not imply that each broken trajectory can arise in this way, i.e. given a trajectory in $\mathcal{M}^f_{y_k,y_{k+1}}$ we cannot say that there exists a sequence of trajectories converging to it. The assertion of this fact is given by the complementary operation to compactification called gluing. The analysis involved in the concept of gluing is quite technical so we shall refer to [11] for details. We start off by stating the fundamental theorem of this section:

Theorem 6.1. *Given a compact set of broken trajectories, $K \subset \mathcal{M}^f_{x,y} \times \mathcal{M}^f_{y,z}$, we can find $\rho(K) > 0$ and a smooth map, called the gluing operator, $\#$*

$$\begin{aligned} \# : K \times [\rho(K), +\infty) &\to \mathcal{M}^f_{x,y} \\ (u, v, \rho) &\mapsto u\#_\rho v \end{aligned}$$

such that $\#_\rho : K \hookrightarrow \mathcal{M}^f_{x,y}$ is an embedding (i.e. injective map with $D\#_\rho$ injective as well) for each $\rho \geqslant \rho(K)$ and for $\widehat{K} \subset \widehat{\mathcal{M}}^f_{x,y} \times \widehat{\mathcal{M}}^f_{y,z}$, $\#$ induces a smooth embedding

$$\widehat{\#} : \widehat{K} \times [\rho(\widehat{K}), +\infty) \to \widehat{\mathcal{M}}^f_{x,z}$$

such that $\widehat{u\#_\rho v} \xrightarrow{C^\infty_{loc}} (\widehat{u}, \widehat{v})$ as $\rho \to +\infty$. Moreover, any sequence of unparametrised trajectories, $\widehat{w_n} \in \widehat{\mathcal{M}^f_{x,z}}$ such that $\widehat{w_n} \to (\widehat{u}, \widehat{v})$ is in fact in the image of $\widehat{\#}$, i.e. the gluing map $\widehat{\#}$ provides an "inverse" for the compactification process.

The construction of this operator is accomplished in 3 steps:

1. We construct a pre-gluing map, $\#^o_\rho : \mathcal{M}^f_{x,y} \times \mathcal{M}^f_{y,z} \to \mathcal{P}^{1,2}_{x,z}$ such that $u\#^o_\rho v(\overline{\mathbb{R}})$ and $u(\overline{\mathbb{R}}) \bigcup v(\overline{\mathbb{R}})$ are close in the metric sense and complying with $u\#^o_\rho v \xrightarrow{C^\infty_{loc}} (\widehat{u}, \widehat{v})$ as $\rho \to +\infty$.

2. We then associate to $u\#^o_\rho v$ a trajectory in $\mathcal{M}^f_{x,z}$ in a non-canonical manner. This is the crucial step in constructing $\#$. It is by an application of the Banach contraction mapping principle that we shall show that if ρ is large enough, then there is in fact a unique such choice. More precisely we find a unique section, γ such that $u\#_\rho v = \exp_{u\#^o_\rho v} \gamma \in \mathcal{M}^f_{x,z}$.

3. Finally we simply need to verify that this mapping is indeed an embedding.

Construction of the Pre-gluing map, $\#^o_\rho$

We start off by defining the simplest gluing operation. Let $K \subset \mathcal{M}^f_{x,y} \times \mathcal{M}^f_{y,z}$ be compact, then given $(u, v) \in K$ we may choose $(\tilde{u}, \tilde{v}) \in C^\infty_{x,y} \times C^\infty_{y,z}$ which are asymptotically constant at y, i.e.

$$\begin{aligned}\tilde{u}(t) &= y, \quad t \geqslant \tilde{T}, \text{ and}\\ \tilde{v}(t) &= y, \quad t \geqslant -\tilde{T},\end{aligned}$$

where this $\tilde{T}$ is dependent on $\tilde{u}$ and $\tilde{v}$, such that u and v are in neighbourhood of $\tilde{u}$ and $\tilde{v}$ respectively, i.e. there exists $\xi \in H^{1,2}(\tilde{u}^*TM)$ and $\eta \in H^{1,2}(\tilde{v}^*TM)$ such that $u = \exp_{\tilde{u}}(\xi)$ and $v = \exp_{\tilde{v}}(\eta)$. Since we assumed K is compact, we may find a finite subcover and pick T to be the maximum of all such $\tilde{T}$. We may then define

$$\tilde{u}\#_\rho\tilde{v}(t) = \begin{cases} \tilde{u}(t+\rho), & t \leqslant 0 \\ \tilde{v}(t-\rho), & t \geqslant 0 \end{cases}$$

and so $\tilde{u}\#_\rho\tilde{v} \in C^\infty_{x,z}$ for $\rho \geqslant T + 1$. Note that smoothness follows from the fact that $\tilde{u}$ and $\tilde{v}$ are each smooth and are asymptotically constant at y hence $\tilde{u}\#_\rho\tilde{v}$ is constant on interval $(-1, 1)$. So given smooth cut-off functions, $\beta^\pm : \overline{\mathbb{R}} \to [0, 1]$ where

$$\beta^+(t) = \begin{cases} 0, & t \leqslant 0 \\ 1, & t \geqslant 1 \end{cases}, \quad \beta^-(t) = \begin{cases} 1, & t \leqslant -1 \\ 0, & t \geqslant 0 \end{cases}$$

we define:

Definition 6.1 (Pre-gluing map)**.**

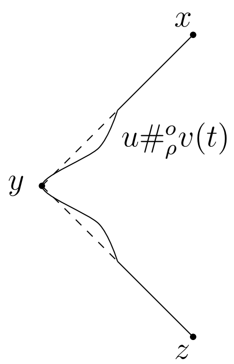
$$\#^o : K \times [\rho_0, \infty) \to \mathcal{P}^{1,2}_{x,y}$$

$$u\#^o_\rho v(t) = \exp_{\tilde{u}\#_\rho\tilde{v}(t)}(\beta^-(t)(\exp^{-1}_{\tilde{u}} u)(t+\rho) + \beta^+(t)(\exp^{-1}_{\tilde{v}} v)(t-\rho))$$

More explicitly,

$$u\#^o_\rho v(t) = \begin{cases} u_\rho(t), & t \leqslant -1 \\ \exp_y(\beta^-(t)\exp^{-1}_y(u_\rho) + \beta^+(t)\exp^{-1}_y(v_{-\rho}))(t), & |t| \leqslant 1 \\ v_{-\rho}(t), & t \geqslant 1 \end{cases}$$

for $\rho \geqslant T$ where we used the notation $u_\rho(t) = u(t + \rho)$. Moreover, since $\xi \in H^{1,2}(\tilde{u}^*TM)$, $\eta \in H^{1,2}(\tilde{v}^*TM)$ and $\beta^\pm$ are smooth, we have that $u\#^o_\rho v(t) \in C^\infty(\overline{\mathbb{R}}, M) \subset \mathcal{P}^{1,2}_{x,y}$.



$\#_\rho^o$ is smooth in u, v. Using that $T_{u\#_\rho^o v}\mathcal{P}_{x,y}^{1,2} = H^{1,2}((u\#_\rho^o v)^*TM)$, $T_u\mathcal{M}_{x,y}^f = \ker DF_u$ and $T_v\mathcal{M}_{y,z}^f = \ker DF_v$ (from the results of transversality) we get its derivative:

$$\begin{aligned} D\#_\rho^o : \ker DF_u \times \ker DF_v &\to H^{1,2}((u\#_\rho^o v)^*TM) \\ (\xi, \eta) &\mapsto \nabla_2 \exp(\beta^-(\nabla_2 \exp^{-1}\xi)_\rho + \beta^+(\nabla_2 \exp^{-1}\eta)_{-\rho}) \end{aligned}$$

Construction of actual gluing map, $\#_\rho$
For simplicity we shall use the notations $\chi = (u, v, \rho) \in K \times [\rho_o, \infty)$, $w_\chi = u\#_\rho^o v \in \mathcal{P}_{x,y}^{1,2}$ and F_γ to be the trivialised operator, F at $\gamma \in C^\infty(\overline{\mathbb{R}}, M)$, more precisely,

$$\begin{aligned} F_\gamma : H^{1,2}(\gamma^*TM) &\to L^2(\gamma^*TM) \\ \eta &\mapsto (\nabla_2 \exp_\gamma(\eta))^{-1} \circ F \circ (\exp_\gamma)(\eta) \end{aligned}$$

We mention here that F_{w_χ} is well-defined since w_χ is smooth by construction. So if we find an appropriate η such that $F_{w_\chi}(\eta) = 0$. then we can conclude that $\exp_{w_\lambda}\eta$ is a zero of F, i.e. we have found a trajectory in $\mathcal{M}_{x,z}^f$.

In order to find this section we need to construct a normal bundle, i.e. a vector bundle that is orthogonal to the tangent bundle and this bundle obviously depends on the embedding, i.e. it is non-canonical. We now want to restrict our attention to these glued trajectories only, so we define the following pullback Hilbert bundles via $\#^o : K \times [\rho_0, \infty) \to \mathcal{P}_{x,z}^{1,2}$ by,

$$H := \#^{o*}H^{1,2}(\mathcal{P}_{x,z}^{1,2*}TM) \equiv \#^{o*}T\mathcal{P}_{x,z}^{1,2} \;\; \textit{and} \;\; L := \#^{o*}L^2(\mathcal{P}_{x,z}^{1,2*}TM) \;\; \textit{i.e.}$$

$$\begin{tikzcd} H \arrow[r, "\#^{o*}"] \arrow[d, "pr_1"'] & T\mathcal{P}_{x,z}^{1,2} \arrow[d, "\tau"] \\ K \times [\rho_o, \infty) \arrow[r, "\#^o"] & \mathcal{P}_{x,z}^{1,2} \end{tikzcd}$$

Here the inner product on this bundle is naturally induced by the Riemannian metric, g. Same for L. We shall denote the fibres at χ by H_χ and L_χ. The inclusion map $i : K \hookrightarrow \mathcal{M}^f_{x,y} \times \mathcal{M}^f_{y,z}$ also induces a pullback bundle on $K \times [\rho_0, \infty)$ and so induces another Hilbert bundle denoted by:

$$(H^{1,2})^2 := i^*(H^{1,2}(\mathcal{P}^{1,2*}_{x,y}TM)\big|_{\mathcal{M}^f_{x,y}} \times H^{1,2}(\mathcal{P}^{1,2*}_{y,z}TM)\big|_{\mathcal{M}^f_{y,z}})$$

Here we restrict to the Sobolev sections of the pullback bundles by trajectories in $\mathcal{M}^f_{x,y}$ and $\mathcal{M}^f_{y,z}$. These pullback bundles enable us to identify the map, F_{w_χ} defined above with a smooth bundle map, $F_\chi : H_\chi \to L_\chi$. In essence, what we have done is construct a natural way to transfer Banach bundles on $\mathcal{P}^{1,2}_{x,z}$ to bundles on $K \times [\rho_o, \infty)$ via our pre-gluing map. We see from these notations that

$$F_{w_\chi}(0) = F(w_\chi)$$

from the fact that $\nabla_2 \exp(0)^{-1} = \mathrm{Id}$, and the fibre derivative of F_χ at 0 is

$$D_\chi := D_2F_\chi(0) \equiv DF_{w_\chi}(0) : H_\chi \to L_\chi$$

and we know that $DF_{w_\chi}(0)$ is simply the trivialised Fredholm operator of the form:

$$\frac{\partial}{\partial t} + A(t)$$

we analysed in the Fredholm section.

Definition 6.2. *With respect to the bundles, $(H^{1,2})^2$ and H we define an extension of the linearisation of $\#^o_\rho$ as follows:*

$$\begin{tikzcd}
(H^{1,2})^2 \arrow[r, "\#"] \arrow[d, "pr_1"'] & H \arrow[dl, "pr_1"] \\
K \times [\rho_o, \infty) &
\end{tikzcd}$$

$$(\xi\#\eta)(t) = \begin{cases} \xi_\rho(t), & t \leqslant -1 \\ \nabla_2 \exp(\beta^-(\nabla_2 \exp^{-1}\xi)_\rho + \beta^+(\nabla_2 \exp^{-1}\eta)_{-\rho}), & |t| \leqslant 1 \\ \eta_{-\rho}(t), & t \geqslant 1 \end{cases}$$

which naturally coincides with $D\#^o$.

Our aim is to now find a normal space to $T\mathcal{M}^f_{x,z}$. We already have this tangent bundle as a subbundle of $H^{1,2}(\mathcal{P}^{1,2*}_{x,z}TM)\big|_{\mathcal{M}^f_{x,z}}$ so we need a concept of orthogonality to distinguish between these tangent and normal vectors.

The key idea is to apply the linearised operator # then project onto the tangent bundle orthogonally and show that the complement space indeed is a normal bundle. As we mentioned earlier we also have to appeal to the contraction mapping principle but we first need to construct an appropriate contraction map.

Definition 6.3. *From our analysis in the transversality section, $D_u = DF_u$ and $D_v = DF_v$ are surjective Fredholm operators for $(u,v) \in \mathcal{M}^f_{x,y} \times \mathcal{M}^f_{y,z}$, and their kernels define the tangent spaces so*

$$(\ker)^2 = \bigcup_{\chi \in K \times [\rho_0, \infty)} \ker D_u \times \ker D_v \subset (H^{1,2})^2$$

defines a subbundle of dimension $\operatorname{ind}(D_u) + \operatorname{ind}(D_v)$*, so we define*

$$L^\perp_\chi = \{v_\chi \in H^{1,2}_{w_\chi} = H_\chi : \langle v_\chi, \xi \#_\rho \eta \rangle^{0,2}_\chi = 0 \;\; for \;\; all \;\; (\xi, \eta) \in \ker D_u \times \ker D_v\}$$

i.e. $L^\perp_\chi$ consists of vectors in H_χ which are orthogonal (w.r.t the L^2 inner product) to the range of $\#_\rho|_{(\ker)^2}$.

Theorem 6.2. *There exists $\rho_1 \geqslant \rho_0$ such that for $\rho \geqslant \rho_1$ and $(u,v) \in K$ the Fredholm operator, D_χ is surjective and*

$$\phi_\chi := P \circ \#_\chi : \ker D_u \times \ker D_v \xrightarrow{\simeq} \ker D_\chi \subset H_\chi$$

is an isomorphism, where P is the orthogonal projection onto $\ker D_\chi$ *given by $H_\chi = \ker D_\chi \oplus (\ker D_\chi)^\perp$ where orthogonal decomposition is with respect to $\langle \cdot, \cdot \rangle_{0,2}$.*

We conclude from the above theorem that

$$\begin{aligned} \dim(\ker D_\chi) &= \dim(\ker D_u) + \dim(\ker D_v) \\ &= \mu(x) - \mu(y) + \mu(y) - \mu(z) \\ &= \mu(x) - \mu(z) \end{aligned}$$

Hence since D_χ is Fredholm,

$$L^\perp = \bigcup_{\chi \in K \times [\rho, \infty)} L^\perp_\chi$$

is a finite codimensional vector bundle such that we have the bundle decomposition

$$H = L^\perp \oplus R(\#|_{(\ker)^2}).$$

Lemma 6.3. *There exists $c > 0$ and $\hat{\rho} > 0$ such that*

$$||D_\chi \xi||_{L^2_\chi} \geqslant c \cdot ||\xi||_{L^2_\chi}$$

for all $\rho \geqslant \hat{\rho}$ and for all $\xi \in L^\perp_\chi$

See [11] for the proof of this technical lemma.

proof of theorem 6.2 . We argue by contradiction. Suppose ϕ_χ is not onto then there exists $\xi \in \ker(D_\chi)$ such that $\xi \notin P(R(\#_\chi))$. Since $\mathrm{span}(\xi) \subset \ker(D_\chi)$ so by the decomposition

$$H = \mathrm{span}(\xi) \oplus (\mathrm{span}(\xi))^\perp$$

we have that $R(\#_\chi) \subset (\mathrm{span}(\xi))^\perp$ hence $\langle R(\#_\chi), \xi\rangle = 0$ i.e $\xi \in L^\perp_\chi$. From the lemma this implies $\xi = 0$, hence a contradiction.
Surjectivity of D_u and D_v from transversality gives

$$\mathrm{ind}(D_\chi) = \mu(x) - \mu(z) = \dim(\ker D_u) + \dim(\ker D_v) \geqslant \dim(\ker D_\chi) \geqslant \mathrm{ind}(D_\chi)$$

hence D_χ is surjective and so counting dimensions shows that ϕ_χ is indeed an isomorphism.

$\square$

In particular we are led to

$$H = L^\perp \oplus \ker D, \text{ where } \ker D := \bigcup_{\chi \in K \times [\rho, \infty)} \ker D_\chi$$

since if there exists $\xi \in \ker D$ such that $\xi \notin R(\#|_{(\ker)^2})$ then as in the above proof $\xi = 0$. Hence we also see that $R(\#|_{(\ker)^2}) = \ker D$. So $L^\perp$ is indeed an appropriate normal bundle.
As mentioned earlier $w_\chi = u\#^o_\rho v$ is not in $\mathcal{M}^f_{x,z}$, so we need to find a unique section of the bundle, $H \to K \times [\rho, \infty)$ such that $\exp_{w_\chi} \gamma(\chi) \in \mathcal{M}^f_{x,z}$. According to our fundamental theorem we also need that $\exp_{w_\chi} \gamma(\chi) \xrightarrow{C^\infty_{loc}} (u, v)$ as $\rho \to \infty$ with respect to the compactness property. Since $w_\chi \xrightarrow{C^\infty_{loc}} (u, v)$ as $\rho \to \infty$ by construction, so it is sufficient to show $\gamma(\chi) \to 0$ sufficiently fast so that the above convergence holds. Since $H_\chi = \ker D_\chi \oplus L^\perp_\chi$ and D_χ is onto, $D_\chi\big|_{L^\perp_\chi} : L^\perp \xrightarrow{\simeq} L$ is an isomorphism so we denote by G its right inverse.
Similar to lemma 6.3, we obtain the estimate:

Lemma 6.4. *There exists $\rho_2 \geqslant \rho_1$ and a constant $d > 0$ such that*

$$||G_\chi \xi||_{1,2} \leqslant d||\xi||_{0,2}$$

for each $\chi \in K \times [\rho_2, \infty)$ and each $\xi \in L_\chi$

Considering the Taylor series expansion of F_χ for s near 0, we have

$$F_\chi(s) = F_\chi(0) + D_\chi(0) \cdot s + N_\chi(0, s)$$

such that $N_\chi(0, s)/||s||_{1,2} \to 0$ as $||s||_{1,2} \to 0$. A direct computation for x, y near 0 gives:

$$N_\chi(0, x) - N_\chi(0, y) = (D_\chi(y) - D_\chi(0))(x - y) + N_\chi(y, x - y)$$

So we obtain the estimate

$$||N_\chi(0, x) - N_\chi(0, y)||^\chi_{0,2} \leqslant C\ (||x||^\chi_{1,2} + ||y||^\chi_{1,2})\ ||x - y||^\chi_{1,2}$$

where C is a constant. The fact that C is independent of χ is once again due to the compactness of K and M. Together with the above lemma, we have the inequality:

$$\begin{aligned} ||G_\chi N_\chi(0, x) - G_\chi N_\chi(0, y)||^\chi_{1,2} &\leqslant d\ ||N_\chi(0, x) - N_\chi(0, y)||^\chi_{0,2} \\ &\leqslant d \cdot C(||x||^\chi_{1,2} + ||y||^\chi_{1,2})\ ||x - y||^\chi_{1,2} \end{aligned}$$

and

$$||G_\chi F_\chi(0)||^\chi_{1,2} \leqslant d\ ||F_\chi(0)||^\chi_{0,2} = d\ ||F(w_\chi)||_{0,2}$$

for $\rho \geqslant \rho_2$ of course.
The following lemma shows that as $\rho \to \infty$, our approximate glued trajectories indeed converge to actual trajectories, i.e. they tend to zeroes of the map, F.

Lemma 6.5. *There exists $m > 0$ and $\alpha > 0$ such that*

$$||F(w_\chi)||^\chi_{0,2} \leqslant \alpha e^{-m\rho}$$

for all $\rho \geqslant \rho_0$ and $(u, v) \in K$.

The proof is essentially to observe that for $|t| > 1$, w_χ correspond to actual trajectories namely, u_ρ and $v_{-\rho}$, hence $F(u_\rho) = F(v_{-\rho}) = 0$. So we need to analyse this trajectory for $|t| \leqslant 1$. By our choice of asymptotically constant curves, $\tilde{u}$ and $\tilde{v}$, we may reduce the problem in local coordinates at y and appeal to lemma 2.8 to get the required estimate.
We summarise the estimates we have collected so far:

$$||G_\chi F_\chi(0)||^\chi_{1,2} \leqslant \tilde{\alpha} e^{-m\rho}\ \ for\ \rho \geqslant \rho_0$$

$$||G_\chi N_\chi(0, x) - G_\chi N_\chi(0, y)||^\chi_{1,2} \leqslant \tilde{d}(||x||^\chi_{1,2} + ||y||^\chi_{1,2})\ ||x - y||^\chi_{1,2}$$

for $x, y \in B(0, \epsilon)$ for small $\epsilon > 0$ and where $G_\chi = (D_\chi\big|_{L^\perp})^{-1}$.

Lemma 6.6. *There exists a unique $\gamma(\chi) \in B(0,\epsilon)\bigcap L_\chi^\perp$ such that*

$$F_\chi(\gamma(\chi)) = 0 \text{ and } ||\gamma(\chi)||_{1,2}^\chi \leqslant d\ e^{-m\rho}$$

for constants $d, m > 0$ and $\rho \geqslant \rho_0$.

Theorem 6.7 (Banach Contraction mapping principle)**.** *Given a non-empty complete metric space, (X, d) and a contraction map, $T : X \to X$ then T has a unique fixed point, i.e. there is a unique $x \in X$ such that $T(x) = x$ and moreover it is given by $x = \lim_{n\to\infty} T(x_n)$ for any choice of x_0 and setting $x_{k+1} = T(x_k)$.*

proof of lemma 6.6. We may choose ρ large enough so that $||G_\chi F_\chi(0)||_{1,2}^\chi \leqslant \frac{\epsilon}{2}$ then we define

$$\begin{aligned} \phi : H_\chi &\to H_\chi \\ x &\mapsto -G_\chi F_\chi(0) - G_\chi N_\chi(0, x) \end{aligned}$$

From the second inequality above we get the estimate:

$$||\phi(x) - \phi(y)||_{1,2}^\chi \leqslant 2C\epsilon||x - y||_{1,2}^\chi$$

By choosing ϵ smaller if needed we may assume ϕ is a contraction map. Then appealing to the contraction mapping principle, we find unique $\gamma(\chi) \in B(0,\epsilon)\bigcap L_\chi^\perp$ such that $\phi(\gamma(\chi)) = \gamma(\chi)$ and together with $F_\chi(\gamma(\chi)) = F_\chi(0) + D_\chi(0)\cdot\gamma(\chi) + N_\chi(0, \gamma(\chi))$ this leads to:

$$G_\chi F_\chi(\gamma(\chi)) = 0 \Rightarrow F_\chi(\gamma(\chi)) = 0$$

Moreover,

$$\begin{aligned} ||\gamma(\chi)||_{1,2}^\chi &\leqslant ||\gamma(\chi) - \phi(0)||_{1,2}^\chi + ||\phi(0)||_{1,2}^\chi \\ &\leqslant \frac{1}{2}||\gamma(\chi)||_{1,2}^\chi + ||\phi(0)||_{1,2}^\chi \end{aligned}$$

i.e $||\gamma(\chi)||_{1,2}^\chi \leqslant ||G_\chi F_\chi(0)||_{1,2}^\chi \leqslant d\ e^{-m\rho}$ □

By the uniqueness of $\gamma(\chi)$, we can now define the gluing map by:

$$u\#_\rho v := \exp_{u\#_\rho^o v} \gamma(\chi)$$

and from the above lemma $F_{w_\chi}(\gamma(\chi)) = 0$ hence $F(\exp_{w_\chi}\gamma(\chi)) = 0$ so it is indeed a trajectory in $\mathcal{M}_{x,z}^f$.

The Embedding property

Concerning the embedding property we refer to [11] for the proofs of the following theorems.

Theorem 6.8. *There exists $\rho(\widehat{K}) > 0$ such that the unparametrised gluing map,*

$$\begin{aligned} \hat{\#} : \widehat{K} \times [\rho(\widehat{K}), \infty) &\to \widehat{\mathcal{M}}^f_{x,z} \\ (\hat{u}, \hat{v}, \rho) &\mapsto \hat{u}\hat{\#}_\rho \hat{v} \end{aligned}$$

is a smooth embedding. Here $\hat{u}$ is once again identified with $u(0)$ such that $f(u(0)) = a$ for some $a \in [f(y), f(x)]$ and for $\hat{u}\hat{\#}_\rho\hat{v}$ we may choose an appropriate $\tau_{u,v,\rho}$ such that $f(u\#_\rho v(\tau_{u,v,\rho})) = a$. (Same for v with some $b \in [f(z), f(y)]$)

Theorem 6.9. *Given broken trajectories $(u, v) \in \widehat{\mathcal{M}}^f_{x,y} \times \widehat{\mathcal{M}}^f_{y,z}$, for any sequence $\rho_n \to \infty$, we have*

$$\hat{u}\hat{\#}_{\rho_n}\hat{v} \xrightarrow{C^\infty_{loc}} (\hat{u}, \hat{v})$$

Moreover, for any sequence, $\hat{w}_n \in \mathcal{M}^f_{x,z}$ such that $\hat{w}_n \xrightarrow{C^\infty_{loc}} (\hat{u}, \hat{v})$, there exists N such that for $k \geqslant N$ $\hat{w}_n \in R(\hat{\#})$.

Sketch. The proof of the first part is essentially to choose appropriate reparametrisation times, τ_n such that $f(\hat{u}\hat{\#}_{\rho_n}\hat{v} \bullet \tau_n) = a$ then as in the compactness section we obtain:

$$\hat{u}\hat{\#}_{\rho_n}\hat{v} \bullet \tau_n \xrightarrow{C^\infty_{loc}} \hat{w} \in \mathcal{M}^f$$

Since

$$|\gamma(\chi_n)(\tau_n)| \leqslant c \cdot e^{-\alpha\rho_n} \to 0 \quad as\ \rho_n \to \infty$$

then from our construct of the pre-gluing map and we obtain

$$\hat{u}\hat{\#}_{\rho_n}\hat{v} \bullet \tau_n \to \hat{u}\hat{\#}^o_{\rho_n}\hat{v} \bullet \tau_n \to \hat{u}(\tau_n - \rho_n)$$

here convergence is pointwise and so $\hat{w} = \hat{u}$ by uniqueness of limit. Choosing another appropriate sequence of reparametrisation times we get the result for $\hat{v}$.
For the second part, it suffices to observe that if $\hat{w}_n \xrightarrow{C^\infty_{loc}} (\hat{u}, \hat{v})$ then it eventually lies in a neighbourhood where lemma 6.6 works. Then by the existence and uniqueness of a fixed point and the fact $\hat{w}_n$ is a solution, it must be in the range of $\#$. □

So we have finally proved the fundamental theorem 6.1 stated at the beginning of this section. Since $\widehat{\mathcal{M}}^f_{x,z}$ is a $\mu(x) - \mu(z) - 1$ dimensional manifold, so for critical points of relative Morse index 2, $\widehat{\mathcal{M}}^f_{x,z}$ is a boundaryless 1-dimensional manifold, so that each of its connected component is either diffeomorphic to S^1 or $(-1, 1)$. If it is diffeomorphic to the circle then there are no broken trajectories since it is in fact a closed manifold. On the other hand if it is diffeomorphic to $(-1, 1)$ then we know that it has a compactification by broken trajectories corresponding to ends $\{0\}$ and $\{1\}$. In fact there are exactly two distinct broken trajectories since working in a neighbourhoods $(-1, -1+\epsilon)$ and $(1-\epsilon, 1)$ and appealing to the existence and uniqueness of a solution in each neighbourhood we deduce that both glued trajectories must be distinct. We make this observation into a definition which will be crucial to prove that $\partial^2 = 0$.

Definition 6.4. *Let $\widetilde{\mathcal{M}}^f_{x,z}$ denote the set of simply broken trajectories with $\mu(x) - \mu(z) = 2$ and define the equivalence relation*

$$(\hat{u}_1, \hat{v}_1) \sim (\hat{u}_2, \hat{v}_2) \Leftrightarrow (\hat{u}_1 \hat{\#}_{\rho_1} \hat{v}_1) \sim (\hat{u}_2 \hat{\#}_{\rho_n} \hat{v}_2) \text{ in } \widehat{\mathcal{M}}^f_{x,z}$$

where a pair of broken trajectories are equivalent if when glued in $\widehat{\mathcal{M}}^f_{x,z}$ they belong to the same pathwise connected component or put differently they are equivalent if they correspond to the ends of the same connected component. From the above observation, each equivalence class contains exactly 2 elements. This equivalence relation is called the cobordism equivalence since the broken trajectories correspond to disjoint boundaries of a connected component of $\mathcal{M}^f_{x,z}$ and are one dimension lower. Also note that this set is finite.

We now still need to consider the $h^{\alpha\beta}$-trajectories and λ-parametrised trajectories.
In the $h^{\alpha\beta}$-trajectories situation there are two additional possible type of gluing that needs to be considered namely gluing of trajectories where one depends on the homotopy and the other one on either f^α or f^β and the case when both trajectories depend on the homotopy. We shall once again restrict to relative Morse index 2. For the first case, there are two possible types of broken trajectories, $\mathcal{M}^{f_\alpha}_{x_\alpha,y_\alpha} \times \mathcal{M}^{h^{\alpha\beta}}_{y_\alpha,y_\beta}$ and $\mathcal{M}^{h^{\alpha\beta}}_{x_\alpha,x_\beta} \times \mathcal{M}^{f_\beta}_{x_\beta,y_\beta}$. The analysis is similar as in the time-independent case. The difference is essentially in the definition of the pre-gluing map. For the first type of broken trajectory we define:

$$u\#^o_\rho v_h(t) = \begin{cases} u_{2\rho}(t), & t \leqslant -\rho+1 \\ \exp_{y_\alpha}(\beta^-_\rho[\exp^{-1}_{y_\alpha}(u)]_{2\rho} + \beta^+_\rho[\exp^{-1}_{y_\alpha}(v_h)])(t), & |t-\rho| \leqslant 1 \\ v_h(t), & t \geqslant -\rho+1 \end{cases}$$

and second type:

$$u_h\#^o_\rho v(t) = \begin{cases} u_h(t), & t \leqslant \rho-1 \\ \exp_{x_\beta}(\beta^-_{-\rho}[\exp^{-1}_{x_\beta}(u_h)] + \beta^+_{-\rho}[\exp^{-1}_{x_\beta}(v)]_{-2\rho})(t), & |t-\rho| \leqslant 1 \\ v_{-2\rho}(t), & t \geqslant \rho+1 \end{cases}$$

Notice here that a time-shifting by the gluing parameter, ρ is only carried out at the time independent trajectory. Then by analogy we obtain the corresponding gluing theorem:

Theorem 6.10. *For a compact set of simply broken trajectories* $\widehat{K} \subset \widehat{\mathcal{M}}^{f_\alpha}_{x_\alpha,y_\alpha} \times \mathcal{M}^{h^{\alpha\beta}}_{y_\alpha,y_\beta}$ *we can find* ρ_0 *and a smooth embedding map,* $\#$ *such that*

$$\begin{aligned} \# : \widehat{K} \times [\rho_0, \infty) &\hookrightarrow \mathcal{M}^{h^{\alpha\beta}}_{x_\alpha,y_\beta} \\ (\hat{u}, v_h, \rho) &\mapsto (\hat{u}\#_\rho v_h) \end{aligned}$$

and once again as $\rho_n \to \infty$ *we have* $\hat{u}\#_\rho v_h \xrightarrow{C^\infty_{loc}} (\hat{u}, v_h)$. *[Same for* $\widehat{K} \subset \mathcal{M}^{h^{\alpha\beta}}_{x_\alpha,x_\beta} \times \widehat{\mathcal{M}}^{f_\beta}_{x_\beta,y_\beta}$ *]*

Coming back to the second situation, i.e. when $(u_{\alpha\beta}, u_{\beta,\gamma}) \in \mathcal{M}^{h^{\alpha\beta}}_{x_\alpha,x_\beta} \times \mathcal{M}^{h^{\beta\gamma}}_{x_\beta,x_\gamma}$ we define in this situation the pre-gluing map by

$$(u_{\alpha\beta}\#^o_R v_{\beta\gamma})(t) = \begin{cases} u_{\alpha\beta}(t+R), & t \leqslant -1 \\ \exp_{x_\beta}(\beta^-[\exp^{-1}_{x_\beta}(u_{\alpha\beta})]_R + \beta^+[\exp^{-1}_{x_\beta}(v_{\beta\gamma})]_{-R})(t), & |t| \leqslant 1 \\ v_{\beta\gamma}(t-R), & t \geqslant 1 \end{cases}$$

Moreover in this case the gluing map, $\#_R$ also determines the codomain, $\mathcal{M}^{h^{\alpha\gamma}_R}_{x_\alpha,x_\gamma}$ and as such we need to define an appropriate homotopy:

$$h^{\alpha\gamma}_R(t) = \begin{cases} h^{\alpha\beta}(t+R,\cdot), & t \leqslant 0 \\ h^{\beta\gamma}(t-R,\cdot), & t \geqslant 0 \end{cases}$$

We again have the same embedding property for the gluing map in this situation. An important consequence of this, which we shall use to prove independence of our chain homomorphism corresponding to different Morse functions, is stated as follows:

Theorem 6.11. *For isolated h-trajectories, i.e.*

$$\mu(x_\alpha) = \mu(x_\beta) = \mu(x_\gamma)$$

there is an $R > 0$ *such that the gluing map*

$$\#_R : \mathcal{M}^{h^{\alpha\beta}}_{x_\alpha,x_\beta} \times \mathcal{M}^{h^{\beta\gamma}}_{x_\beta,x_\gamma} \xrightarrow{\simeq} \mathcal{M}^{h^{\alpha\gamma}}_{x_\alpha,x_\gamma}$$

is an isomorphism.

To conclude this section, we state the gluing theorem for λ-trajectories.

Theorem 6.12. *Given a compact set, $\widehat{K} \subset \mathcal{M}^H_{x_\alpha, y_\beta} \times \widehat{\mathcal{M}}^{f^\beta}_{y_\beta, z_\beta}$ of mixed simply broken trajectories and λ-trajectories, $(\lambda, u_\lambda, \hat{v})$ then there exists $\rho_0 > 0$ and a smooth embedding map,*

$$\begin{aligned} \#^H : \widehat{K} \times [\rho_0, \infty) &\hookrightarrow \mathcal{M}^H_{x_\alpha, z_\beta} \\ ((\lambda, u_\lambda), \hat{v}, \rho) &\mapsto (\lambda, u_\lambda) \#^H_\rho \hat{v} = (\tilde{\lambda}, w_{\tilde{\lambda}}) \end{aligned}$$

such that for any sequence $\rho_n \to \infty$, we have $w_{\tilde{\lambda}} \xrightarrow{C^\infty_{loc}} (u_\lambda, \hat{v})$ and $\tilde{\lambda} \to \lambda$. [Same for $\widehat{\mathcal{M}}^{f^\alpha}_{x_\alpha, y_\alpha} \times \mathcal{M}^H_{y_\alpha, z_\beta}$]

7 Orientation of Space of Trajectories

The analysis carried out so far enables us to define the Morse Homology over the finite ring, $\mathbb{Z}_2 = \{\overline{0}, \overline{1}\}$ but in order to extend our homology theory to the ring of integers, $\mathbb{Z}$ we need the concept of orientation. Recall that an orientation of an n-manifold is just a choice of a volume form, i.e. a nowhere vanishing section of the bundle, $\Lambda^n(T^*M)$. To define an analog concept of orientation on our space of trajectories, we first observe that each trajectory, $u \in C^\infty(\overline{\mathbb{R}}, M)$ has an associated Fredholm operator, $D_2F(u)$. We shall define an orientation concept using these Fredholm operators. Since this section is quite long, we first give an overview of the main ideas involved:

We start by defining an orientation concept for a general topological space which has a Fredholm operator associated to each point. This is done by constructing a determinant bundle which provides an analog to the top exterior power of the cotangent bundle. An orientation is then defined a non-vanishing section of this line bundle. Then looking at the trivial case, there exists an orientation that we can define on $\Theta_{F_A} \in \Sigma$ since it is contractible. We define a gluing map of these Fredholm operators in different classes in such a way that it induces an orientation on the target class which is independent of the actual gluing map and which obeys the rule of associativity. We then move on to the non-trivial case where we have that each trajectory, u has an associated Fredholm operator, D_u. We construct suitable trivialisations that enable us to transfer to the trivial case in a coherent manner. Then we show that the concept of gluing of operators developed in the trivial case is actually compatible with the gluing map constructed in the previous section. Finally we show that there indeed exists a choice of orientation that is coherent w.r.t all the equivalence classes.

Definition 7.1 (Quillen's Determinant Bundle)**.** *Given finite dimensional vectors spaces, E and G, we consider their exterior maximum power derivative, i.e. $\Lambda^{\max} E = \Lambda^{\dim(E)} E$ which is a 1-dimensional vector space with basis $\{e_1 \wedge \ldots \wedge e_n\}$ where, $\{e_i\}_{i=1}^n$ is a basis for E and define their determinant bundle by*

$$\mathrm{Det}(E, G) := (\Lambda^{\max} E) \otimes (\Lambda^{\max} G)^*$$

where V^* simply denotes the dual of V. For a Fredholm operator, $F \in \mathcal{F}(X, Y)$ we define

$$\mathrm{Det}(F) = \mathrm{Det}(\ker(F), \mathrm{coker}(F))$$

Notice that due the Fredholm property, $\mathrm{Det}(F)$ is well-defined. So given a continuous function, $f : Z \to \mathcal{F}(X, Y)$ where Z is a topological space, we can define a fibre, $\text{Det}(f(z))$ at each z. However we cannot immediately deduce that this indeeds form a line bundle on Z.

Remark:
If there exists an open neighbourhood, $U \subset Z$ such that $\dim(\ker(f(z)))$ is constant for $z \in U$, then since the Fredholm index is locally constant it follows that $\dim(\text{coker}(f(z)))$ is constant as well. Then

$$\bigcup_{z \in Z} \{z\} \times \ker(f(z)) \to U \quad \text{and} \quad \bigcup_{z \in Z} \{z\} \times \text{coker}(f(z)) \to U$$

are vector subbundles of $Z \times X$ and $Z \times Y$ respectively, where the cokernel is naturally identify with the complement of the $R(f(z))$. Since for each $z \in U$, $\ker(f(z)) \cong \mathbb{R}^k$, where k is independent of z, we naturally obtain a smooth trivialisation, $\phi : \pi^{-1}(U) \xrightarrow{\cong} U \times \mathbb{R}^k$, similarly for $\text{coker}(f(z))$. So $\text{Det}(f)$ indeed defines a line bundle on Z.

If $f(z)$ does not have a locally constant dimensional kernel, then we use the following trick:
Since $f(z)$ is Fredholm, it has a finite codimensional range, say with basis $\{y_1, ..., y_n\}$ so we define a linear map, $\psi : \mathbb{R}^n \to Y$ by $\psi(e_i) = y_i$, then

$$\begin{aligned} \hat{f}_\psi(z) : \mathbb{R}^n \times X &\to Y \\ (h,k) &\to \psi(h) + f(z) \cdot k \end{aligned}$$

is Fredholm since ψ is trivially a compact operator (see property 3 of Fredholm operators) and also $\hat{f}_\psi(z)$ is surjective hence $\text{coker}(\hat{f}_\psi(z))$ is trivial. Since surjectivity is a regular property (proof is similar to isomorphism being regular) and the Fredholm index is locally constant, we can find a neighbourhood, $U(z)$ of z such that $\hat{f}_\psi|_{U(z)}$ is a surjective Fredholm operator of constant index for any $y \in U(z)$, we further define

$$\begin{aligned} f_\psi(y) : \mathbb{R}^n \times X &\to \mathbb{R}^n \times Y \\ (h,k) &\mapsto (0, \hat{f}_\psi(y)) \end{aligned}$$

so $\ker(f_\psi(z)) = \ker(\hat{f}_\psi(z))$ and $\text{coker}(f_\psi(y))) \cong \mathbb{R}^n$. We get the following exact sequence:

$$0 \to \ker(f(y)) \xrightarrow{d_1} \ker f_\psi(y) \xrightarrow{d_2} \mathbb{R}^n \xrightarrow{d_3} \text{coker}\, f(y) \to 0$$

where $d_1(k) = (0, k)$, $d_2(h, k) = h$, and $d_3(h) = \phi(h) \mod \mathrm{R}(f(y))$

$$\begin{aligned}\text{since } \ker(d_1) &= 0\\ R(d_1) &= 0 \oplus \ker(f(y)) = \ker(d_2)\\ R(d_2) &= \{h \in \mathbb{R}^n : \exists\ k\ s.t\ \hat{f}_\psi(y)(h, k) = 0\} = \ker(d_3)\\ R(d_3) &= \mathrm{coker}(f(y))\end{aligned}$$

We conclude with the following lemma:

Lemma 7.1. *Given an exact sequence,*

$$0 \to E_1 \xrightarrow{d_1} E_2 \to \ldots \xrightarrow{d_{n+1}} E_n \to 0$$

there is a canonical isomorphism, $\phi : \bigotimes_{i\ even}(\Lambda^{\max} E_i) \xrightarrow{\simeq} \bigotimes_{i\ odd}(\Lambda^{\max} E_i)$

Sketch of proof. Choose a basis for E_1 and extend by exactness to a basis of E_2 and repeat. Then there is a natural choice of an isomorphism which is in fact independent on the choice of basis. □

$$\Lambda^{\max} \ker(f(y)) \otimes \Lambda^n \mathbb{R}^n \xrightarrow{\simeq} \Lambda^{\max} \ker(f_\psi(y)) \otimes \Lambda^{\max} \mathrm{coker}(f(y))$$

Multiplying both sides by $(\Lambda^{\max} \mathrm{coker}(f(y)))^* \otimes\ (\Lambda^n \mathbb{R}^n)$ and using that $(\Lambda^{\max} V) \otimes (\Lambda^{\max} V)^* \cong \mathbb{R}$ by $\xi \otimes \eta^* \mapsto \eta^*(\xi)$ and $V \otimes W \cong W \otimes V$, we get:

$$\Lambda^{\max} \ker f(y) \otimes (\Lambda^{\max} \mathrm{coker}\, f(y))^* \xrightarrow{\simeq} \Lambda^{\max} \ker f_\psi(y) \otimes (\Lambda^n \mathbb{R}^n)^*$$

$$i.e. \qquad \mathrm{Det}(f(y)) \cong \mathrm{Det}\, f_\psi(y)$$

so $\mathrm{Det}(f(y))$ defines a line bundle on Z, since by the above remark $\mathrm{Det}(f_\psi(y))$ does. We can now define an orientation of f.

Definition 7.2. *An orientation of* $f : Z \to \mathcal{F}(X, Y)$ *is a non-vanishing section of the determinant bundle,* $\mathrm{Det}(f) \to Z$.

Returning back to our situation we have that each trajectory, u has an associated Fredholm operator, namely DF_u which we can now use to construct an orientation on this space of trajectories. As in the section of Fredholm analysis, we will first develop the orientation concept for the trivial bundle, $\overline{\mathbb{R}} \times \mathbb{R}^n$ and use a trivialisation to transfer it to the non-trivial case in way that does not depend on the choice of trivialisation. The main hurdle in our situation is that we need an orientation concept that is compatible under the gluing operation. We will henceforth use the same notation as in chapter 3. From lemma 3.4, we may choose asymptotic representatives for each contractible class, $\Theta_{F_A} = \Theta(A^-, A^+)$.

Definition 7.3. *Let $K, L \in \Sigma$, where $K = \frac{\partial}{\partial t} + A$ and $L = \frac{\partial}{\partial t} + B$ with A, B being asymptotically constant with $K^+ := A^+ = B^- =: L^-$, we define a gluing of operators K and L by:*

$$K\#_\rho L = F_{A_\rho} \in \Theta(K^-, L^+)$$

$$A_\rho(t) = \begin{cases} A(t+\rho), & t \leqslant 0 \\ A(t-\rho), & t \geqslant 0 \end{cases}$$

$\forall \rho \geqslant \rho_0(K, L)$ such that $A(\cdot+\rho)|_{[-1,+\infty]} = A^+ = B^- = B(\cdot-\rho)|_{[-\infty,1]}$ under the asymptotic assumption.

It follows from our definition and theorem 3.5, that

$$\begin{aligned} \mathrm{ind}(K\#_\rho L) &= \mu(A^-) - \mu(B^+) = \mu(A^-) - \mu(A^+) + \mu(B^-) - \mu(B^+) \\ &= \mathrm{ind}(K) + \mathrm{ind}(L) \end{aligned}$$

Clearly the class, $\Theta_{K\#_\rho L}$ does not depend on parameter, ρ but only on the asymptotic behaviour of the operators. If we define the index and gluing operators on the equivalence classes, Θ_{F_A} then the above relation can be written as

$$+ \ \circ \ (\mathrm{ind}, \mathrm{ind}) = \mathrm{ind} \circ \#$$

In words, the sum of the indices is equal to the index of the glued (asymptotically constant) operators. Hence we consider the determinant bundle on the equivalence classes, Θ_{F_A} defined by

$$\mathrm{Det}\ \ F \to \Theta(K^+, K^-)$$

where $F : \Theta(K^-, K^+) \to \mathcal{F}(X, Y)$ (which we showed is continuous) such that $F(A) = F_A$, more explicitly, $A \mapsto \frac{\partial}{\partial t} + A(t)$ with $A^\pm = K^\pm$. Since $\Theta(K^+, K^-)$ is contractible any fibre bundle over it is trivial, in particular, Det F is a trivial bundle, i.e. there exists a non-vanishing global section, hence these determinant bundles are indeed orientable.

As mentioned above we need to show that given arbitrary orientations of Θ_K and Θ_L, these induce an orientation of $\Theta_{K\#_\rho L}$ and furthermore, this orientation does not depend on the choice of class representatives.

From lemma 3.4, we know that each equivalence class is contractible, so given $K^0, K^1 \in \Theta_K$, we can define a homotopy, $K^\tau : [0,1] \times \Theta_K \to \Theta_K$ (similarly for Θ_L).

From our previous construction, given $K^\tau, L^\tau \in \mathcal{F}(H^{1,2}(\mathbb{R}), L^2(\mathbb{R}))$ with $K^{\tau+} = L^{\tau-}$, we can find a linear maps, $\psi^\tau : \mathbb{R}^k \to L^2$ such that:

$$\mathrm{Det}(K^\tau_{\psi^\tau}) \cong \mathrm{Det}(K^\tau) \text{ and } \mathrm{Det}(L^\tau_{\psi^\tau}) \cong \mathrm{Det}(L^\tau)$$

Moreover, we can assume that $\psi^\tau(a) \in H^{1,2}(\mathbb{R})$ has compact support for each $a \in \mathbb{R}^n$ since we know that $\hat{K^\tau}_{\psi^\tau}$ is a surjective Fredholm operator and hence there exists $\epsilon > 0$ such if $||\hat{K^\tau}_{\psi^\tau} - F||_{\mathcal{L}} < \epsilon$ then F is a surjective Fredholm operator. So it suffices to replace $\psi^\tau(a)$ by $\beta\psi^\tau(a)$ for an appropriate cut-off function, β such that we can take $F = \hat{K^\tau}_{\beta\psi^\tau}$. Note that since $[0,1]$ is compact and K^τ is continuous, we can choose β and ϵ independent of τ.

Definition 7.4. *With consideration to the extension to surjective Fredholm operators, we need to modify definition 7.3, with compact support assumption on ψ^τ, and again due to the compactness of $[0,1]$ and continuity of K^τ, we can choose ρ_0 large enough such that for $\rho > \rho_0(\tau)$,*

$$\begin{aligned} \hat{K}^\tau_{\psi^\tau} \#_\rho \hat{L}^\tau_\psi &: \mathbb{R}^k \times \mathbb{R}^k \times X \to Y \\ (a, b, s) &\mapsto (K^\tau \#_\rho L^\tau) \cdot s + \psi^\tau(a)(\cdot + \rho) + \psi^\tau(b)(\cdot - \rho) \end{aligned}$$

Theorem 7.2. *Noting that $\mathbb{R}^k \times \mathbb{R}^k \times X$ is a Hilbert space, we define an orthogonal projection, P^τ_ρ onto finite dimensional subspace,* $\ker(\hat{K}^\tau_{\psi^\tau} \#_\rho \hat{L^\tau}_{\psi^\tau})$ *such that for $\rho > \rho_2 > \rho_0$, $\hat{K}^\tau_{\psi^\tau} \#_\rho \hat{L}^\tau_{\psi^\tau}$ is surjective and the following map is an isomorphism,*

$$\begin{aligned} \phi^\tau_\rho = P^\tau_\rho \circ \#_\rho : \ker \hat{K}^\tau_{\psi^\tau} \times \ker \hat{L}^\tau_{\psi^\tau} &\xrightarrow{\simeq} \ker(\hat{K^\tau}_{\psi^\tau} \#_\rho \hat{L}^\tau_{\psi^\tau}) \\ ((a, u), (b, v)) &\mapsto P^\tau_\rho(a, b, u_\rho + v_{-\rho}). \end{aligned}$$

Proof. The proof is identical to theorem 6.2, by replacing D_u, D_v *and* D_χ by $\hat{K}^\tau_{\psi^\tau}$, $\hat{L}^\tau_{\psi^\tau}$ and $\hat{K}^\tau_{\psi^\tau} \#_\rho \hat{L}^\tau_{\psi^\tau}$. □

Hence this induces the isomorphism :

$$(\Lambda^{\max} \ker \hat{K}^\tau_{\psi^\tau}) \otimes (\Lambda^{\max} \ker \hat{L}^\tau_{\psi^\tau}) \xrightarrow{\simeq} \Lambda^{\max} \ker(\hat{K}^\tau_{\psi^\tau} \#_\rho \hat{L}^\tau_{\psi^\tau}) \tag{3}$$

from previous construction, we also have:

$$\begin{aligned} K^\tau_{\psi^\tau} \#_\rho L^\tau_{\psi^\tau} :&\mathbb{R}^k \times \mathbb{R}^k \times H^{1,2} \to \mathbb{R}^k \times \mathbb{R}^k \times L^2 \\ &(a, b, u) \mapsto (0, 0, (\hat{K}^\tau_{\psi^\tau} \#_\rho \hat{L}^\tau_{\psi^\tau})(a, b, u)) \end{aligned}$$

which is equivalent to $K^\tau_{\psi^\tau} \#_\rho L^\tau_{\psi^\tau} \equiv (K^\tau \#_\rho L^\tau)_{\psi^\tau_\rho \oplus \psi^\tau_{-\rho}}$. We also see that:

$$\mathrm{coker}(K^\tau_{\psi^\tau} \#_\rho L^\tau_{\psi^\tau}) \cong \mathbb{R}^k \times \mathbb{R}^k \quad \text{for } \rho \geqslant \rho_2$$

Using the canonical isomorphism:

$$(\Lambda^{\max}\mathbb{R}^k)^* \otimes (\Lambda^{\max}\mathbb{R}^k)^* \xrightarrow{\cong} (\Lambda^{\max}\mathbb{R}^{2k})^*$$
$$\omega \otimes \eta \mapsto \omega \wedge \eta$$

and multiplying (3) by $(\Lambda^{\max}\mathbb{R}^k)^* \otimes (\Lambda^{\max}\mathbb{R}^k)^*$ we get

$$(\Lambda^{\max}\ker \hat{K}^{\tau}_{\psi^\tau}) \otimes (\Lambda^{\max}\mathbb{R}^k)^* \otimes (\Lambda^{\max}\ker \hat{L}^{\tau}_{\psi^\tau})^* \otimes (\Lambda^{\max}\mathbb{R}^k)^* \xrightarrow{\simeq} \mathrm{Det}(K_\psi \#_\rho L_\psi)$$

$$\Rightarrow \mathrm{Det}(K^\tau_{\psi^\tau}) \otimes \mathrm{Det}(L^\tau_{\psi^\tau}) \xrightarrow{\simeq} \mathrm{Det}(K^\tau_{\psi^\tau} \#_\rho L^\tau_{\psi^\tau})$$

So we obtain an isomorphism:

$$\mathrm{Det}\, K^\tau \otimes \mathrm{Det}\, L^\tau \xrightarrow{\simeq} \mathrm{Det}(K^\tau \#_\rho L^\tau)$$

The above also gives us a vector bundle isomorphisms on $[0,1] \ni \tau$, hence this shows that induced orientation is actually independent on our class representatives. So to summarise we have defined a gluing operation for asymptotically constant operators which is compatible with our orientation concept and which does not depend on the actual representatives. Using similar techniques as above, we can also prove that the orientation induced by our gluing map is an associative operation as well. We now give the conclusion of our analysis as a theorem :

Theorem 7.3. *Given asymptotically constant operators, K, L such that $K^+ = L^-$ and orientations, o_K and o_L of the determinant bundles, Θ_K and Θ_L respectively, then for all $\rho > \rho_0$*

$$\#_\rho : (K, L) \mapsto K \#_\rho L$$

induces an orientation on $\Theta_{K\#L}$ denoted by $o_K \# o_L$ which is independent on the representatives, K, L and furthermore,

$$(o_K \# o_L) \# o_M = o_K \# (o_L \# o_M)$$

So far, we have been able to define an orientation concept on trivial bundles which is compatible w.r.t the gluing operation. As stated earlier, we now want to transfer our results to the non-trivial bundles, i.e. the pullback bundles given by the smooth trajectories in M. One of the problem we run into is that if 2 trajectories have the same endpoints, then we need to know how the pullback tangent spaces at the endpoints are related in order to define a compatible orientation at these points. More explicitly, say $u(\pm\infty) = v(\pm\infty)$ then by reparametrising we have that $u \circ \bar{v}$ is a loop, where $\bar{v}(t) = v(-t)$. If $(u \circ \bar{v})^* TM \cong \mathbb{S}^1 \times \mathbb{R}^n$ then we can find a canonical orientation, however it may also happen that this bundle is not trivial, i.e. we obtain a *Möbius* band. Moreover, we also have to take into account that each Fredholm operator, $D_2F(u)$ is associated with a trajectory, u, so need to define an appropriate equivalence relation taking into consideration the trajectories. We denote $\Sigma_{u^*TM,u^*\nabla}$ as in definition 3.6 simply by Σ_{u^*TM}.

Definition 7.5. *Let $K \in \Sigma_{u^*TM}$ and $L \in \Sigma_{v^*TM}$, then we define an equivalence relation by $(u,K) \sim (v,L)$ if and only if $u(\pm\infty) = v(\pm\infty)$ and $K^{\pm} = L^{\pm} \in \mathrm{End}(T_{u(\pm\infty)}M)$.*
Given trivialisations,

$$\phi_u : u^*TM \xrightarrow{\cong} \overline{\mathbb{R}} \times \mathbb{R}^n$$

similarly for ϕ_v. We say ϕ_u and ϕ_v is an admissible pair for $(u,K) \sim (v,L)$ if

1. $\phi_u(-\infty) = \phi_v(-\infty)$
2. $\phi_u(+\infty) \cdot \phi_v(+\infty)^{-1} = \mathrm{diag}(\pm 1, 1, ..., 1) \in GL(n,\mathbb{R})$ *and*
3. *the trivialised operators are equivalent, $\phi_u K \phi_u^{-1} \sim \phi_v L \phi_v^{-1}$, i.e. they agree at infinity.*

Remark: There always exists an admissible pair. If $\phi : (u \circ \bar{v})^*TM \cong \mathbb{S}^1 \times \mathbb{R}^n$, then letting ϕ_u be the restriction of ϕ to the half loop defined by u and ϕ_v be the restriction to the other half loop and with an appropriate reparametrisation, we see that ϕ_u, ϕ_v form an admissible pair with $\phi_u(+\infty) = \phi_v(+\infty)$. On the other hand, if we have a *Möbius* band, then due to the "twist", we can simply fix $\phi_u(-\infty) = \phi_v(-\infty)$ then "untwist" at the other endpoint by multiplying by $\mathrm{diag}(-1,1,...,1)$, i.e. either $\phi_u(+\infty) = \mathrm{diag}(-1,1,...,1)\ \phi_v(+\infty)$ or $\mathrm{diag}(-1,1,...,1)\ \phi_u(+\infty) = \phi_v(+\infty)$. (We can think of this as adding a twist to the *Möbius* band in an appropriate manner so that it undoes the original twist).
These admissible trivialisations allow us to transfer the orientation problem on the non-trivial bundle to the trivial case, more precisely, these trivialisations induce an orientation on the non-trivial bundle which is independent of the actual choice of admissible trivialisation. We will assume throughout the rest of this section that K and L are equivalent.

Theorem 7.4. *Let (ϕ,ψ) and (ϕ',ψ') be admissible pairs for $(u,K) \sim (v,L)$ and let o_K and o_L be arbitrary orientations of $\mathrm{Det}(K)$ and $\mathrm{Det}(L)$, respectively. We may trivialise the sections by $\phi(o_K) = o_{\phi K \phi^{-1}}$ and $\psi(o_L) = o_{\psi L \psi^{-1}}$. Then*

$$\phi(o_K) = \psi(o_L) \cdot \alpha, \text{ where } \alpha \text{ is a strictly positive function on } \Theta_{\phi K \phi^{-1}}$$

if and only if,

$$\phi'(o_K) = \psi'(o_L) \cdot \beta, \text{ where } \beta \text{ is a strictly positive function on } \Theta_{\phi' K \phi'^{-1}}$$

$$i.e. \ \ \phi(o_K) \simeq \psi(o_L) \Leftrightarrow \phi'(o_K) \simeq \psi'(o_L)$$

Hence we conclude from this theorem that an admissible pair induces a unique orientation on an equivalence class, $[u, K]$ i.e. it suffices to orient any $L \in [u, K]$, then this will orient all the other elements. Equivalently, for any admissible pair, (ϕ, ψ),

$$o_K \simeq o_L \Leftrightarrow \phi(o_K) \simeq \psi(o_L).$$

Sketch of proof. Suppose $\phi(o_K) \simeq \psi(o_L)$. We want to show that $\phi'(o_K) \simeq \psi'(o_L)$ equivalently $\phi'\phi^{-1}\phi(o_K) \simeq \psi'\psi^{-1}\psi(o_L)$. So it is sufficient to show that $\det(\psi\psi'^{-1}\phi'\phi^{-1})(\pm\infty) > 0$ and that $\alpha(o_K) \simeq o_K$ for $\alpha(t) \in GL^+$ i.e. it is invertible and has positive determinant, where $L = \alpha \cdot K \cdot \alpha^{-1}$ (the existence of this α is from the fact that they are equivalent). By our admissible assumption, $(\psi\psi'^{-1}\phi'\phi^{-1})(-\infty) = \mathrm{Id}$ and since it varies continuously and is never singular hence it is in the connected component $GL^+(n)$ and hence $(\psi\psi'^{-1}\phi'\phi'^{-1})(+\infty) = \mathrm{Id}$ as well. For the second part, we may assume that $\alpha(\pm\infty) = \mathrm{Id}$ by row reducing. Furthermore we may assume $K = \frac{\partial}{\partial t} + \mathrm{Id}$ and that

$$\alpha(t) = \begin{pmatrix} \begin{matrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{matrix} & 0 \\ 0 & \mathrm{Id} \end{pmatrix}$$

where $\theta(t) \in C^\infty(\overline{\mathbb{R}}, \mathbb{S}^1)$ is asymptotically constant. (See [11] for details of these assumptions)
Then

$$\alpha(K) = \alpha \cdot K \cdot \alpha^{-1} = K + \dot{\theta} \cdot \begin{pmatrix} \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} & 0 \\ 0 & \mathrm{Id} \end{pmatrix}$$

Since $||\dot{\theta}||_{0,2} < +\infty$ and K is an isomorphism by reparametrising $\alpha(t)$ we may assume that $||K - \alpha \cdot K \cdot \alpha^{-1}||_{\mathcal{L}} < ||\dot{\theta}||_{0,2}$ is small enough so that $\alpha(K)$ is an isomorphism and so induces an isomorphism on the orientations.

□

We now need to induce coherent orientations w.r.t gluing of paths, i.e. given orientations for 2 classes, we need to induce a compatible orientation on the

glued trajectory. So we need to look once again at our pre-glued trajectories, $u\#^o_\rho v(t)$.
So let $u(+\infty) = v(-\infty) = y$, then since our manifold is compact, we can always find a normal neighbourhood, $U(y)$ then by means of a coordinate chart, $\Gamma = G \circ \exp_y^{-1}$ where G is an isomorphism of the tangent space at y and $\mathbb{R}^n$ we have:

$$\Gamma : TM\big|_{U(y)} \xrightarrow{\simeq} U(y) \times \mathbb{R}^n$$

By assuming that $u(-1, +\infty], v[-\infty, 1) \subset U(y)$, the above trivialisation by restriction to the trajectories of u and v in $U(y)$ induces trivialisations:

$$\overline{\phi} : u|^*_{(1,+\infty]}TM \xrightarrow{\simeq} (-1, +\infty] \times \mathbb{R}^n \ , \ \overline{\psi} : v|^*_{[-\infty,+1)}TM \xrightarrow{\simeq} [-\infty, +1) \times \mathbb{R}^n$$

then we may extend these to trivialisations ϕ and ψ on u^*TM and v^*TM respectively. This can be done for instance by choosing a smooth function, β which is 1 on $[-\infty, -2]$ and 0 on $[-1, \infty]$ with $\dot{\beta} < 0$ on $(-2, -1)$, then defining $\phi^{-1} = \beta \cdot \phi_u^{-1} + \overline{\phi}^{-1}$ for a smooth trivialisation ϕ_u that agrees with $\overline{\phi}$ at -1. So we define gluing for trivialisations, ϕ and ψ as follows:

$$\phi\#^o_\rho\psi : (u\#^o_\rho v)^*TM \xrightarrow{\simeq} \overline{\mathbb{R}} \times \mathbb{R}^n$$

$$\phi\#^o_\rho\psi(t) = \begin{cases} \phi(t + \rho), & t \leqslant -1 \\ \Gamma_{u\#^o_\rho v(t)}, & |t| \leqslant 1 \\ \phi(t - \rho), & t \geqslant +1 \end{cases}$$

In essence, for the ends of u and v close enough to y we glue them by means of the pre-gluing map which as we saw in the last section coincides with u and v for $|t| > 1$ then since Γ is compatible with the charts, ϕ and ψ by the above construction, so we have a natural definition for trivialisation of the pullback bundle by the pre-glued trajectory that coincides with ϕ and ψ.

Definition 7.6. *Let $K \in \Sigma_{u*TM}$, $L \in \Sigma_{v*TM}$ and we denote their trivialisations by K_ϕ and L_ψ respectively. Observe that*

$$(K_\phi)_\rho = \phi(t + \rho)(\frac{\partial}{\partial t} + A_K(t + \rho))\phi(t + \rho)^{-1} = (K_\rho)_\phi$$

In order to define a gluing of operators as in the trivial case, we want them to be asymptotically constant at those ends, so we choose asymptotically constant representatives by:

$$K_{\phi as} = \frac{\partial}{\partial t} + \beta^- \cdot A_{K_\phi} + \beta^+ \cdot A^+_{K_\phi} \in \Theta_{K_\phi}$$

$$L_{\psi as} = \frac{\partial}{\partial t} + \beta^- \cdot A^-_{L_\psi} + \beta^+ \cdot A_{L_\psi} \in \Theta_{L_\psi}$$

and clearly $K_{\phi as}(+\infty) = L_{\psi as}(-\infty)$. So since we already know how to glue trivial asymptotic operators in a way that is compatible with orientation (theorem 7.3), so we have an isomorphism:

$$\operatorname{Det} K_{\phi as} \times \operatorname{Det} L_{\psi as} \xrightarrow{\cong} \operatorname{Det}(K_{\phi as} \# L_{\psi as}) \qquad (*)$$

We may simply define gluing of operators by gluing of their constant representatives since in view of theorem 7.4 this is sufficient to orient

$$\Theta_{K_\phi \#^o_\rho L_\psi} = \Theta_{K_{\phi as} \#_\rho L_{\psi as}},$$

i.e. we write $K_\phi \#^o_\rho L_\psi = K_{\phi as} \#_\rho L_{\psi as}$ for $\rho \geqslant 1$. By $(*)$, the induced orientation does not depend on our choice of asymptotic constant representatives. So we can now transform back to the non-trivial bundle by,

$$K \#^o_\rho L := (\phi \#^o_\rho \psi)^{-1} (K_\phi \#^o_\rho L_\psi)(\phi \#^o_\rho \psi) \in \Sigma_{(u \#^o_\rho v)^* TM}$$

To summarise:
We have constructed a gluing operation

$$\#^o_\rho : [u, K] \times [v, L] \to [u \#^o_\rho v, K^-, L^+] = [u, K] \#^o_\rho [v, L]$$

which induces a canonical orientation $o(u \#^o_\rho v)$ on the class $[(u \#^o_\rho v), (K \#^o_\rho L)]$ by $o([u, K])$ and $o([v, L])$ from $(*)$. Moreover from the trivial case, we also have the associativity property:

$$(o([u, K]) \# o([u, L])) \# o([u, M]) = o([u, K]) \# (o([u, L]) \# o([u, M])).$$

We now state the results for the trajectories solving $F(u) = \dot{u} + \nabla f \circ u = 0$ with the associated Fredholm operators,

$$D_u : H^{1,2}(u^* TM) \to L^2(u^* TM)$$

which is simply the linearisation of our operator, F in local coordinates at u and in a trivialisation of the Banach bundle, L^2 (as already seen in the gluing section). Also recall that we showed that these Fredholm operators are actually surjective, i.e. $\operatorname{coker}(D_u)^* \cong 1^*$, for both the time-dependent and time-independent trajectories.
Note that in above analysis, we defined a gluing operator, $\#^o$ for Fredholm operators, we still need to verify that this orientation is indeed compatible with the actual trajectory gluing, $\#_\rho$ since the glued trajectory already has the associated Fredholm operator, $D_{u \#_\rho v}$, i.e. we need to show that the orientation associated to $[u \#_\rho v, D_{u \#_\rho v}]$ agrees with $[u \#^o_\rho v, D_u \#^o_\rho D_v]$.

Theorem 7.5. *Considering the gluing map,*

$$\mathcal{M}_{x,y} \times \mathcal{M}_{y,z} \ni (u, v) \mapsto u\#_\rho v \in \mathcal{M}_{x,y}$$

given orientations, $o[u, D_u]$ and $o[v, D_v]$, the isomorphism

$$D\#_\rho : \mathrm{Det}(D_u) \otimes \mathrm{Det}(D_v) \cong \ker D_u \times \ker D_v \xrightarrow{\simeq} \ker D_{u\#_\rho v} \cong \mathrm{Det}(u\#_\rho v)$$

induces the same orientation $o[u\#_\rho v, D_{u\#_\rho v}]$ as $o[u, D_u]\#o[v, D_v]$ and moreover,

$$[u\#_\rho v, D_{u\#_\rho v}] = [u\#^o_\rho v, D_u\#^o_\rho D_v]$$

The proof consists of first showing that trajectories $u\#_\rho v$ and $u\#^o_\rho v$ are homotopic and that this homotopy induces an isomorphism of $\mathrm{Det}(D_{u\#_\rho v})$ and $\mathrm{Det}(D_{u\#^o_\rho v})$. Hence $[u\#_\rho v, D_{u\#_\rho v}] = [u\#^o_\rho v, D_{u\#^o_\rho v}]$ have the same orientation. Then we verify that $[u\#^o_\rho v, D_{u\#^o_\rho v}] = [u\#^o_\rho v, D_u\#^o_\rho D_v]$ by constructing a homotopy of isomorphisms to our above defined gluing operator. The reader to is referred to [11] for details of these constructions. A similar analysis leads to the corresponding result for the time-dependent trajectories.

With regards to the λ-parametrised trajectories, we have to consider $G^{\alpha\beta}(\lambda, \gamma)$ (defined in the transversality section). Although $DG^{\alpha\beta}$ is a surjective Fredholm operator, $D_2G^{\alpha\beta}$ is not always surjective. So we once again use lemma 7.1 and the exact sequence:

$$0 \to \ker(D_2G^{\alpha\beta}) \xrightarrow{d_1} \ker(DG^{\alpha\beta}) \xrightarrow{d_2} T_\lambda[0,1] \cong \mathbb{R} \xrightarrow{d_3} \mathrm{coker}(D_2G^{\alpha\beta}) \to 0$$

to obtain the isomorphism:

$$\Omega : \mathrm{Det}(D_2G^{\alpha\beta}) \xrightarrow{\simeq} \mathrm{Det}(DG^{\alpha\beta}) \cong \Lambda^{\max}(DG^{\alpha\beta})$$

Hence w.r.t the Fredholm operator, $D_2F_{u_\lambda} = D_{u_\lambda} \in \Sigma_{u^*_\lambda TM}$, where u_λ is a λ-parametrised trajectory, we once again have that the orientation induced by Ω and that induced by the gluing operator we defined are compatible:

Theorem 7.6. *Considering the gluing map,*

$$\mathcal{M}^H_{x_\alpha,y_\beta} \times \mathcal{M}^{f^\beta}_{y_\beta,z_\beta} \ni ((\lambda, u_\lambda), v) \mapsto (\lambda, u_\lambda)\#^H_\rho v = (\tilde{\lambda}, w_{\tilde{\lambda}}) \in \mathcal{M}^H_{x_\alpha,y_\beta}$$

given orientations, $o[u_\lambda, D_{u_\lambda}]$ and $o[v, D_v]$ then the isomorphism induced by Ω induces the same orientation as the one we defined, i.e.

$$o[w_{\tilde{\lambda}}, D_{w_{\tilde{\lambda}}}] = o[u_\lambda, D_{u_\lambda}]\#o[v_\lambda, D_{v_\lambda}]$$

and moreover,

$$[w_{\tilde{\lambda}}, D_{w_{\tilde{\lambda}}}] = [u_\lambda\#^o_\rho v, D_{u_\lambda\#^o_\rho D_v}].$$

The proof is similar to the previous one but by taking into consideration the surjective operator, $D^{\alpha\beta}$ then using Ω to transfer the orientation to $D_2 G^{\alpha\beta}$. See [11].

We now want to orient the equivalence classes in such way that their orientations are compatible with gluing. The picture to have in mind here is that $\mathcal{M}^f_{x,y} \times \mathcal{M}^f_{y,z}$ and $\mathcal{M}^f_{x,w} \times \mathcal{M}^f_{w,z}$ both consist of broken trajectories that can be glued into the space $\mathcal{M}^f_{x,z}$, so we need to verify that there exists choices of orientations such that the induced orientations are coherent, i.e. there is a choice of orientations on these four spaces of trajectories so that when glued the induced orientations are the same on $\mathcal{M}^f_{x,z}$.

Definition 7.7. *Denote by*

$$\Lambda := \{[u, K] : u \in C^\infty(\overline{\mathbb{R}}, M),\ \ K \in \Sigma_{u^*TM}\}$$

the set of all equivalence classes of Fredholm operators along compact curves joining critical points of some Morse function, f. Then any map, σ which picks an orientation $\sigma[u, K]$ of each class $[u, K]$ is called coherent if it is compatible with the gluing operation, i.e.

$$\sigma[u, K]\#\sigma[v, L] = \sigma[u\#v, K\#L]$$

We denote $\mathcal{C}_\Lambda = \{\sigma : \sigma \text{ coherent}\}$. Let

$$\Gamma = \{f : \Lambda \to \{\pm 1\} : f([u, K]\#[v, L]) = f([u, K]) \cdot f([v, L])\}$$

be the group acting on $\mathcal{C}_\Lambda$ by

$$\begin{aligned} &\Gamma \times \mathcal{C}_\Lambda \to \mathcal{C}_\Lambda \\ &(f, \sigma) \cdot ([u, K]) = f([u, K]) \cdot \sigma[u, K] \end{aligned}$$

Note that Γ is indeed a group with group operation, $f \circ g = f \cdot g$, inverses, $f^{-1} = -f$ and $\mathrm{Id} = 1$.

Theorem 7.7. *$\mathcal{C}_\Lambda \neq \emptyset$, i.e. there exists coherent orientations and Γ acts freely and transitively, i.e. all the stabilisers of $\mathcal{C}_\Lambda$ are trivial and any two coherent orientations are related by some $f \in \Gamma$.*

Proof. $\underline{\mathcal{C}_\Lambda \neq \emptyset}$ We give a constructive proof. Consider the constant trajectory, $u_0(t) = x_0 \in M$ then the associated Fredholm operator, $K_0 = D_{u_0} \in \Sigma_{u_0^*TM}$ is given by

$$D_{u_0} = \frac{\partial}{\partial t} + A$$

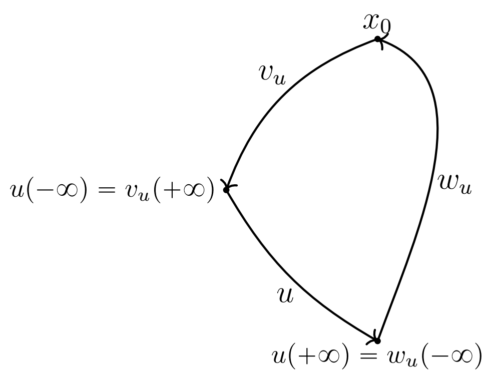
with $A \in C^\infty(\mathrm{End}(u_0^* TM))$ being the Hessian of f at critical point, x_0. Hence A is an isomorphism, so $\mathrm{Det}(D_{u_0}) = \Lambda^0\{0\} \otimes (\Lambda^0\{0\})^* = \mathbb{R} \otimes \mathbb{R}^*$ so we may orient $[u_0, D_{u_0}]$ by $1 \otimes 1^*$. Let

$$S^{\pm} = \{[u, K] \in \Lambda : u(\pm\infty) = x_0\}$$

and fix an orientation, $\sigma[u, K]$ for each element of S^-. We also have $S^+ \cong S^-$ by $[u(t), K^-, K^+] \mapsto [u(-t), K^+, K^-]$. So the relation

$$\sigma[u(-t), K^+, K^-] \# \sigma[u(t), K^-, K^+] = \sigma[u_0, K_0]$$

(since $K^+ = K_0$) fixes an orientation for each $[u(t), K^-, K^+]$ in S^+.



Now considering

$$[v_u, K_0, K^-] \in S^- \text{ with } v_u(+\infty) = u(-\infty)$$
$$[w_u, K^+, K_0] \in S^+ \text{ with } w_u(-\infty) = u(+\infty)$$

By the associativity rule, we already have an orientation on $[u, K]$ given by

$$\sigma[v_u, K_0, K^-] \# \sigma[u, K] \# \sigma[w_u, K^+, K_0] = \sigma[u_0].$$

So by repeating this process we have actually constructed a coherent orientation, σ. To summarise, we start at a critical point and choose arbitrary orientations for curves starting at that point, then this fixes an orientation for curves ending at that point. Using the rules of associativity, we construct a coherent orientation on Λ.

<u>Γ acts freely</u> By the above construction and the definition of Γ, it is clear that

$$f([u, K])\sigma[u, K] = \sigma[u, K] \Leftrightarrow f([u, K]) = 1$$

<u>Γ acts transitively</u> Let $\sigma_1, \sigma_2 \in \mathcal{C}_\Lambda$ and let $\sigma_1 = f \cdot \sigma_2$, where $f : \Lambda \to \{\pm 1\}$, then a direct calculation and using that $f^2 = 1$ shows that:

$$\sigma_1[u, K]\#\sigma_1[v, L] = f[u\#v, K\#L] \cdot f[u, K] \cdot f[v, L] \cdot \sigma_1[u, K]\#\sigma_1[v, L]$$

$$\Rightarrow f[u, K] \cdot f[v, L] = f[u\#v, K\#L] \quad i.e. \quad f \in \Gamma$$

$\square$

This concludes the section of orientation of our Banach space of trajectories. We are now in position to define the Morse homology over the ring of integers.

8 Morse Homology

In this chapter, we shall use all the results obtained from our analysis in previous chapters to define the Morse homology and show its independence on the choice of Morse function. As a result we obtain Poincaré Duality theorem as a simple corollary of this invariance. In order to make appropriate sign choices in our construction, we need to consider, in contrast to coherent orientations, canonical orientations of the Fredholm classes, $[u, K] \in \Lambda$ for isolated trajectories. We shall use the natural notations u, u_h and (λ, u_λ) for the each three types of trajectories.

1. Time independent case, $\mathcal{M}^f_{x,y}$ with $\mu(x) - \mu(y) = 1$

   $\mathcal{M}^f_{x,y}$ is a 1-dimensional manifold with $T_u\mathcal{M}^f_{x,y} \equiv \ker D_u$ and recall also that D_u is onto hence it has a trivial cokernel. We may thus find a canonical orientation by

   $$0 \neq -\nabla f \circ u = \dot{u} \in \ker D_u \cong \operatorname{Det} D_u$$

   since $F(u(t)) = 0$ implies $\dot{u}$ is in the kernel of the linearisation of F. Also $\dot{u}$ is invariant under reparametrisation by the 1-dimensional time-shifting. So we denote this canonical orientation on $\widehat{\mathcal{M}}^f_{x,y}$ by $[\dot{\hat{u}}]$. (The other possible choice naturally being $-\dot{\hat{u}}$)

2. Time independent case, $\mathcal{M}^{h^{\alpha\beta}}_{x_\alpha,y_\beta}$ with $\mu(x_\alpha) - \mu(y_\beta) = 0$

   We still have that D_u is onto and since the relative index is zero, $\ker D_u$ is trivial as well. Hence $\operatorname{Det} D_{u_h} \cong \mathbb{R} \otimes \mathbb{R}^*$. So we choose the canonical orientation, $[1 \otimes 1^*]$.

3. λ-parametrised case, $\mathcal{M}^{H^{\alpha\beta}_\lambda}_{x_\alpha,y_\beta}$ with $\mu(x_\alpha) - \mu(y_\beta) + 1 = 0$

   Although we have that $DG^{\alpha\beta}$ is onto, $D_2G^{\alpha\beta}$ in general is not. So we once again use the canonical isomorphism constructed in the previous chapter,

   $$\Omega : \operatorname{Det} D_2G^{\alpha\beta} \xrightarrow{\simeq} \Lambda^{\max} \ker DG^{\alpha\beta}$$

   where $D_2G^{\alpha\beta} = D_{u_\lambda} \in \Sigma_{u^*_\lambda TM}$. Since $\mathcal{M}^{H^{\alpha\beta}_\lambda}_{x_\alpha,y_\beta}$ is a 0-dimensional manifold and $DG^{\alpha\beta}$ is onto so it also has a trivial kernel and hence we get the canonical orientation $1 \otimes 1^*$ on $T_{(\lambda,u_\lambda)}\mathcal{M}^{H^{\alpha\beta}_\lambda}_{x_\alpha,y_\beta}$. The index formula from theorem 4.9 also implies that $\dim(\ker D_{u_\lambda}) = 0$ and $\dim(\operatorname{coker} D_{u_\lambda}) = 1$. Recall that $DG^{\alpha\beta} = D_1G^{\alpha\beta} + D_2G^{\alpha\beta}$ and it is onto so $\operatorname{coker}(D_{u_\lambda}) \cong R(D_1G^{\alpha\beta})$. Hence we obtain the canonical orientation $1 \otimes (D_1G^{\alpha\beta} \cdot \frac{\partial}{\partial\lambda})^*$ on $[u_\lambda, D_{u_\lambda}]$ in $\mathcal{P}^{1,2}_{x,y}$.

These canonical choice of orientations define the coefficients (in $\mathbb{Z}$) that arise w.r.t the boundary map, ∂.

Definition 8.1. *Given a fixed coherent orientation, $\sigma \in C_\Lambda$, we define the characteristic signs, $\tau(u)$ associated to the isolated trajectories considered above by:*

1. $\tau(\hat{u})\sigma[\hat{u}] = [\dot{\hat{u}}]$
2. $\tau(u_h)\sigma[u_h] = [1 \otimes 1^*]$
3. $\tau(u_H)\sigma[u_H] = [1 \otimes (D_1G^{\alpha\beta} \cdot \frac{\partial}{\partial\lambda})^*]$

So $\tau(u)$ are the signs (± 1) that arise by comparing a coherent orientation and the canonical orientation. We are now in position to define the Morse Chain Complex. From now on we shall denote all three types of trajectories by u as long as there is no risk of confusion.

Definition 8.2 (Morse Complex). *Let $f \in C^\infty(M, \mathbb{R})$ be a Morse function and denote by $\operatorname{Crit}_k(f)$ its critical points of index k and the free abelian group over $\mathbb{Z}$ generated by $\operatorname{Crit}_k(f)$ by C_k. Given a coherent orientation $\sigma \in C_\Lambda$ we define the boundary maps,*

$$\begin{aligned} \partial_k : C_k(f) &\to C_{k-1}(f) \\ x &\mapsto \sum_{y \in \operatorname{Crit}_{k-1}(f)} \langle x, y \rangle\, y \end{aligned}$$

where

$$\langle x, y \rangle = \begin{cases} \sum\limits_{\hat{u} \in \widehat{\mathcal{M}}^f_{x,y}} \tau(\hat{u}), & \text{if } \mu(x) - \mu(y) = 1 \\ 0, & \text{otherwise} \end{cases}$$

So $\langle x, y \rangle$ in fact counts the number of (unparametrised) trajectories joining x and y (which is finite by compactness of M) with coefficients determined by the characteristic signs. The chain complex is written out as,

$$\cdots \to C_{k+1} \xrightarrow{\partial_{k+1}} C_k \xrightarrow{\partial_k} C_{k-1} \xrightarrow{\partial_{k-1}} \cdots \xrightarrow{\partial_1} C_0$$

Note that here $C_k = 0$ for $k \geqslant (n+1)$.

We shall prove the fundamental theorem of this article:

Theorem 8.1. *For any fixed $\sigma \in C_\Lambda$, given Morse functions, f^α and f^β, there is a canonical isomorphism*

$$\Phi_*^{\beta\alpha} : H_*(f^\alpha) \xrightarrow{\simeq} H_*(f^\beta)$$

which satisfies the relations,

1. $\Phi_*^{\beta\alpha} \circ \Phi_*^{\gamma\beta} = \Phi_*^{\gamma\alpha}$
2. $\Phi_*^{\alpha\alpha} = \mathrm{Id}$
3. $\Phi_*^{\beta\alpha} = (\Phi_*^{\alpha\beta})^{-1}$

Hence our homology theory is in fact independent on the choice of Morse function. We first need to check that our boundary map indeed satisfies $\partial^2 = 0$. A direct computation gives:

$$\begin{aligned}
\partial^2 x &= \partial \left[\sum_{y \in \mathrm{Crit}_{k-1} f} \langle x, y \rangle \, y \right] \\
&= \sum_{z \in \mathrm{Crit}_{k-2} f} \sum_{y \in \mathrm{Crit}_{k-1} f} \langle x, y \rangle \, \langle y, z \rangle z = \sum_{z \in \mathrm{Crit}_{k-2} f} \sum_{(\hat{u},\hat{v}) \in \widetilde{M}_{x,z}} \tau(\widetilde{u}) \tau(\widetilde{v}) z \\
&= \sum_{z \in \mathrm{Crit}_{k-2} f} \sum_{[(\hat{u},\hat{v})] \in \widetilde{M}_{x,z}/\sim} \sum_{(\hat{u},\hat{v}) \in [(\hat{u},\hat{v})]} \tau(\widetilde{u}) \tau(\widetilde{v}) z
\end{aligned}$$

where the last equality follows from the fact that $\widetilde{M}_{x,z}$ can be decomposed into equivalence classes where each class consists of exactly two broken trajectories corresponding to the opposite ends of the connected components of $\mathcal{M}_{x,z}^f$. Hence if we were working over the finite field, $\mathbb{Z}_2$ then we could immediately conclude that $\partial^2 = 0$. To reach the same conclusion, we need to show that

$$\tau(\widehat{u}_1)\tau(\widehat{v}_1) = -\tau(\widehat{u}_2)\tau(\widehat{v}_2)$$

for $\tau(\widehat{u}_1)\tau(\widehat{v}_1)$ and $\tau(\widehat{u}_2)\tau(\widehat{v}_2)$ corresponding the opposite ends on a connected component. By the theory of coherent orientations, we have

$$\begin{aligned}
\sigma[\widehat{u}_1]\#\sigma[\widehat{v}_1] &= \sigma[\widehat{u}_1\#\widehat{v}_1] \\
&= \sigma[\widehat{u}_2\#\widehat{v}_2] \\
&= \sigma[\widehat{u}_2]\#\sigma[\widehat{v}_2]
\end{aligned}$$

where the second equality is due to the fact that $u_1\#v_1$ and $u_2\#v_2$ belong to the same connected component of $\mathcal{M}_{x,z}^f$ and as such have the same orientation induced by σ. From our definition of the characteristic signs we also have the relation:

$$[\dot{\widehat{u}}_i]\#[\dot{\widehat{v}}_i] = \tau(\widehat{u}_i)\tau(\widehat{v}_i)\sigma[\widehat{u}_i]\#\sigma[\widehat{v}_i] \text{ for } i = 1, 2$$

So we need to look at how the gluing operation, $\#_\rho$ transforms the canonical orientations. Recall the isomorphism given by :

$$\widehat{\mathcal{M}}^f_{x,y} \times \mathbb{R} \xrightarrow{\simeq} \mathcal{M}^f_{x,y}$$
$$(\widehat{u}, \tau) \to \widehat{u}_\tau = \widehat{u}(\cdot + \tau)$$

Given broken trajectories in $\widehat{\mathcal{M}}^o_{x,z}$ (connected component), we can find $\epsilon > 0$ and a gluing parameter, ρ_0 such that the following diagram commutes:

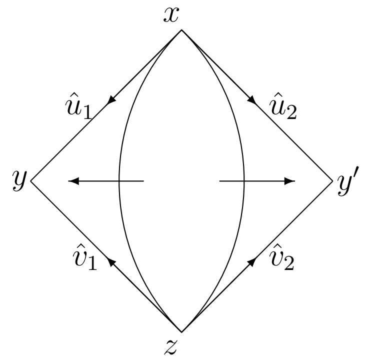
$$\begin{tikzcd}
\{\hat{u}\}_{(-\epsilon,\epsilon)} \times \{\hat{v}\}_{(-\epsilon,\epsilon)} \arrow[r, "\#_{\rho_0}"] & \mathcal{M}^o_{x,z} \arrow[d, "\cdot/\mathbb{R}"] \\
(\rho_0 - \epsilon, \rho_0 + \epsilon) \arrow[u, "\varphi"] \arrow[r, "\psi"] & \widehat{\mathcal{M}}^o_{x,z}
\end{tikzcd}$$

where, $\varphi(\rho) = (\hat{u}_{\rho-\rho_0}, \hat{v}_{\rho_0-\rho})$ (this corresponds to a shifting by ρ as in the definition of the pre-gluing map) and $\varphi(\rho) = \hat{u}\widehat{\#_\rho}\hat{v}$. Moreover, we also consider the diffeomorphism given by:

$$\phi : \widehat{\mathcal{M}}^o_{x,z} \xrightarrow{\simeq} (-1, 1)$$

We now look at the canonical orientations:

Given $\frac{\partial}{\partial s} \in T_{s_0}(-1, 1)$, we obtain $e = \phi_*^{-1}\frac{\partial}{\partial s} \in T_{\phi(\rho_0)}\widehat{\mathcal{M}}^o_{x,z}$. Then the time-shifting invariance induces the orientation vector $(e, \frac{\partial}{\partial \tau})$ on $\mathcal{M}^o_{x,z}$. On the other hand, the vector $\frac{\partial}{\partial \rho} \in T_{\rho_0}(\rho_0 - \epsilon, \rho_0 + \epsilon)$ (note that this corresponds to the direction ρ increases, i.e. convergence to broken trajectories) is mapped to the orientation vector, $\psi_*\frac{\partial}{\partial \rho} \in T_{\psi(\rho_0)}\widehat{\mathcal{M}}^o_{x,z}$ and to the orientation vector $(\dot{\widehat{u}}, -\dot{\widehat{v}})$ on $\{\hat{u}\}_{(-\epsilon,\epsilon)} \times \{\hat{v}\}_{(-\epsilon,\epsilon)}$ by φ. So by commutativity, $\#_{\rho_0}$ identifies $(\dot{\widehat{u}}, -\dot{\widehat{v}})$ with $(\psi_*\frac{\partial}{\partial \rho}, \frac{\partial}{\partial \tau})$. So to study the change in orientation given by $\#_{\rho_0}$ it now suffices to compare e and $\psi_*\frac{\partial}{\partial \rho}$ or equivalently, $\frac{\partial}{\partial s}$ and $(\phi \circ \psi)_*\frac{\partial}{\partial \rho}$.
If $\phi(\hat{u}_1\widehat{\#_\rho}\hat{v}_1) \to 1$ as $\rho \to \infty$ and $\phi(\hat{u}_2\widehat{\#_\rho}\hat{v}_2) \to -1$ as $\rho \to \infty$.
then $\frac{\partial}{\partial s}$ is identified with $\frac{\partial}{\partial \rho}$ for $(\hat{u}_1, \hat{v}_1)$ and $-\frac{\partial}{\partial s}$ is identified with $\frac{\partial}{\partial \rho}$ for $(\hat{u}_2, \hat{v}_2)$.
This can be viewed as:



where the arrows on the broken trajectories correspond to the canonical orientation $(\dot{\hat{u}}, -\dot{\hat{v}})$ and the opposite pointing arrows correspond to the direction of convergence of unparametrised trajectories as $\rho \to \infty$, i.e. $\frac{\partial}{\partial \rho}$. So we deduce that the gluing operation induces opposite signs on canonical orientations, i.e.

$$[\dot{\hat{u}}_1]\#[\dot{\hat{v}}_1] = -[\dot{\hat{u}}_2]\#[\dot{\hat{v}}_2].$$

hence $\partial \circ \partial = 0$.

The next step is to define a chain homomorphism between the chain complexes associated to two Morse function, f^α and f^β. It is here that the $h^{\alpha\beta}$-trajectories will prove to be useful. We define for these trajectories:

$$\begin{aligned} &\langle \cdot, \cdot \rangle : C_k(f^\alpha) \times C_k(f^\beta) \to \mathbb{Z} \\ &\langle x_\alpha, x_\beta \rangle = \begin{cases} \sum\limits_{u \in \mathcal{M}^{h^{\alpha\beta}}_{x_\alpha, x_\beta}} \tau(u), & \text{if } \mu(x_\alpha) - \mu(x_\beta) = 0 \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

By analogy to our definition of the boundary maps we define the chain homomorphism by:

$$\begin{aligned} &\Phi^{\beta\alpha}_k : C_k(f^\alpha) \to C_k(f^\alpha) \\ &\Phi^{\beta\alpha}_k(x_\alpha) = \sum_{x_\beta \in \mathrm{Crit}_k(f^\beta)} \langle x_\alpha, x_\beta \rangle \, x_\beta \end{aligned}$$

We need to verify that the diagram:

$$\begin{tikzcd} C_k(f^\alpha) \arrow[r, "\partial^\alpha"] \arrow[d, "\Phi^{\beta\alpha}_k"'] & C_{k-1}(f^\alpha) \arrow[d, "\Phi^{\beta\alpha}_{k-1}"] \\ C_k(f^\beta) \arrow[r, "\partial^\beta"] & C_{k-1}(f^\beta) \end{tikzcd}$$

commutes, i.e. $\partial^\beta \circ \Phi_k^{\beta\alpha} = \Phi_k^{\beta\alpha} \circ \partial^\alpha$. So from this relation we see that Φ induces a well-defined map in homology.

Sketch of proof. A direct computation gives

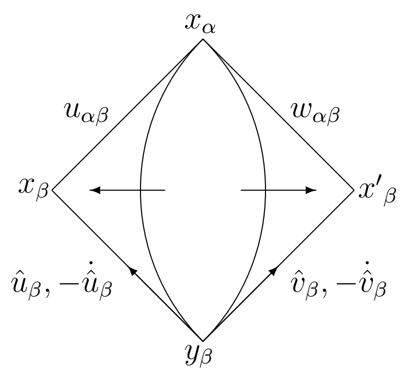
$$\begin{aligned}(\partial^\beta \circ \Phi_k^{\beta\alpha} - \Phi_k^{\beta\alpha} \circ \partial^\alpha)\, x_\alpha = \sum_{\mu(y_\beta)=k-1} \Bigg[& \sum_{\mu(x_\beta)=k} \sum_{u_{\alpha\beta} \in \mathcal{M}^h_{x_\alpha, x_\beta}} \sum_{\hat{u}_\beta \in \widehat{\mathcal{M}}^{f^\beta}_{x_\beta, y_\beta}} \tau(u_{\alpha\beta}) \cdot \tau(\hat{u}_\beta) \\ & - \sum_{\mu(y_\alpha)=k} \sum_{v_{\alpha\beta} \in \mathcal{M}^h_{y_\alpha, y_\beta}} \sum_{\hat{u}_\alpha \in \widehat{\mathcal{M}}^{f^\alpha}_{x_\alpha, y_\alpha}} \tau(u_{\alpha\beta}) \cdot \tau(\hat{u}_\beta) \Bigg]\, y_\beta\end{aligned}$$

We need to show the terms in the square brackets vanish. In this situation we get two different possible broken trajectories namely:

1. $(u_{\alpha\beta}, \hat{u}_\beta) \in \mathcal{M}^{h^{\alpha\beta}}_{x_\alpha, y_\beta} \times \widehat{\mathcal{M}}^{f^\beta}_{x_\beta, y_\beta}$ and
2. $(\hat{u}_\alpha, v_{\alpha\beta}) \in \widehat{\mathcal{M}}^{f^\alpha}_{x_\alpha, y_\alpha} \times \mathcal{M}^{h^{\alpha\beta}}_{y_\alpha, y_\beta}$

depending on $h^{\alpha\beta}$ of course. Considering the first case, we have 2 types of cobordism equivalence:

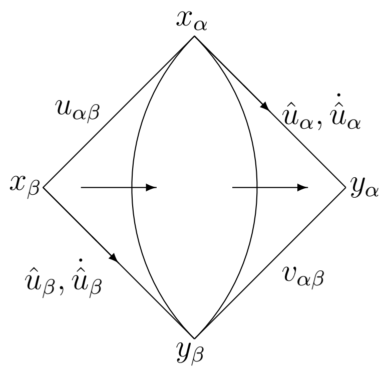
1. $(u_{\alpha\beta}, \hat{u}_\beta) \sim (w_{\alpha\beta}, \hat{v}_\beta)$, where $(w_{\alpha\beta}, \hat{v}_\beta) \in \mathcal{M}^{h^{\alpha\beta}}_{x_\alpha, x'_\beta} \times \widehat{\mathcal{M}}^{f^\beta}_{x'_\beta, y_\beta}$ (this means that they are the boundaries of some connected component)
   Our canonical choice of orientation gives the orientation $(1 \otimes 1^*, \dot{\hat{u}}_\beta)$ on $\mathcal{M}^{h^{\alpha\beta}}_{x_\alpha, y_\beta} \times \widehat{\mathcal{M}}^{f^\beta}_{x_\beta, y_\beta}$. So we need to see whether this corresponds to $\frac{\partial}{\partial\rho}$ or $-\frac{\partial}{\partial\rho}$ under gluing. A similar analysis as in the proof that $\partial^2 = 0$ can be carried out. The difference being that for mixed broken trajectory a time-shifting by -2ρ is only carried out for u_β (in $(u_{\alpha\beta}, u_\beta)$) when glued. So we get the picture:



then as in the previous proof, we get opposite orientations, i.e.

$$\tau(u_{\alpha\beta})\tau(\hat{u}_\beta) = -\tau(w_{\alpha\beta})\tau(\hat{v}_\beta)$$

2. $(u_{\alpha\beta}, \hat{u}_\beta) \sim (\hat{u}_\alpha, v_{\alpha\beta})$, where $(\hat{u}_\alpha, v_{\alpha\beta}) \in \widehat{\mathcal{M}}^{f^\alpha}_{x_\alpha,y_\alpha} \times \mathcal{M}^{h^{\alpha\beta}}_{y_\alpha,y_\beta}$
Now we get the picture:



In this case we get :

$$\tau(u_{\alpha\beta})\tau(\hat{u}_\beta) = \tau(v_{\alpha\beta})\tau(\hat{u}_\alpha)$$

Since there is a one to one correspondence between mixed broken trajectories as the boundaries of a connected component, we can conclude this proof. More precisely, for type 1)-1) the product of characteristic signs cancel out in the first term of the above equation i.e. $\partial^\beta \circ \Phi^{\beta\alpha}_k$ since they have opposite signs and for type 1)-2) the corresponding terms from the first and second term cancel out again. (same for 2)-2) and 2)-1).)

□

So given Morse functions, f^α and f^β we have constructed a chain homomorphism, $\Phi^{\beta\alpha}$ (depending on $h^{\alpha\beta}$) which induces a homomorphism in homology, i.e.

$$\Phi^{\beta\alpha}_* : H_*(f^\alpha) \to H_*(f^\beta)$$

where $H_k(f) := \ker \partial_k / \operatorname{Im}(\partial_{k+1})$.
We now need to show that this homomorphism is in fact canonical, i.e. it is independent of our homotopy, $h^{\alpha\beta}$. It is here that the λ-parametrised trajectories will prove to be useful.

Lemma 8.2. *Given two regular finite homotopies, $h_0^{\alpha\beta}$ and $h_1^{\alpha\beta}$ together with the associated chain homomorphisms, $\Phi_0^{\beta\alpha}$ and $\Phi_1^{\beta\alpha}$, we may find a family of chain homomorphisms,*

$$\Psi_k^{\beta\alpha} : C_k(f^\alpha) \to C_{k+1}^{f^\beta}$$

such that

$$\Phi_{1,k}^{\beta\alpha} - \Phi_{0,k}^{\beta\alpha} = \partial_{k+1}^{\beta} \circ \Psi_k^{\beta\alpha} - \Psi_{k-1}^{\beta\alpha} \circ \partial_k^{\alpha}$$

i.e. we have the (non-commutative) diagram:

$$\begin{tikzcd}
\cdots \arrow[r, "\partial_{k+2}^{\alpha}"] & C_{k+1}(f^\alpha) \arrow[r, "\partial_{k+1}^{\alpha}"] & C_k(f^\alpha) \arrow[r, "\partial_k^{\alpha}"] \arrow[dl, "\Psi_k^{\beta\alpha}"'] & C_{k-1}(f^\alpha) \arrow[r, "\partial_{k-1}^{\alpha}"] \arrow[dl, "\Psi_{k-1}^{\beta\alpha}"'] & \cdots \\
\cdots \arrow[r, "\partial_{k+2}^{\beta}"] & C_{k+1}(f^\beta) \arrow[r, "\partial_{k+1}^{\beta}"] & C_k(f^\beta) \arrow[r, "\partial_k^{\beta}"] & C_{k-1}(f^\beta) \arrow[r, "\partial_{k-1}^{\beta}"] & \cdots
\end{tikzcd}$$

Proof. Considering the λ-homotopy of homotopies, $H_\lambda^{\alpha\beta}$ between $h_0^{\alpha\beta}$ and $h_1^{\alpha\beta}$. We once again define by analogy to the definition of ∂.

$$\begin{aligned}
\Psi_k^{\beta\alpha} : C_k(f^\alpha) &\to C_{k+1}(f^\alpha) \\
x_\alpha &\mapsto \sum_{\mu(z_\beta)=k+1} \langle x_\alpha, z_\beta \rangle \, z_\beta
\end{aligned}$$

where, $\langle x_\alpha, z_\beta \rangle = \sum_{(\lambda,u_\lambda)\in\mathcal{M}^H_{x_\alpha,z_\beta}} \tau(u_\lambda)$ and $\mathcal{M}^H_{x_\alpha,z_\beta} \subset [0,1] \times \mathcal{P}^{1,2}_{x_\alpha,z_\beta}$ and $\mu(x_\alpha) - \mu(z_\beta) + 1 = 0$, hence it is a 0-dimensional space and by compactness results is a finite set, so Ψ_k is well-defined. A direct computation gives:

$$\partial^\beta \circ \Psi_k(x_\alpha) = \sum_{\mu(x_\beta)=k} \Big[\sum_{\mu(z_\beta)=k+1} \sum_{(\lambda,u_\lambda)\in\mathcal{M}^H_{x_\alpha,z_\beta}} \sum_{\hat{u}_\beta \in \widehat{M}^{f^\beta}_{z_\beta,x_\beta}} \tau(u_\lambda)\tau(\hat{u}_\beta) \Big] x_\beta \tag{4}$$

$$\Psi_{k-1} \circ \partial^\alpha(x_\alpha) = \sum_{\mu(x_\beta)=k} \Big[\sum_{\mu(y_\alpha)=k-1} \sum_{(\lambda,v_\lambda)\in\mathcal{M}^H_{y_\alpha,x_\beta}} \sum_{\hat{u}_\alpha \in \widehat{M}^{f^\alpha}_{x_\alpha,y_\alpha}} \tau(v_\lambda)\tau(\hat{u}_\alpha) \Big] x_\beta \tag{5}$$

$$(\Phi_1 - \Phi_0)(x_\alpha) = \sum_{\mu(x_\beta)=\mu(x_\alpha)} \Big[\sum_{u_1 \in \mathcal{M}^{h_1}_{x_\alpha,x_\beta}} \tau(u_1) \; - \sum_{u_0 \in \mathcal{M}^{h_0}_{x_\alpha,x_\beta}} \tau(u_2) \Big] x_\beta \tag{6}$$

This one dimensional manifold is not necessarily closed hence it may have boundaries so this leads us to consider 4 possible cases. Each of its connected component can be diffeomorphic to:

1. $[0,1]$, i.e. it has two boundaries in $\mathcal{M}^{h_0}_{x_{\alpha\beta}} \bigcup \mathcal{M}^{h_1}_{x_{\alpha\beta}}$ corresponding to $\lambda = 0, 1$.

2. S^1, in which case there is no boundary or broken trajectories,

3. $(0,1)$, i.e. it has a compactification by 2 distinct broken trajectories, or

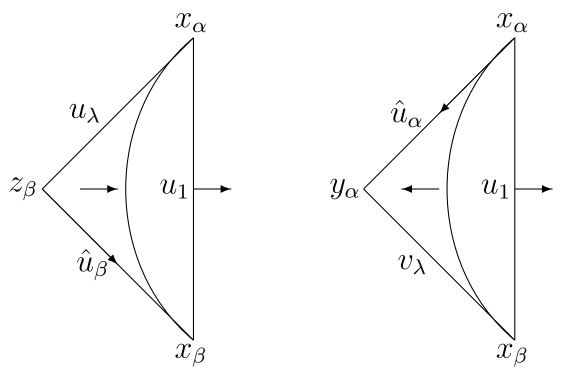
4. $(0,1]$ and $[0,1)$, i.e. it has 1 actual boundary and 1 broken trajectory.

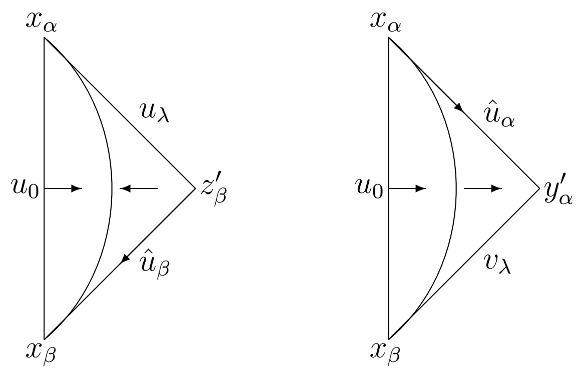
Case 2) is trivial.
Case 3) is the similar to the previous theorem. Subtracting (5) from (4) we see that terms corresponding to mixed broken trajectories cancel out.
For case 1), our choice of canonical orientation, $1 \otimes 1^*$ under the isomorphism Ω induces orientations pointing towards the interior of the connected component at $\lambda = 0$ and outwards at $\lambda = 1$. So if the boundary consists of trajectories, $u_{\alpha\beta}, v_{\alpha\beta} \in \mathcal{M}^{h_0}_{x_{\alpha\beta}}$ then the orientations at the boundaries are in opposite directions hence we get $\tau(u_{\alpha\beta}) = -\tau(v_{\alpha\beta})$. We get the same result if both boundaries are in $\mathcal{M}^{h_1}_{x_{\alpha\beta}}$.
On the other hand, if the trajectories are in different space of trajectories then their orientations agree (both pointing either to the left or to the right) hence we obtain

$$\tau(u_{\alpha\beta}) = \tau(v_{\alpha\beta}).$$

So in equation (6), the terms corresponding to this case cancel out leaving out only terms that have exactly one (proper) boundary.

For case 4), as mentioned above we already know the canonical orientation at $\lambda = 0$ and $\lambda = 1$ so we need to look at the orientation at the broken trajectory. There are two possible types of broken trajectories namely $(u_\lambda, \hat{u}_\beta)$ and $(\hat{u}_\alpha, v_\lambda)$. We picture these 4 possible situations as follows:





which give the following relations:

1. $\tau(u_\lambda)\tau(\hat{u}_\beta) = -\tau(u_0)$
2. $\tau(\hat{u}_\alpha)\tau(v_\lambda) = \tau(u_0)$
3. $\tau(u_\lambda)\tau(\hat{u}_\beta) = \tau(u_1)$
4. $\tau(\hat{u}_\alpha)\tau(v_\lambda) = -\tau(u_1)$

Subtracting equation (5) from equation (4) and using the above relations, we see that it is equal to the reduced equation (6) from case 1). This completes the proof.

□

So given $c \in \mathrm{Crit}_k(f^\alpha)$ such that $\partial^\alpha c = 0$, i.e. $c \in \ker(\partial^\alpha)$ then $(\Phi_1 - \Phi_0)(c) \in R(\partial^\beta_{k+1})$ so we conclude that

$$\Phi_{1*} = \Phi_{0*}$$

i.e. any chain homomorphism induces the same map in homology (independent of the homotopy chosen).

Lemma 8.3. *The above induced map in homology satisfies the following composition rule :*

$$\Phi_*^{\gamma\beta} \circ \Phi_*^{\beta\alpha} = \Phi_*^{\gamma\alpha} : H_*(f^\alpha) \to H_*(f^\beta).$$

and so it follows that these maps are isomorphisms with $\left(\Phi_*^{\gamma\beta}\right)^{-1} = \Phi_*^{\beta\gamma}$ *and* $\Phi_*^{\alpha\alpha} = \mathrm{Id}$.

Proof. Let $x_\alpha \in \mathrm{Crit}_k(f^\alpha)$, $x_\beta \in \mathrm{Crit}_k(f^\beta)$, $x_\gamma \in \mathrm{Crit}_k(f^\gamma)$ so that $\mu(x_\alpha) = \mu(x_\beta) = \mu(x_\gamma)$, then from theorem 6.11, there is a bijection between $\mathcal{M}^{h^{\alpha\beta}}_{x_\alpha,x_\beta} \times \mathcal{M}^{h^{\beta\gamma}}_{x_\beta,x_\gamma}$ and $\mathcal{M}^{h^{\alpha\gamma}}_{x_\alpha,x_\gamma}$. Note here that the homotopy $h^{\alpha\gamma}$ is determined by the gluing map, $\#_R$. Due to the canonical orientation, $1 \otimes 1^*$ on these spaces of trajectories it follows that $\tau(u_{\alpha\beta})\tau(u_{\beta\gamma}) = \tau(u_{\alpha\beta}\#_R u_{\beta\gamma}) = \tau(u_{\alpha\gamma})$. So we get,

$$\begin{aligned}
\Phi^{\gamma\beta} \circ \Phi^{\beta\alpha}(x_\alpha) &= \Phi^{\gamma\beta}\left(\sum_{x_\beta} \sum_{u_{\alpha\beta} \in \mathcal{M}^{h^{\alpha\beta}}_{x_\alpha,x_\beta}} \tau(u_{\alpha\beta}) x_\beta\right) \\
&= \sum_{x_\gamma}\sum_{x_\beta} \sum_{u_{\beta\gamma} \in \mathcal{M}^{h^{\beta\gamma}}_{x_\beta,x_\gamma}} \sum_{u_{\alpha\beta} \in \mathcal{M}^{h^{\alpha\beta}}_{x_\alpha,x_\beta}} \tau(u_{\alpha\beta})\tau(u_{\beta\gamma}) x_\gamma \\
&= \sum_{x_\gamma}\sum_{x_\beta} \sum_{u_{\beta\gamma} \in \mathcal{M}^{h^{\beta\gamma}}_{x_\beta,x_\gamma}} \sum_{u_{\alpha\beta} \in \mathcal{M}^{h^{\alpha\beta}}_{x_\alpha,x_\beta}} \tau(u_{\alpha\gamma}) x_\gamma \\
&= \Phi^{\gamma\alpha}(x_\alpha)
\end{aligned}$$

The third equality follows from theorem 6.11 and the above relation of characteristic signs. So we obtain

$$\Phi^{\gamma\beta}_* \circ \Phi^{\beta\alpha}_*([x_\alpha]) = \Phi^{\gamma\alpha}_*([x_\alpha])$$

It is important to note here that this homotopy depends on x_α, x_β and x_γ, so we cannot immediately say that this relation holds for all critical points. However, we may repeat this process for each triple of critical points using appropriate homotopies but from the previous lemma the map induced in homology is independent of the homotopy chosen so indeed the above relation in homology holds for all critical points. $\square$

So we have proved the fundamental theorem of this section. Hence from now on, we may simply denote the homology groups by $H_\bullet(M)$. Using the group, Γ defined in the orientation section we can also show that the Morse homology does not depend on the chosen coherent orientation as well.(See [11] for the details) To conclude this section we define the Morse cohomology and prove the Poincaré Duality theorem.
Given a chain complex, $(C_\bullet(f), \partial_\bullet)$ we define the cochain complex, $(C^\bullet(f), \delta^\bullet)$ as its dual, i.e.

$$\cdots \leftarrow C^{k+1} \xleftarrow{\delta^k} C^k \xleftarrow{\delta^{k-1}} C^{k-1} \xleftarrow{\delta^{k-2}} \cdots \xleftarrow{\delta^0} C^0$$

$$C^k(f) = \operatorname{Hom}(C_k(f), \mathbb{Z}) \text{ and } \delta^k(c_k^*)(c_{k+1}) = c_k^*(\partial_{k+1} c_{k+1})$$

We define the cohomology group by $H^k := \ker \delta^k / \operatorname{Im} \delta^{k-1}$.

Theorem 8.4. *Poincaré Duality Theorem*
For a closed orientable n-manifold, M, there is a natural isomorphism between the k^{th} homology group and the $(n-k)^{th}$ cohomology groups, i.e.

$$H_k(M) \xrightarrow{\cong} H^{n-k}(M) \quad k = 0, 1, ..., n$$

Proof. Given any Morse function, f we may define the homology groups, $H_k(M)$. We notice that $-f$ is also a Morse function and so defines $H_k(M)$ as well, due to independence of the Morse homology on the function. We also observe that there is natural 1-to-1 correspondence,

$$\begin{aligned} \mathrm{Crit}_k(-f) &\to \mathrm{Crit}_{n-k}(f) \\ x &\mapsto x \end{aligned}$$

This is easily seen from the fact f and $-f$ have the same critical points and that if the Hessian of $-f$ at x has r negative eigenvalues then the Hessian of f at x has $n-k$ negative eigenvalues. So there are natural isomorphisms:

$$C_k(-f) \cong C_{n-k}(f) \cong C^{n-k}(f)$$

and by definition, the boundary maps, ∂^{-f} agree with the coboundary maps δ^f giving $H_k(-f) \cong H^{n-k}(f)$. Together with the fact that $H_k(-f) \cong H_k(f)$, this proves the stated isomorphism. □

Having now defined the Morse homology and cohomology, we now want to look at two crucial operations that comes with cohomology structure namely the cup and cap product.

9 Cup Product in Morse Cohomology

In this section, we shall construct by analogy to the techniques covered in the previous chapters Y-shaped space of trajectories, also referred to as Feynman diagrams.

Definition 9.1. *Let $\xi_1 \in \mathrm{Vec}(\overline{\mathbb{R}}^-)$ and $\xi_2, \xi_3 \in \mathrm{Vec}(\overline{\mathbb{R}}^+)$ be smooth vector bundles on the half-lines with global trivialisations, ϕ_i for $i = 1, 2, 3$, we then define*

$$H^{1,2}(\xi_1 \oplus \xi_2 \oplus \xi_3) := \phi_*^{-1}(H^{1,2}(\overline{\mathbb{R}}^-, \mathbb{R}^n) \oplus H^{1,2}(\overline{\mathbb{R}}^+, \mathbb{R}^n) \oplus H^{1,2}(\overline{\mathbb{R}}^+, \mathbb{R}^n))$$

where $\phi_*^{-1} = (\phi_{1*}^{-1}, \phi_{2*}^{-1}, \phi_{3*}^{-1})$. This defines a Banach space with a norm induced by $||(s_1, s_2, s_3)||_{1,2} = \sum_{i=1}^{3} ||s_i||_{1,2}$. Here $\overline{\mathbb{R}}^+ = [0, +\infty]$ is given a manifold structure by the chart:

$$h : \overline{\mathbb{R}}^+ \to [0, 1]$$
$$h(t) = \begin{cases} \frac{t}{\sqrt{1+t^2}}, & t \geqslant 0 \\ 1, & t = +\infty \end{cases}$$

(Similarly for $\overline{\mathbb{R}}^-$.)

Given three distinct Morse functions, f_1, f_2, f_3 on M we denote their critical points by x_i, y_i for $i = 1, 2, 3$ respectively. Let $h_i \in C^\infty(\overline{\mathbb{R}}, M)$ such that $h_i(-\infty) = x_i$ and $h_i(+\infty) = y_i$. From now on, we shall restrict h_1 to $\overline{\mathbb{R}}^-$ and h_2, h_3 to $\overline{\mathbb{R}}^+$.
We define

$$\widehat{\mathcal{P}}_{x_1,y_2,y_3} := \left\{(\exp \circ s_1, \exp \circ s_2, \exp \circ s_3) : s_i \in H^{1,2}(h_i^*\mathcal{O}),\ i = 1, 2, 3\right\}$$

where $\mathcal{O}$ once again denotes the injectivity neighbourhood associated to the exponential map.
Since we endowed the half-lines by the natural submanifold topology all the analysis carried out in the section on space of trajectories can be carried out in the same way. So by strict analogy to theorem 2.6, we can prove that $\widehat{\mathcal{P}}_{x_1,y_2,y_3}$ is a Banach manifold with chart $\{\bigoplus_{i=1}^{3} H^{1,2}(h_i^*\mathcal{O}), \bigoplus_{i=1}^{3} \exp_{h_i}\}_{(h_1,h_2,h_3)\in C^\infty(M)}$.
We shall denote this manifold simply by $\widehat{\mathcal{P}}$ when there is no risk of confusion.
Now consider the map, G which is only defined locally in coordinate charts by

$$G : \widehat{\mathcal{P}}_{x_1,y_2,y_3} \to \mathbb{R}^n \times \mathbb{R}^n$$
$$(\gamma_1, \gamma_2, \gamma_3) \mapsto (\hat{\gamma}_1(0) - \hat{\gamma}_2(0), \hat{\gamma}_1(0) - \hat{\gamma}_3(0))$$

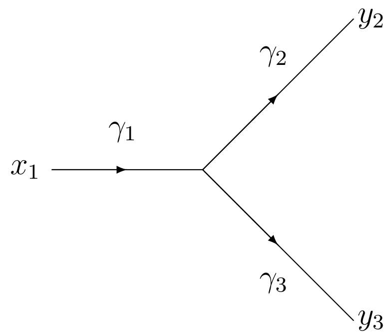
where $\hat{\gamma}_i(0) = \varphi(\gamma_i(0))$ for a chart map, φ. This map is locally given by,

$$\begin{aligned} G_{loc} : H^{1,2}(h_1^*\mathcal{O} \oplus h_2^*\mathcal{O} \oplus h_3^*\mathcal{O}) &\to T_{h_1(0)}M \times T_{h_1(0)}M \cong \mathbb{R}^n \times \mathbb{R}^n \\ (\eta_1, \eta_2, \eta_3) &\mapsto (\eta_1(0) - \exp_{h_1(0)}^{-1} \exp_{h_2(0)} \eta_2(0), \\ &\qquad \eta_1(0) - \exp_{h_1(0)}^{-1} \exp_{h_3(0)} \eta_3(0)) \end{aligned}$$

where we chose φ to be a normal coordinates chart at $h_1(0)$. Since this map is simply an evaluation map and is linear in the first variable, it is easy to see that $DG(\gamma_1, \gamma_2, \gamma_3)$ is surjective for we can always choose a section that takes any specific value at a point. So by the Implicit function theorem, $G^{-1}(0)$ is a submanifold of $\widehat{\mathcal{P}}$ of codimension $2n$ i.e.

$$\mathcal{P} := G^{-1}(0) = \{(\gamma_1, \gamma_2, \gamma_3) \in \widehat{\mathcal{P}} : \gamma_1(0) = \gamma_3(0) = \gamma_3(0)\}$$

Geometrically, $\mathcal{P}$ consists of Y-shaped structures:



This manifold is usually called a graph moduli space, where γ_1 is an incoming edge and γ_2 and γ_3 are outgoing edges. The endpoints and the point of intersection are called vertices.

Definition 9.2. *We define a section map, F by*

$$\begin{aligned} F : \widehat{\mathcal{P}} &\to L^2(\widehat{\mathcal{P}}^*TM) \\ (\gamma_1, \gamma_2, \gamma_3) &\mapsto (\dot{\gamma}_1 + \nabla f_1 \circ \gamma_1, \dot{\gamma}_2 + \nabla f_2 \circ \gamma_2, \dot{\gamma}_3 + \nabla f_3 \circ \gamma_3) \end{aligned}$$

This map is indeed well-defined since it can be written explicitly as $F = F_1 \oplus F_2 \oplus F_3$ where each F_i is defined as in corollary 2.8.1.
We now want to prove that this map is indeed a Fredholm map and we need to compute its Fredholm index. From our analysis in the Fredholm section, it suffices to carry out the proofs for the trivialised linearisation of F and we

may then transfer the results to the non-trivial operator in a way independent of the chosen trivialisation. So from now on, we may assume

$$DF = \oplus_{i=1}^{3} DF_i : \oplus_{i=1}^{3} H^{1,2}(\mathbb{R}^{\pm}, \mathbb{R}^n) \to \oplus_{i=1}^{3} L^2(\mathbb{R}^{\pm}, \mathbb{R}^n)$$

$$DF(s_1, s_2, s_3) = (\dot{s}_1 + A_1 s_1, \dot{s}_2 + A_2 s_2, \dot{s}_3 + A_3 s_3)$$

$$\text{where } A_1 : \overline{\mathbb{R}}^{-} \to M(n, \mathbb{R}) \text{ and } A_2, A_3 : \overline{\mathbb{R}}^{+} \to M(n, \mathbb{R})$$

The fact that DF is a Fredholm operator follows immediately from theorem 3.1 since the proof remains unchanged if we replace $\mathbb{R}$ by the half-lines $\mathbb{R}^+$ and $\mathbb{R}^-$ and so the proof can be carried out in each component. We now need to compute the index of DF, so we start with the map DF_1. For simplicity of notation we denote this map by K and A_1 by A. Given any $\delta > 0$, we define a function, $\beta_\delta \in C^\infty(\mathbb{R}, [0,1])$ such that

$$\beta_\delta(t) = \begin{cases} 0, & t = 0 \\ 1, & |t| \geqslant \delta \end{cases}$$

and let $\hat{K}_\delta(t) := \frac{\partial}{\partial t} + \beta_\delta(t) A(t)$. By a simple computation we have the estimate:

$$||\hat{K}_\delta - K||_{\mathcal{L}} \leqslant ||A||_\infty \cdot ||\beta_\delta - 1||_{1,2}$$

So by choosing δ small enough, we have that $\hat{K}_\delta$ is a Fredholm operator as well and has the same index as K. Hence w.l.o.g we may start off by assuming $A(0) = 0$ and define a continuous odd extension by

$$\hat{A}(t) = \begin{cases} -A(-t), & t \geqslant 0 \\ A(t), & t < 0 \end{cases}$$

so that $\hat{A}(+\infty) = -A(-\infty)$ and $\hat{A}(-\infty) = A(-\infty)$.
We also define

$$\hat{D} : H^{1,2}(\mathbb{R}, \mathbb{R}^n) \to L^2(\mathbb{R}, \mathbb{R}^n)$$
$$\eta \mapsto \dot{\eta} + \hat{A}\eta$$

From theorem 3.5, it follows that

$$\operatorname{ind}(\hat{D}) = \mu(A(-\infty)) - \mu(-A(-\infty)) = 2\mu(A(-\infty)) - n.$$

We decompose $H^{1,2}(\mathbb{R}, \mathbb{R}^n) := H_e^{1,2}(\mathbb{R}, \mathbb{R}^n) \oplus H_o^{1,2}(\mathbb{R}, \mathbb{R}^n)$ where

$$H_e^{1,2}(\mathbb{R}, \mathbb{R}^n) = \{\eta \in H^{1,2}(\mathbb{R}, \mathbb{R}^n) : \eta(-t) = \eta(t)\} \text{ and}$$
$$H_o^{1,2}(\mathbb{R}, \mathbb{R}^n) = \{\eta \in H^{1,2}(\mathbb{R}, \mathbb{R}^n) : \eta(-t) = -\eta(t)\}$$

Similarly we decompose $L^2(\mathbb{R},\mathbb{R}^n) = L^2_o(\mathbb{R},\mathbb{R}^n) \oplus L^2_e(\mathbb{R},\mathbb{R}^n)$. Since by construction $\hat{D}$ maps odd functions to even functions and even functions to odd functions, we can decompose the operator $\hat{D}$ as well.

$$\hat{D} = \hat{D}_e \oplus \hat{D}_o : H^{1,2}_e(\mathbb{R},\mathbb{R}^n) \oplus H^{1,2}_o(\mathbb{R},\mathbb{R}^n) \to L^2_o(\mathbb{R},\mathbb{R}^n) \oplus L^2_e(\mathbb{R},\mathbb{R}^n)$$

Lemma 9.1. *The diagram,*

$$\begin{tikzcd} H^{1,2}_e(\mathbb{R},\mathbb{R}^n) \arrow[r, "\hat{D}_e"] \arrow[d, "\alpha"'] & L^2_o(\mathbb{R},\mathbb{R}^n) \arrow[d, "\beta"'] \\ H^{1,2}(\mathbb{R}^-,\mathbb{R}^n) \arrow[r, "D_2"] & L^2(\mathbb{R}^-,\mathbb{R}^n) \end{tikzcd}$$

is commutative where α and β are the restriction maps and D_2 is the restriction of $\hat{D}_e$ and moreover, the maps α and β are isomorphisms.

Proof. It is clear that if $\eta \in L^2_o(\mathbb{R})$ then $\beta(\eta) \in L^2(\mathbb{R}^-)$ so this map is well-defined. Since $L^2_o(\mathbb{R})$ consists of odd functions, it follows that β is injective. Given any $\xi \in L^2(\mathbb{R}^-)$, we can naturally extend it to an even function, $\overline{\xi}$ so that $||\overline{\xi}||_{0,2} = \sqrt{2}||\xi||_{0,2} < \infty$ hence β is onto as well. Using the same argument, we deduce that α is injective. However, we still need to verify surjectivity since it is not obvious that the even extension is necessarily weakly differentiable.
We need to show that given $\eta(t) \in H^{1,2}(\mathbb{R}^-)$, its even extension,

$$\hat{\eta}(t) = \begin{cases} \eta(t), & t \leqslant 0 \\ \eta(-t), & t > 0 \end{cases}$$

has weak derivative

$$\xi(t) = \begin{cases} \dot{\eta}(t), & t \leqslant 0 \\ -\dot{\eta}(-t), & t > 0 \end{cases}$$

then it follows that $\hat{\eta} \in H^{1,2}_e(\mathbb{R})$ and $\alpha(\hat{\eta}) = \eta$. Let $\{g_i\}_{i=1}^3$ be a smooth partition of unity such that

$$g_1(t) = \begin{cases} 1, & t \leqslant -\delta \\ 0, & t \geqslant 0 \end{cases}$$

and $g_2(t) = g_1(-t)$ so w.l.o.g we may assume g_3 is even. Then a direct computation gives

$$\int_{\mathbb{R}} \dot{\phi}\hat{\eta}\,\mathrm{dt} = \int_{\mathbb{R}} \dot{\left(\sum_{i=1}^{3} g_i\phi\right)} \hat{\eta}\,\mathrm{dt} = \sum_{i=1}^{3}\int_{\mathbb{R}} \left(\dot{\phi g_i}\right)\hat{\eta}\,\mathrm{dt}$$

Since ϕg_1 and ϕg_2 are test functions on $\mathbb{R}^-$ and $\mathbb{R}^+$ respectively and $\eta(t) \in H^{1,2}(\mathbb{R}^-)$, so we get

$$\begin{aligned}\sum_{i=1}^{2}\int_{\mathbb{R}} \left(\dot{\phi g_i}\right)\hat{\eta}\,\mathrm{dt} = &-\int_{-\infty}^{-\delta} \phi(t)\dot{\eta}(t)\,\mathrm{dt} + \int_{\delta}^{\infty}\phi(t)\dot{\eta}(-t)\,\mathrm{dt}\\ &-\int_{-\delta}^{0}\phi(t)g_1(t)\dot{\eta}(t)\,\mathrm{dt} + \int_{0}^{\delta}\phi(t)g_2(t)\dot{\eta}(-t)\,\mathrm{dt}\end{aligned}$$

Also $||g_1||_\infty = ||g_2||_\infty = 1$ and $||\phi||_\infty$ and $\operatorname{ess\,sup}_{t\in[-\delta,0]}|\dot{\eta}|$ are both bounded, hence we see that as $\delta \to 0$ the last two integrals converge to zero and the first two integrals converge to our desired expression. So it suffices to show that as $\delta \to 0$,

$$\int_{-\delta}^{\delta}\left(\dot{\phi g_3}\right)\hat{\eta}\,\mathrm{dt} = \int_{-\delta}^{\delta}\dot{\phi} g_3\hat{\eta}\,\mathrm{dt} + \int_{-\delta}^{\delta}\phi\dot{g}_3\hat{\eta}\,\mathrm{dt} \to 0$$

The first integral clearly converges to zero by a similar argument as above. The trouble with the second term is that when $\delta \to 0$, $\dot{g}_3 \to +\infty$ so we cannot immediately conclude that it converges to zero. However we may assume that $||\dot{g}_3||_\infty \leqslant C/\delta$ for some constant $C > 0$. (For instance we can choose $g_3(t) = \exp(1 - \delta^2/(\delta^2 - x^2))$ on $(-\delta, \delta)$.)
Moreover, from the fundamental theorem of calculus we have that

$$\int_{-\delta}^{\delta}\phi(0)\dot{g}_3(t)\hat{\eta}(0) = 0$$

since $g_3(\delta) = g_3(-\delta) = 0$. Hence a direct computation gives

$$\begin{aligned}\left|\int_{-\delta}^{\delta}\phi(t)\dot{g}_3(t)\hat{\eta}(t)\,\mathrm{dt}\right| &= \left|\int_{-\delta}^{\delta}\dot{g}_3(t)(\phi(t)\hat{\eta}(t) - \phi(0)\hat{\eta}(0))\,\mathrm{dt}\right|\\ &\leqslant \int_{-\delta}^{\delta}|\dot{g}_3(t)|\,\mathrm{dt}\cdot\sup_{t\in[-\delta,\delta]}|\phi(t)\hat{\eta}(t) - \phi(0)\hat{\eta}(0)|\\ &\leqslant 2C\cdot\sup_{t\in[-\delta,\delta]}|\phi(t)\hat{\eta}(t) - \phi(0)\hat{\eta}(0)| \to 0 \ \text{ as } \delta \to 0.\end{aligned}$$

Here we have used the continuity of $\hat{\eta}$ which follows from the Sobolev embeding theorem. This completes the proof. □

By the property of Fredholm operators, $\operatorname{ind} \hat{D}_e = \operatorname{ind} D_2$ (since $\operatorname{ind}(\alpha) = \operatorname{ind}(\beta) = 0$).
Now considering,

$$\begin{array}{ccc} H_o^{1,2}(\mathbb{R}, \mathbb{R}^n) & \xrightarrow{\hat{D}_o} & L_e^2(\mathbb{R}, \mathbb{R}^n) \\ \alpha \big\downarrow & & \beta \big\downarrow \\ H_o^{1,2}(\mathbb{R}^-, \mathbb{R}^n) & \xrightarrow{D_3} & L^2(\mathbb{R}^-, \mathbb{R}^n) \end{array}$$

where $H_o^{1,2}(\mathbb{R}^-, \mathbb{R}^n) = \{\eta \in H^{1,2}(\mathbb{R}^-) : \eta(0) = 0\}$ and D_3 is the restriction of $\hat{D}_o$. β is again an isomorphism by the same argument as in the above lemma. By repeating the same steps as above, we are once again led to estimating

$$\int_{-\delta}^{\delta} \phi \dot{g}_3 \hat{\eta} \, \mathrm{dt} = \int_{-\delta}^{0} (\phi(t) + \phi(-t)) \dot{g}_3(t) \eta(t) \, \mathrm{dt} \, .$$

Since $\eta(0) = 0$ and it is continuous at $t = 0$ (by Sobolev embedding theorem), given any $\epsilon > 0$, there exists $\delta > 0$ such that $|\eta(t)| < \epsilon$ for $t \in [0, \delta]$. This gives the estimate

$$\left| \int_{-\delta}^{\delta} \phi \dot{g}_3 \hat{\eta} \, \mathrm{dt} \right| \leqslant 2 ||\phi||_\infty \cdot \frac{C}{\delta} \cdot \epsilon \cdot \delta \to 0.$$

Hence we once again obtain that the restriction map, α is an isomorphism and so $\operatorname{ind}(\hat{D}_o) = \operatorname{ind}(D_3)$. Observe that $H_o^{1,2}(\mathbb{R}^-)$ has codimension n in $H^{1,2}(\mathbb{R}^-)$. Hence we have $\operatorname{ind}(D_3) = \operatorname{ind}(D_2) - n$.
Here we used the fact that decreasing the dimension of the domain of a Fredholm operator by h reduces its index by h as well. This can easily be seen as follows: suppose the dimension of the kernel decreases by k then it follows that the dimension of the cokernel increases by $h - k$ and hence the Fredholm index decreases by h.
So we calculate $\operatorname{ind}(\hat{D}) = \operatorname{ind}(\hat{D}_o) + \operatorname{ind}(\hat{D}_e) = 2 \operatorname{ind}(D_2) - n$ implying that $\operatorname{ind}(DF_1) = \operatorname{ind}(D_2) = \mu(A(-\infty))$. This proves the index formula for the incoming edge. Now looking at DF_2, we consider the extension given by:

$$\hat{A}(t) = \begin{cases} A_2(t), & t \geqslant 0 \\ -A_2(-t), & |t| < 0 \end{cases}$$

Using the same notation as earlier, we get

$$\begin{aligned} \operatorname{ind}(\hat{D}) &= \operatorname{ind}(-A_2(+\infty)) - \operatorname{ind}(A(+\infty)) \\ &= n - \mu(A(+\infty)) - \mu(A(+\infty)) \\ &= n - 2\mu(A(+\infty)). \end{aligned}$$

The same analysis leads to $2\,\mathrm{ind}(DF_2) - n = \mathrm{ind}(\hat{D}) = n - 2\mu(A_2(+\infty))$ i.e. $\mathrm{ind}(DF_2) = n - \mu(A_2(+\infty))$. Similarly we get $\mathrm{ind}(DF_3) = n - \mu(A_3(+\infty))$. So,

$$\mathrm{ind}(DF) = \sum_{i=1}^{3} \mathrm{ind}(DF_i) = \mu(A_1(-\infty)) - \mu(A_2(+\infty)) - \mu(A_3(+\infty)) + 2n.$$

Restricting our domain to $\mathcal{P}$ which has codimension $2n$ in $\widehat{\mathcal{P}}$ and using the same argument as above, we get

$$\mathrm{ind}(DF|_{\mathcal{P}}) = \mathrm{ind}(DF) - 2n = \mu(A_1(-\infty)) - \mu(A_2(+\infty)) - \mu(A_3(+\infty)).$$

Remark: Note in fact this result can be extended directly to the case when we have k edges, k_1 of which are incoming and $k - k_1$ are outgoing. Definition 9.1 can be naturally adapted for k vector bundles on the half-lines. We then define our above function, G by $G(\gamma_1, ..., \gamma_k) = (\hat{\gamma}_1(0) - \hat{\gamma}_2(0), ..., \hat{\gamma}_1(0) - \hat{\gamma}_k(0))$ and denote the critical points of incoming edges by x_i and those of the outgoing edges by y_j. We see that proof of the Fredholm index is still unchanged and we get for the corresponding section map, F

$$\mathrm{ind}(DF) = \sum_{j=1}^{k_1} \mu(x_j) + \sum_{l=k_1+1}^{k} [n - \mu(y_l)]$$

which gives

$$\begin{aligned}\mathrm{ind}(DF\big|_{G^{-1}(0)}) &= \sum_{j=1}^{k_1} \mu(x_j) + \sum_{l=k_1+1}^{k} [n - \mu(y_l)] - n(k-1)\\ &= \sum_{j=1}^{k_1} \mu(x_j) - \sum_{l=k_1+1}^{k} \mu(y_l) - n(k_1 - 1)\end{aligned}$$

This leads to a more general operation on cohomology called the Massey product.

From now on we shall restrict to the submanifold, $\widehat{\mathcal{P}}$. The transversality argument can again be carried out by strict analogy. Considering each component of F we find generic sets of Riemannian metrics, $\Sigma_1, \Sigma_2, \Sigma_3$ then by the Baire Category theorem $\Sigma := \bigcap_{i=1}^{3} \Sigma_i$ is a generic set as well. Hence we see that the 0-section (more precisely $0 \oplus 0 \oplus 0$) is a regular value of F and by the Implicit function theorem we have that $\mathcal{M}^{f_1,f_2,f_3}_{x_1,y_2,y_3} = F^{-1}(0)$ is a Banach submanifold (without boundary) of $\mathcal{P}$ for generic metric, $g \in \Sigma$ and has dimension $\mu(x_1) - \mu(y_2) - \mu(y_3)$.

We denote the space of Y-trajectories for critical points, x_1, y_2 and y_3 by $\mathcal{M}^{f_1,f_2,f_3}_{x_1,y_2,y_3}$. We now need to define a compactification of this Banach manifold by broken trajectories.

Theorem 9.2. *Let $\mu(x_1) - \mu(y_2) - \mu(y_3) = 1$ so that $\mathcal{M}^{f_1,f_2,f_3}_{x_1,y_2,y_3}$ is a 1-dimensional manifold. Then given any sequence $\{u_n \oplus v_n \oplus w_n\}_{n=1}^{\infty} \subset \mathcal{M}^{f_1,f_2,f_3}_{x_1,y_2,y_3}$ if there does not exist a subsequence which converges in the $H^{1,2}$-norm ($||(s_1, s_2, s_3)||_{1,2} = \sum_{i=1}^{3} ||s_i||_{1,2}$) then there exists one critical point $x' \in \mathrm{Crit}(f_1)$, $y' \in \mathrm{Crit}(f_2)$ or $z' \in \mathrm{Crit}(f_3)$ (distinct from x_1, y_2 and y_3 respectively) and a subsequence, n_k such that exactly one of the following holds:*

1. $(u_{n_k}, v_{n_k}, w_{n_k}) \xrightarrow{C^{\infty}_{loc}} (u, v, w) \in \mathcal{M}^{f_1,f_2,f_3}_{x',y_2,y_3}$

2. $(u_{n_k}, v_{n_k}, w_{n_k}) \xrightarrow{C^{\infty}_{loc}} (u, v, w) \in \mathcal{M}^{f_1,f_2,f_3}_{x_1,y',y_3}$

3. $(u_{n_k}, v_{n_k}, w_{n_k}) \xrightarrow{C^{\infty}_{loc}} (u, v, w) \in \mathcal{M}^{f_1,f_2,f_3}_{x_1,y_2,z'}$

Proof. The proof relies essentially on suitably modifying this problem so that we can use the results from the section on compactness. Considering the vertex where the edges meet, we have a sequence of points $\{u_n(0)\}_{n=1}^{\infty}$ in M and due to the compactness of M we may extract a convergent subsequence, $u_{n_j}(0) \to \bar{x} \in M$. We may view u_{n_j}, v_{n_j} and w_{n_j} as elements of $\mathcal{M}^{f_1}$,$\mathcal{M}^{f_2}$ and $\mathcal{M}^{f_3}$ respectively. In order to apply lemma 5.2 and lemma 5.4, we need to fix the endpoints as well. In general we cannot say that $u_{n_j} \in \mathcal{M}^{f_1}_{x_1,y_1}$ for all $j \in \mathbb{N}$ since for different js, we may have that the trajectories do not converge to the same critical point y_1. But since there are only finitely many critical points, we can extract a further subsequence such that indeed $\{u_{n_l}\}_{l=1}^{\infty} \subset \mathcal{M}^{f_1}_{x_1,y_1}$. By extracting further subsequences for the two other edges, we have

$$(u_{n_k}, v_{n_k}, w_{n_k}) \in \mathcal{M}^{f_1}_{x_1,y_1} \times \mathcal{M}^{f_2}_{x_2,y_2} \times \mathcal{M}^{f_3}_{x_3,y_3} \text{ for every } k \in \mathbb{N}$$

So by lemma 5.2, we have

$$u_{n_k} \xrightarrow{C^{\infty}[-R,R]} u \in C^{\infty}(\mathbb{R}, M) \text{ for any } R > 0$$

Similarly for v_{n_j} and w_{n_j}.

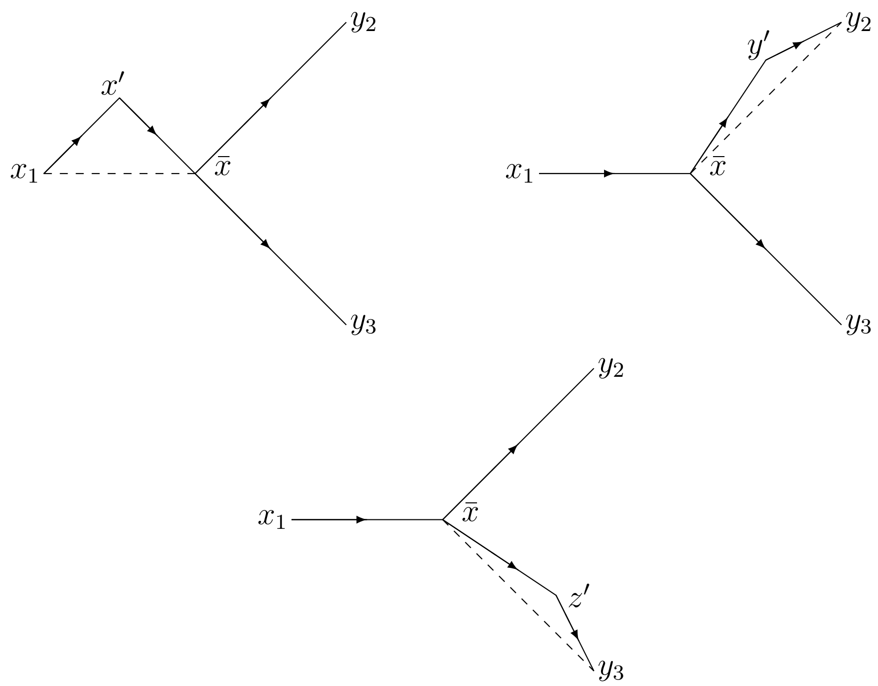
If we have that $u(-\infty) = x_1$, $v(+\infty) = y_2$ and $w(+\infty) = y_3$ then together with the fact that $u(0) = v(0) = w(0) = \bar{x}$ and lemma 5.4 this imply that in fact we have $H^{1,2}$-convergence in the space of Y-trajectories in contradiction to our hypothesis. So w.l.o.g suppose $u(-\infty) \neq x_1$, then by theorem 5.5 there exist $x', y \in \mathrm{Crit}(f_1)$ such that $u \in \mathcal{M}^{f_1}_{x',y}$. So we have convergence to a broken trajectory i.e.

$$u_{n_k} \to (u', u) \in \mathcal{M}^{f_1}_{x_1,x'} \times \mathcal{M}^{f_1}_{x',y}$$

Since u' is endowed with a 1-dimensional time-shifting invariance and $\mathcal{M}^{f_1,f_2,f_3}_{x_1,y_2,y_3}$ is a 1-dimensional manifold, by dimension counting it follows that we cannot have broken trajectories on the other two edges. So we have

$$u_{n_k} \xrightarrow{C^\infty[-R,0]} u, \quad v_{n_k} \xrightarrow{C^\infty[0,R]} v, \quad w_{n_k} \xrightarrow{C^\infty[0,R]} w \text{ for any } R > 0$$

Hence this shows that 1. holds. Similarly we can consider broken trajectories of the other edges.
We illustrate these three cases:



□

For the more general situation, when we do not impose any dimension restrictions, the proof is carried out exactly the same way and by appealing to theorem 5.5 again we obtain one Y-shaped structure with the end vertices connected to chains of edges of time-independent trajectories, i.e. we have the compactified space,

$$\overline{\mathcal{M}}^{f_1,f_2,f_3}_{x_1,y_2,y_3} = \bigcup_{\substack{\mathrm{Crit}(f_i)\\ i=1,2,3}} \mathcal{M}^{f_1,f_2,f_3}_{x,y,z} \times \mathcal{M}^{f_1}_{x',x} \times \cdots \times \mathcal{M}^{f_3}_{z,z'}$$

We state the following gluing result without proof:

Theorem 9.3. *Let K be a compact subset of $\mathcal{M}^{f_1,f_2,f_3}_{x',y_2,y_3} \times \mathcal{M}^{f_1}_{x_1,x'}$ then there exists a smooth gluing map, $\#$ and a constant $\rho_K > 0$ such that*

$$\# : K \times [\rho_K, \infty) \to \mathcal{M}^{f_1,f_2,f_3}_{x_1,y_2,y_3}$$

and for each $\rho \in [\rho_K, \infty)$, $\#_\rho$ is an embedding. Also as $\rho \to \infty$ we have that

$$(u, v, w)\#_\rho u' \xrightarrow{C^\infty_{loc}} ((u, v, w), u').$$

As seen in the gluing section, this means that that $\mathcal{M}^{f_1,f_2,f_3}_{x_1,y_2,y_3}$ is a 1-dimensional manifold without boundary whenever $\mu(x_1) - \mu(y_2) - \mu(y_3) = 1$ and can be compactified by adding broken trajectories as its boundaries. We shall omit matters of orientation and work over the finite field, $\mathbb{Z}_2$. Since any closed 1-dimensional manifold is either diffeomorphic to the circle or the open interval, $(0, 1)$, hence the number of boundary points is always even i.e. $\equiv 0 \mod 2$. As a result we see that:

$$\begin{aligned}
&\sum_{\substack{\mu(x')-\mu(y_2)\\ -\mu(y_3)=0}} \sum_{\substack{\mu(x')=\\ \mu(x_1)-1}} \left|\mathcal{M}^{f_1,f_2,f_3}_{x',y_2,y_3}\right| \cdot \left|\widehat{\mathcal{M}}^{f_1}_{x_1,x'}\right| \\
+ &\sum_{\substack{\mu(x_1)-\mu(y')\\ -\mu(y_3)=0}} \sum_{\substack{\mu(y')=\\ \mu(y_2)+1}} \left|\mathcal{M}^{f_1,f_2,f_3}_{x_1,y',y_3}\right| \cdot \left|\widehat{\mathcal{M}}^{f_2}_{y',y_2}\right| \\
+ &\sum_{\substack{\mu(x_1)-\mu(y_2)\\ -\mu(z')=0}} \sum_{\substack{\mu(z')=\\ \mu(y_3)+1}} \left|\mathcal{M}^{f_1,f_2,f_3}_{x_1,y_2,z'}\right| \cdot \left|\widehat{\mathcal{M}}^{f_3}_{z',y_3}\right| = 0
\end{aligned} \tag{7}$$

Note that since we are summing over 0-dimensional Y-trajectories and due to the compactness of M, these sums are all finite. We are now ready to define the cup product operation.

Definition 9.3. *We define a bilinear map on the cochains as follows:*

$$\begin{aligned}
\smile : C^k(f_2) \otimes C^j(f_3) &\to C^{k+j}(f_1) \\
y^* \smile z^* &= \sum_{\substack{\mu(x)-\mu(y)\\ -\mu(z)=0}} \left|\mathcal{M}^{f_1,f_2,f_3}_{x,y,z}\right| x^*
\end{aligned}$$

where x^*, y^* and z^* are the duals of critical points x, y and z of f_1, f_2 and f_3 respectively.

Lemma 9.4.

$$\delta(y^* \smile z^*) = (\delta y^*) \smile z^* + y^* \smile (\delta z^*)$$

Proof. Since the map is bilinear it suffices to show that the relation holds for the dual of critical points. So for $p \in \mathrm{Crit}_{k+j+1}(f_1)$ we compute directly,

$$\begin{aligned}
\delta(y^* \smile z^*)(p) &= y^* \smile z^*(\partial p) \\
&= y^* \smile z^* \left(\sum_{\mu(p)-\mu(x)=1} \left|\widehat{\mathcal{M}}^{f_1}_{p,x}\right| x \right) \\
&= \sum_{\substack{\mu(x)-\mu(y)\\-\mu(z)=0}} \sum_{\mu(p)-\mu(x)=1} \left|\mathcal{M}^{f_1,f_2,f_3}_{x,y,z}\right| \cdot \left|\widehat{\mathcal{M}}^{f_1}_{p,x}\right| \\
(\delta y^*) \smile z^*(p) &= \sum_{\mu(r)-\mu(y)=1} \left|\widehat{\mathcal{M}}^{f_2}_{r,y}\right| (r^* \smile z^*)(p) \\
&= \sum_{\substack{\mu(x)-\mu(r)\\-\mu(z)=0}} \sum_{\mu(r)-\mu(y)=1} \left|\mathcal{M}^{f_1,f_2,f_3}_{x,r,z}\right| \cdot \left|\widehat{\mathcal{M}}^{f_2}_{r,y}\right| \\
y^* \smile (\delta z^*)(p) &= \sum_{\mu(s)-\mu(z)=1} \left|\widehat{\mathcal{M}}^{f_3}_{s,z}\right| (y^* \smile s^*)(p) \\
&= \sum_{\substack{\mu(x)-\mu(y)\\-\mu(s)=0}} \sum_{\mu(s)-\mu(z)=1} \left|\mathcal{M}^{f_1,f_2,f_3}_{x,y,s}\right| \cdot \left|\widehat{\mathcal{M}}^{f_3}_{s,z}\right|
\end{aligned}$$

So from equation (7), the asserted identity holds (over $\mathbb{Z}_2$).

□

We deduce immediately from this lemma that if $[y^*] \in H^k(f_2)$ and $[z^*] \in H^j(f_3)$ then $\delta(y^* \smile z^*) = 0$. Hence since the cohomology does not depend on the Morse chosen function $\smile$ induces a cup product operation in cohomology, i.e.

$$\smile : H^k(M) \otimes H^j(M) \to H^{k+j}(M).$$

We now move on to define the cap product in Morse cohomology. Using the notation of definition 9.3, we define

$$\begin{aligned}
\frown : C_{k+j}(f_1) \otimes C^k(f_2) &\to C_j(f_3) \\
x \frown y^* &:= \sum_{\substack{\mu(x)-\mu(y)\\-\mu(z)}=0} \left|\mathcal{M}^{f_1,f_2,f_3}_{x,y,z}\right| z
\end{aligned}$$

To show that this bilinear map indeed induces a well-defined map in cohomology, we need a similar relation as in lemma 9.4. Instead of deriving this relation from scratch, we make the observation that:

$$z^*(x \frown y^*) = (y^* \smile z^*)(x) \text{ for any } z \in C^j(f_3)$$

Then:

$$\begin{aligned} z^*(\partial(x \frown y^*)) &= \delta(z^*)(x \frown y^*) \\ &= y^* \smile \delta(z^*)(x) \\ &= \delta(y^* \smile z^*)(x) + (\delta y^*) \smile z^*(x) \\ &= (y^* \smile z^*)(\partial x) + z^*(x \frown \delta y^*) \\ &= z^*(\partial x \frown y^*) + z^*(x \frown \delta y^*) \end{aligned}$$

So we get the relation

$$\partial(x \frown y^*) = \partial x \frown y^* + x \frown \delta y^*$$

which allows us to conclude that $\frown$ induces the cap product operation,

$$\frown : H_{k+j}(M) \otimes H^k(M) \to H^j(M).$$

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