On Symplectic Cobordisms Between Contact Manifolds

\[(M_+, \xi_+)\]

\[(M_-, \xi_-)\]

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Slides available at:
http://www.math.hu-berlin.de/~wendl/publications.html
Prologue

The following famous quotation is due to George Orwell:

\[ \text{All animals are equal, but some animals are more equal than others.} \]

The following is not:

\[ \text{Most contact manifolds are non-fillable, but some are more non-fillable than others.} \]
Outline

• Part 1: On Symplectic Fillings

• Part 2: On Symplectic Cobordisms

• Part 3: A Hierarchy of Obstructions

• Part 4: Open Books and Fiber Sums

• Part 5: Non-Exact Cobordisms
  (or some low-tech proofs of results that used to seem hard)
Part 1
On Symplectic Fillings

Definitions

\((W, \omega)\) compact, symplectic, \(\partial W = M\).
Assume \(\eta\) is a Liouville vector field, i.e.

\[ \mathcal{L}_\eta \omega = \omega, \]

defined near \(\partial W\) and pointing transversely outward. Then

\[ \lambda := \iota_\eta \omega \]

satisfies \(d\lambda = \omega\) and is a positive contact form on \(M\), defining a contact structure \(\xi = \ker \lambda\).

\((W, \omega)\) is a **strong** (symplectic) filling of \((M, \xi)\).

\[(W, \omega)\]

\(((−\epsilon, 0] \times M, d(e^t\alpha))\)

\((W, \omega)\) is an **exact** filling of \((M, \xi)\) \iff \(\eta\) (or equivalently \(\lambda\)) exists globally.
Gromov ’85, Eliashberg ’89
$(M, \xi)$ overtwisted $\Rightarrow$ not fillable.

Proof requires technology:
e.g. holomorphic curves, Seiberg-Witten, Heegaard Floer . . .

A modern proof: overtwisted $\Rightarrow$
the ECH contact invariant vanishes.

Recall Embedded Contact Homology:

Assume $\text{dim } M = 3$ and choose:
- Contact form $\alpha$ for $\xi$
- Compatible $J$ on $\mathbb{R} \times M$
Choices \( \rightsquigarrow \)

- Chain complex \( C_\ast(M, \alpha) \) generated by sets of Reeb orbits
- Differential \( \partial : C_\ast(M, \alpha) \to C_\ast(M, \alpha) \) counting embedded \( J \)-holomorphic curves in \( \mathbb{R} \times M \).

\[
ECH_\ast(M, \alpha, J) := H_\ast(C_\ast(M, \alpha), \partial)
\]
matches the Seiberg-Witten Floer homology of \( M \) (Taubes '08).

**ECH contact invariant** := “homology class of the empty orbit set”

\[
c_\text{ech}(\xi) = [\emptyset] \in ECH_\ast(M, \alpha, J).
\]

**Taubes '08 + Kronheimer-Mrowka '97:**
\( c_\text{ech}(\xi) \) is an invariant of \((M, \xi)\), and is nonzero whenever \((M, \xi)\) is strongly fillable.
(\(M, \xi\)) overtwisted \(\Rightarrow\) contains a "Lutz tube" (Eliashberg classification '89)

\(\Rightarrow\) an orbit \(\gamma\) spanned by a unique embedded rigid \(J\)-holomorphic plane. Thus

\[ \partial(\gamma) = \emptyset, \]

so \(c_{ech}(\xi) = [\emptyset] = 0, \Rightarrow\) not fillable. \(\square\)

**Remark 1**
Same argument proves trivial contact homology: \(HC_\ast(M, \xi) = \{1\} \).

**Remark 2**
Conjecturally, \(c_{ech}(\xi)\) is equivalent to the Ozsváth-Szabó contact invariant in Heegaard Floer homology.
D. Gay ’06:
$(M, \xi)$ has Giroux torsion $\geq 1 \Rightarrow$ not fillable.

Recall:
$(M, \xi)$ has *Giroux torsion* $N$ if it contains $[0, 1] \times T^2 \ni (s, \phi, \theta)$ with contact structure
$$\xi_N := \ker [\cos(2\pi Ns) \, d\theta + \sin(2\pi Ns) \, d\phi].$$

**Proof by ECH:** count *holomorphic cylinders*
$$\Rightarrow \partial(\gamma_1 \gamma_2) = \emptyset \Rightarrow c_{\text{ech}}(\xi) = 0. \quad \square$$

(Corresponding Heegaard result by Ghiggini, Honda, Van Horn-Morris ’07.)


Part 2
On Symplectic Cobordisms

Definitions

$(W, \omega)$ compact, symplectic,

$$\partial W = M_+ \sqcup (-M_-),$$

with Liouville vector field $\eta$ near $\partial W$ pointing outward at $M_+$ and inward at $M_-$. 

Call this a **symplectic cobordism** from $(M_-, \xi_-)$ to $(M_+, \xi_+)$, and write

$$(M_-, \xi_-) \preccurlyeq (M_+, \xi_+).$$
If $\eta$ exists globally, call $(W, \omega)$ an **exact cobordism** and write

$$(M_-, \xi_-) \prec (M_+, \xi_+).$$

Observe $M_- \prec M_+$ implies $M_- \preccurlyeq M_+$.

Each is a **preorder** (reflexive and transitive) on the contact category.
Some facts about cobordisms

Abbreviate $M = (M, \xi)$. Let $M_{\text{ot}}$ denote anything overtwisted.

- $\emptyset \sim M \iff \text{fillable}; \emptyset \prec M \iff \text{exactly fillable}$
- No $M$ satisfies $M \prec \emptyset$. (Stokes theorem)
- All $M$ satisfy $M \preceq \emptyset$. (Etnyre-Honda ’02)
- If $M_- \preceq M_+$ and $M_-$ is fillable, then $M_+$ is also fillable. For example,
  \[ M \preceq M_{\text{ot}} \implies M \text{ not fillable}. \]
- $M_{\text{ot}} \prec M$ for all $M$. (Etnyre-Honda ’02)

Are overtwisted contact manifolds more non-fillable than some others?

Is there a non-fillable $M$ such that

\[ M \not\preceq M_{\text{ot}} \]

for all overtwisted $M_{\text{ot}}$?
Yes:

\[ M \not\approx M_{ot} \Rightarrow \text{by adapting a holomorphic disk argument due to Hofer, } M \text{ always has a contractible Reeb orbit.} \]

There are non-fillable examples without contractible orbits, e.g. \((T^3, \xi_N)\) for \(N \geq 2\) \(\Rightarrow\) Giroux torsion \(N - 1\).

We’ll show:
these \textit{do} admit \textbf{non-exact cobordisms} to some \(M_{ot}\) (a result of Gay ’06 for \(N \geq 3\)).

\textbf{Exercise for bored listeners:}
There are symplectic cobordisms from \((T^3, \xi_{std})\) to \((S^3, \xi_{std})\), but they are \textbf{never exact}.
Part 3
A Hierarchy of Obstructions

**Theorem** (joint with J. Latschev)
For closed contact manifolds \((M, \xi)\) in all dimensions, one can use Symplectic Field Theory to define the *algebraic torsion*

\[
AT(M, \xi) = \inf \left\{ k \geq 0 \mid [\hbar^k] = 0 \in H_{SFT}^*(M, \xi) \right\} 
\in \mathbb{N} \cup \{0, \infty\},
\]
which has the following properties:

1. \(AT(M, \xi) < \infty \Rightarrow \text{not strongly fillable.}\)
2. \(HC_*(M, \xi) = \{1\} \iff AT(M, \xi) = 0\)
3. positive Giroux torsion \(\Rightarrow AT(M, \xi) \leq 1.\)
4. For every integer \(k \geq 0\), there are examples \((M_k, \xi_k)\) with \(AT(M_k, \xi_k) = k\).
5. \((M_-, \xi_-) \prec (M_+, \xi_+) \Rightarrow AT(M_-, \xi_-) \leq AT(M_+, \xi_+).\)

Morally:
“Larger \(AT(M, \xi) \approx \text{closer to fillability.}”
Remark 1
As we’ll see, all examples I know for which $\text{AT}(M) < \infty$ satisfy:

1. ECH contact invariant $= 0$
2. $M \preccurlyeq M_{\text{ot}}$

Hence by Etnyre-Honda, they are (non-exactly!) cobordant to everything.

Remark 2
An analogue of $\text{AT}(M, \xi)$ can be defined via ECH. Heegaard???

The examples $(M_k, \xi_k)$
Part 4
Open Books and Fiber Sums

Initial Goal:
Find more general contact subdomains \((M_0, \xi_0)\) (possibly with boundary) such that
\[(M_0, \xi_0) \hookrightarrow (M, \xi) \Rightarrow c_{\text{ech}}(\xi) = 0.\]

Observation:
Informally, there is a correspondence
(Hofer-Wysocki-Zehnder, Abbas, W.)

pages of supporting open books
\[
\longleftrightarrow
\]
embedded \(J\)-holomorphic curves

\[
\pi : M \setminus B \rightarrow S^1
\]
Two operations on open books (and contact structures)

1. **Blow up a binding component** $\gamma \subset B$:
   Replace $\gamma$ with $\tilde{\gamma} := (\nu \gamma \setminus \gamma)/\mathbb{R}_+ \cong T^2$.
   $\sim$ natural basis $\{\lambda, \mu\} \in H_1(\tilde{\gamma})$.

2. **Binding sum** of $\gamma_1, \gamma_2 \subset B$:
   Blow up both and attach such that $\lambda \mapsto \lambda, \mu \mapsto -\mu$.

$\cong$ **contact fiber sum** along $\gamma_1, \gamma_2$
(Gromov, Geiges)

$\gamma_1 \cup \gamma_2$ replaced by one “interface” torus.
Definitions

*Blown up summed open book* $\equiv$ result of blowing up and/or summing some binding components of an open book.

$\sim$ compact mfd. $M$ (maybe with boundary), and fibration

$$\pi : M \setminus (B \cup I) \to S^1$$

Here:

- $B$ (the *"binding"*) $=$ a link
- $I$ (the *"interface"*) $=$ a disjoint union of 2-tori with homology bases $(\lambda, \pm \mu)$
- $\partial M = 2$-tori with homology bases $(\lambda, \mu)$

*Pages* $\equiv$ connected components of fibers. 
$\pi$ is *irreducible* $\Leftrightarrow$ fibers connected.

*Planar* $\equiv$ irreducible with genus 0 pages.
Any blown up summed open book decomposes into \textit{irreducible subdomains} 

\[ M = M_1 \cup \ldots \cup M_n \]

glued along interface tori.

\textbf{Definition}

The decomposition \textit{supports} a contact structure $\xi$ on $M$ if there is a Reeb vector field $X$ such that:

1. $X$ is positively \textit{transverse} to all pages
2. $X$ is positively \textit{tangent} to all boundaries of pages
3. \textit{Characteristic foliation} at $\mathcal{I} \cup \partial M$ is parallel to $\pm \mu$

\textbf{Proposition}

Unless $B \cup \mathcal{I} \cup \partial M = \emptyset$, a supported contact structure \textit{exists}.

(Otherwise $\pi : M \to S^1$ has closed fibers.)
Examples

Consider simple open books on the tight $S^3$ and $S^1 \times S^2$:

(1) Two copies of $S^3$ with disk pages binding sum $\sim$ tight $S^1 \times S^2$
(2) Two copies of tight $S^1 \times S^2$
Two binding sums $\sim (T^3, \xi_1)$

(3) Two copies of $S^1 \times S^2$
One binding sum $\sim$ overtwisted $S^1 \times S^2$

Definition
A blown up summed open book is *symmetric* if it has exactly two irreducible subdomains, all its pages are diffeomorphic, and it has no binding or boundary.

Examples
(1) and (2) are symmetric, (3) is not.
(4) Four copies of $S^1 \times S^2$
four binding sums in a ring $\sim (T^3, \xi_2)$

(5) One copy of $S^1 \times S^2$, sum one binding component to the other
$\sim$ Stein fillable torus bundle $T^3/\mathbb{Z}_2$

(sorry, I can't draw this)
(6) Three copies of $S^1 \times S^2$, two binding sums and two blow-ups $\sim ([0,3/2] \times T^2, \xi_1)$, i.e. 

*Giroux torsion domain*
$S^3$ summed to $S^1 \times S^2$, remaining binding blown up $\leadsto$ Lutz tube

**Definition**

For $k \geq 0$, a compact contact domain $(M_0, \xi_0)$ with supporting blown up summed open book is a *planar $k$-torsion domain* if:

1. It is **not symmetric**.
2. The **interior** contains a **planar irreducible subdomain**

$$M_0^P \subset \text{int } M_0,$$

the **planar piece**, whose pages have $k + 1$ **boundary components**. We call $M_0 \setminus M_0^P$ the **padding**.
A closed contact 3-manifold has *planar $k$-torsion* if it admits a contact embedding of a planar $k$-torsion domain.

**Some planar torsion domains of the form $S^1 \times \Sigma$**
**Theorem**

If \((M, \xi)\) has **planar** \(k\)-torsion then it is **not** strongly fillable. Moreover,

1. \(c_{\text{ech}}(\xi) = 0\) and \(\text{AT}(M, \xi) \leq k\)

2. **Overtwisted** \(\Leftrightarrow\) planar \(0\)-torsion

3. **Giroux torsion** \(\Rightarrow\) planar \(1\)-torsion

4. The examples \((M_k, \xi_k)\) for \(k \geq 2\) have planar \(k\)-torsion but no Giroux torsion.
Part 5
Non-Exact Cobordisms

Eliashberg '04 (*symplectic capping*): symplectically attaching 2-handles to binding  
\(\sim\) 0-surgery removes the binding

Gay-Stipsicz '09: doing this at some (not all!) binding components \(\sim\)  
symplectic cobordism between two open books

Blown up version  
can attach a round 1-handle  
\[ S^1 \times [0, 1] \times \mathbb{D} \]

to remove an interface torus and cap off pages.
Theorem

If \((M_-, \xi_-)\) has planar \(k\)-torsion for \(k \geq 1\), then \((M_-, \xi_-) \preccurlyeq (M_+, \xi_+)\) for some contact manifold \((M_+, \xi_+)\) with planar \((k-1)\)-torsion.

Moreover, this induces a \(U\)-equivariant map

\[
\text{ECH}_*(M_+, \xi_+) \to \text{ECH}_*(M_-, \xi_-)
\]

taking \(c_{\text{ech}}(\xi_+)\) to \(c_{\text{ech}}(\xi_-)\).

(Last part is known for Heegaard in simple open book case; J. Baldwin '09)
Corollary

\( M \) with \( k \)-torsion is cobordant to something overtwisted, and hence to everything.

\( \Rightarrow \) not fillable and \( c_{\text{ech}}(\xi) = 0. \)

Final Remark

Using such cobordisms, the proof that \( M_{\text{ot}} \) is not fillable can be reduced to the following:

**Lemma**

Suppose \((W, \omega)\) is a compact symplectic manifold with all boundary components either convex or Levi-flat, and it contains an embedded symplectic sphere of self-intersection 0. Then all boundary components of \( W \) are symplectic sphere-bundles.

**Proof** uses closed holomorphic curves; it’s still technology, but it’s simpler technology. Just read McDuff “Rational and Ruled...” 1990, and think about it.