What can have a 3-sphere as its boundary, and why should you ask Isaac Newton?

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Slides available at:

http://www.homepages.ucl.ac.uk/~ucahcwe/publications.html#talks
PART 1: Differential topology

The $n$-dimensional sphere

$$S^n := \left\{ x \in \mathbb{R}^{n+1} \mid x_1^2 + \ldots + x_{n+1}^2 = 1 \right\}$$

= boundary of the $(n + 1)$-dimensional ball

$$B^{n+1} := \left\{ x \in \mathbb{R}^{n+1} \mid x_1^2 + \ldots + x_{n+1}^2 \leq 1 \right\}.$$
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![Diagram of a sphere and a cylinder as examples of surfaces with \(\partial \Sigma = S^1\).]
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![Diagram showing examples of surfaces with \(S^1\) as boundary](image-url)
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![Diagram of surfaces](image)

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Definition

Suppose \(M \subset \mathbb{R}^N\) is a subset, \(U \subset M\) is open.

An \(n\)-dimensional coordinate chart on \(U\) is a set of functions \(x_1, \ldots, x_n : U \to \mathbb{R}\) such that the mapping

\[
(x_1, \ldots, x_n) : U \to \mathbb{R}^n
\]

is bijective onto some open subset of \(\mathbb{R}^n\).
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Two manifolds $M$ and $M'$ are \textit{diffeomorphic} ($M \cong M'$) if there exists a bijection

$$f : M \rightarrow M'$$

such that both $f$ and $f^{-1}$ are everywhere \textit{infinitely differentiable} when expressed in coordinate charts.
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\end{itemize}

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\textbf{Proposition}

If $M \cong M'$, then they have the same dimension, and $M$ compact $\iff$ $M'$ compact.
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- The universe? (dimension = 4? 10? 11?) (compact?)
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Answer: Almost any!

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M := \hat{M} \setminus B_\epsilon(p),
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where \(B_\epsilon(p) := \{x_1^2 + \cdots + x_{n+1}^2 \leq \epsilon\} \subset \mathcal{U}\).
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Conclusion: We asked the wrong question. The answer was too easy!
PART 2: Dynamics

Newton (18th century):

A system of particles moving with $n$ degrees of freedom is described by a path in $\mathbb{R}^n$,

$$q(t) := (q_1(t), \ldots, q_n(t)) \in \mathbb{R}^n.$$
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If the system is conservative, its forces are derived from a potential function \( V(q) \) by

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F(q) = -\nabla V(q).
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Then Newton’s second law gives

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a system of $n$ second-order ordinary differential equations (ODE). Its total energy

$$E = \sum_{j=1}^{n} \frac{1}{2} m_j \dot{q}_j^2 + V(q)$$

is conserved, i.e. $\frac{dE}{dt} = 0$. 
**Hamilton** (19th century):

Pretend $q_i$ and $p_j := m_j \dot{q}_j$ (momentum) are independent variables moving in the "phase space" $\mathbb{R}^{2n}$. 
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H : \mathbb{R}^{2n} \to \mathbb{R} : (q, p) \mapsto \sum_{j=1}^{n} \frac{p_j^2}{2m_j} + V(q),
\]

and Newton’s second-order system becomes Hamilton’s (first-order!) equations:

\[
\begin{align*}
\dot{q}_j &= \frac{\partial H}{\partial p_j}, \\
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Idea: To study motion of systems satisfying constraints, we can treat $(q, p)$ as local coordinates of a point moving in a manifold.
\[ \dot{q}_j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q_j} \]  \hspace{1cm} (*)

**Complication:** A system that satisfies (*) for one particular choice of coordinates might not satisfy it for all other choices.
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**Definition**

A \(2n\)-dimensional manifold \(M\) has a *symplectic structure* if it is covered by special coordinate charts of the form \((q_1, \ldots, q_n, p_1, \ldots, p_n)\) such that for any smooth function \(H : M \rightarrow \mathbb{R}\), all coordinate transformations preserve the form of Hamilton's equations \((*)\).
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**Exercise:** A transformation on \(\mathbb{R}^2\) preserves (*) \iff it is area and orientation preserving.
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\textbf{Simple examples}

- Symplectic: $\mathbb{R}^{2n}$, all orientable surfaces
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- \textbf{Symplectic:} $\mathbb{R}^{2n}$, all orientable surfaces
- \textbf{Not symplectic:} $S^{2n}$ for $n > 1$  
  (can prove using \textit{de Rham cohomology})
Assume $M$ is symplectic, $H : M \to \mathbb{R}$ a smooth function. Then any path $\gamma : \mathbb{R} \to M$ satisfying Hamilton’s equations “conserves energy”:

$$\frac{d}{dt} H(\gamma(t)) = 0,$$
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Any star-shaped hypersurface in $\mathbb{R}^{2n}$ is diffeomorphic to $S^{2n-1}$.
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A submanifold $N$ of a manifold $M$ is a subset $N \subset M$ such that the natural inclusion map $N \hookrightarrow M$ is infinitely differentiable.

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PART 3: Symplectic topology

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Suppose $M$ is a compact 4-manifold with an exact symplectic structure which, at its boundary, looks like a star-shaped hypersurface in $\mathbb{R}^4$. Then $M \cong B^4$. 
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Some preparation from complex analysis

A function \( f = u + iv : \mathbb{C} \to \mathbb{C} \) is analytic / holomorphic if it satisfies the Cauchy-Riemann equations:

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\begin{align*}
\partial_s u(s + it) &= \partial_t v(s + it), \\
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Equivalently: \( \partial_s f + i \partial_t f = 0 \). \( \text{(**)} \)

A map \( f : \mathbb{C} \to \mathbb{C}^n \) satisfying this equation is called a holomorphic curve in \( \mathbb{C}^n \).
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A 2n-dimensional manifold \( M \) has a complex structure if it is covered by special (complex) coordinate charts of the form \( (z_1, \ldots, z_n) : U \to \mathbb{C}^n \) such that all coordinate transformations preserve the form of the Cauchy-Riemann equation \( (**) \).

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Examples: \( \mathbb{C}^n, \text{SL}(n, \mathbb{C}), \mathbb{C} \cup \{\infty\} \cong S^2 \)
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The next best thing... 

An almost complex structure on $\mathbb{C}^n$ is a smooth function

$$J : \mathbb{C}^n \to \{\text{real-linear maps} \ \mathbb{C}^n \to \mathbb{C}^n\} \cong \mathbb{R}^{2n \times 2n}$$

such that for all $p \in \mathbb{C}^n$, $[J(p)]^2 = -1$. 
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A map $f : \mathbb{C} \to \mathbb{C}^n$ is then called a pseudo-holomorphic curve if it satisfies the nonlinear Cauchy-Riemann equation:

$$\partial_s f + J(f) \partial_t f = 0.$$

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This is a nonlinear first-order elliptic partial differential equation (PDE).
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Fundamental lemma:
Every symplectic manifold admits a special class of compatible almost complex structures.
A decomposition of the standard $B^4 \subset \mathbb{R}^4$

Identify $\mathbb{R}^4 = \mathbb{C}^2$ and define

$$J_0(p) := i \quad \text{for all } p \in \mathbb{R}^4.$$ 

We now see two obvious 2-dimensional families of pseudoholomorphic curves:

$$u_w : \mathbb{C} \to \mathbb{C}^2 : z \mapsto (z, w) \quad \text{for } w \in \mathbb{C},$$

$$v_w : \mathbb{C} \to \mathbb{C}^2 : z \mapsto (w, z) \quad \text{for } w \in \mathbb{C}.$$ 

They form two transverse foliations of $\mathbb{C}^2$: 

![Diagram of pseudoholomorphic curves](image-url)
Proof of the main theorem

Given $\partial M = \Sigma \subset \mathbb{R}^4$ star-shaped, construct a symplectic manifold $W$ by surgery:

(1) Remove from $\mathbb{R}^4 = \mathbb{C}^2$ the interior of $\Sigma$;
(2) Attach $M$ along its boundary to $\Sigma$. 

\begin{center}
\includegraphics[width=0.8\textwidth]{proof_of_main_theorem.png}
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1. Remove from $\mathbb{R}^4 = \mathbb{C}^2$ the interior of $\Sigma$;
2. Attach $M$ along its boundary to $\Sigma$.

Choose $J$ matching $J_0$ outside a large ball. Then for large $|w|$, the pseudoholomorphic curves $u_w$ and $v_w$ also exist in $W$. 
Let $\mathcal{M}_u$ and $\mathcal{M}_v$ denote the families of pseudoholomorphic curves in $W$ containing the curves $u_w$ and $v_w$ respectively.
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**Lemma 1** (smoothness):
One can choose $J$ such that $\mathcal{M}_u$ and $\mathcal{M}_v$ are each parametrized by smooth, oriented 2-dimensional manifolds, and within each family, any two distinct curves are disjoint. Moreover, every curve in $\mathcal{M}_u$ intersects every curve in $\mathcal{M}_v$ exactly once, transversely.
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Any bounded sequence of curves in $\mathcal{M}_u$ or $\mathcal{M}_v$ has a convergent subsequence.
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These lemmas concern general properties of solution spaces.

One can prove them without knowing how to solve the PDE, and without knowing what $M$ actually is!
Final step: “turn on the machine...”
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\[ \Sigma \approx S^3 \]
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$\Rightarrow W \cong \mathbb{C}^2$. \qed
That was nearly 30 years ago.

Here is a more recent but similar result...

**Theorem (W. 2010)**
The only exact symplectic fillings of a 3-dimensional torus

\[ \mathbb{T}^3 := S^1 \times S^1 \times S^1 \]

are star-shaped domains in the cotangent bundle of \( \mathbb{T}^2 \).

**Question:**
For a surface \( \Sigma \) of genus \( g \geq 2 \), does the unit cotangent bundle have more than one exact symplectic filling?

*No one has any idea.*