NON-EXACT SYMPLECTIC COBORDISMS BETWEEN CONTACT 3-MANIFOLDS

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Abstract. We show that the pre-order defined on the category of contact manifolds by arbitrary symplectic cobordisms is considerably less rigid than its counterparts for exact or Stein cobordisms: in particular, we exhibit large new classes of contact 3-manifolds which are symplectically cobordant to something overtwisted, or to the tight 3-sphere, or which admit symplectic caps containing symplectically embedded spheres with vanishing self-intersection. These constructions imply new and simplified proofs of several recent results involving fillability, planarity and non-separating contact type embeddings. The cobordisms are built from symplectic handles of the form $\Sigma \times \mathbb{D}$ and $\Sigma \times [-1,1] \times S^1$, which have symplectic cores and can be attached to contact 3-manifolds along sufficiently large neighborhoods of transverse links and pre-Lagrangian tori. We also sketch a construction of J-holomorphic foliations in these cobordisms and formulate a conjecture regarding maps induced on Embedded Contact Homology with twisted coefficients.

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1. Introduction

Many important notions and results in contact topology can be expressed in terms of symplectic cobordisms. For example, the existence of a symplectic cobordism between particular contact manifolds can be used to determine whether one of them is symplectically fillable, assuming the filling properties of the other are already understood. Moreover, cobordisms are intimately associated with various notions of surgery, e.g. Weinstein [Wei91] defined a notion of symplectic handle attachment in which handles with Lagrangian cores can be attached along Legendrian spheres in a contact manifold (M, ξ), giving a Stein cobordism (see [CE12]) to the contact manifold obtained from (M, ξ) by Legendrian surgery. In dimension three, Eliashberg [Eli04], Gay [Gay06] and Gay-Stipsicz [GS12] have defined various other types of surgeries on contact manifolds that yield symplectic cobordisms which are not Stein. In some cases, these cobordisms have also been shown to induce surprisingly well-behaved morphisms for certain contact invariants, e.g. in Heegaard Floer homology [Bal]. The cobordism of [Gay06] and its precursor in [Eli96] yielded breakthroughs in the study of symplectic fillings, resulting in the proof that contact manifolds with Giroux torsion are not strongly fillable. The latter result was recently reinterpreted by the author [Wen] in the wider context of blown up summed open book decompositions, leading to an infinite hierarchy of more general filling obstructions that can be detected via holomorphic curves.

The main purpose of the present article is to show that the various seemingly unrelated constructions of non-Stein cobordisms mentioned above are all special cases of a much more general phenomenon, which arises naturally in the setting of blown up summed open books and produces non-exact symplectic cobordisms in many situations where exact or Stein cobordisms are forbidden. The overall pattern seems to be that while exact and Stein cobordisms are rigidly constrained by a variety of symplectic obstructions (for instance in Symplectic Field Theory [LW11] or Heegaard Floer homology [Kar]), non-exact symplectic cobordisms are quite flexible: they tend to exist whenever there is no obvious reason why they should not.

Our constructions are based on a new notion of generalized symplectic handles, which have symplectic cores and co-cores and can be attached to contact 3-manifolds along "sufficiently wide" neighborhoods of transverse links and pre-Lagrangian tori. The notion of a "sufficiently wide" neighborhood here is somewhat subtle, and indicates that in contrast to Weinstein handle attachment and many other forms of surgery, our surgery is not a truly *local* operation—e.g. the surgery along a transverse knot requires a neighborhood of a size that is only guaranteed to exist in certain situations, notably when the knot is one of several binding components in a supporting open book.

We begin by stating in §1.1 the essential definitions and explaining some existence results for non-exact symplectic cobordisms that follow from the handle construction. As easy applications, these results imply new and substantially simplified proofs of several recent theorems of the author and collaborators on obstructions to symplectic fillings, contact type embeddings and embeddings of partially planar domains. The main results concerning the handle construction itself will be explained in §1.2, together with a simple application to Embedded Contact Homology and a conjectured generalization. In §2 we discuss further applications and examples, providing a unified framework for reproving several important previous results of Eliashberg, Gay, Etnyre and others involving the existence of symplectic cobordisms. The hard work is then undertaken in §3, of which the first several sections construct the symplectic handles described in §1.2, §3.7 completes the proofs of the results stated in the introduction, and §3.8 discusses the construction of a holomorphic foliation in the cobordisms, which we expect should have interesting applications in Embedded Contact Homology and/or Symplectic Field Theory.

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1.1. **Some background and sample results.** In topology, an oriented cobordism from one closed oriented manifold M_- to another M_+ is a compact oriented manifold W such that $\partial W = M_+ \sqcup (-M_-)$. If W has dimension $2n$ and also carries a symplectic structure ω, then it is natural to consider the case where (W, ω) is **symplectically convex** at M_+ and **concave** at M_-: this means there exists a vector field Y near ∂W which points transversely outward at M_+ and inward at M_-, and is a **Liouville** vector field, i.e. $\mathcal{L}_Y \omega = \omega$. In this case the 1-form $\lambda := \iota_Y \omega$ is a primitive of ω and its restriction to each boundary component $M_\pm$ is a (positive) **contact form**, meaning it satisfies $\lambda \wedge (d\lambda)^{n-1} > 0$. The induced **contact structure** on $M_\pm$ is the co-oriented[1] hyperplane field $\xi_\pm := \ker(\lambda|_{TM_\pm})$, and up to isotopy it depends only on the symplectic structure near the boundary. We thus call (W, ω) a **(strong) symplectic cobordism** from (M_-, ξ_-) to (M_+, ξ_+), and when such a cobordism exists, we say that (M_-, ξ_-) is **(strongly) symplectically cobordant** to (M_+, ξ_+). If λ also extends to a global primitive of ω, or equivalently, Y extends to a global Liouville field on W, then we call (W, ω) an **exact symplectic cobordism** from (M_-, ξ_-) to (M_+, ξ_+). Whenever (M_-, ξ_-) and (M_+, ξ_+) are both connected, we shall abbreviate the existence of a *connected* symplectic cobordism from (M_-, ξ_-) to (M_+, ξ_+) by writing

$$(M_-, \xi_-) \preccurlyeq (M_+, \xi_+)$$

for the general case, and

$$(M_-, \xi_-) \prec (M_+, \xi_+)$$

for the exact case.[2]

When $\dim W = 4$, it is also interesting to consider a much weaker notion: without assuming that ω is exact near ∂W, we call (W, ω) a **weak symplectic cobordism** from (M_-, ξ_-) to (M_+, ξ_+) if $\xi_\pm$ are any two positive co-oriented (and hence also oriented) contact structures such that $\omega|_{\xi_\pm} > 0$. We then say that ω **dominates** the contact structures on both boundary components. In order to distinguish strong symplectic cobordisms from this weaker notion, we will sometimes refer to convex/concave boundary components of strong cobordisms as *strongly* convex/concave.

[1]Though contact structures need not be co-orientable in general, all contact structures considered in this paper will be, and we shall regard the co-orientation always as an essential part of the data, though it will usually be suppressed in the notation.

[2]The reason to single out connected cobordisms is that technically, every closed contact 3-manifold (M, ξ) is symplectically cobordant to the standard contact 3-sphere (S^3, ξ_0), namely via the disjoint union of a symplectic cap for (M, ξ) with a symplectic filling of (S^3, ξ_0). We will see however that if the cobordism is required to be connected, then the question of when $(M, \xi) \preccurlyeq (S^3, \xi_0)$ becomes an interesting one, cf. Theorems 4, 4′ and 10.

It is a standard fact that strong or exact symplectic cobordisms can always be glued together along contactomorphic boundary components, thus the relations $\preccurlyeq$ and $\prec$ define preorders on the category of closed and connected contact manifolds, i.e. they are reflexive and transitive. They are neither symmetric nor antisymmetric, as is clear from some simple examples that we shall recall in a moment. Regarding the empty set as a trivial example of a contact manifold, we say that (M, ξ) is **weakly/strongly/exactly fillable** if there exists a weak/strong/exact symplectic cobordism from $\emptyset$ to (M, ξ). For example, the tight 3-sphere (S^3, ξ_0) is exactly fillable, as it is the convex boundary of the unit 4-ball with its standard symplectic structure. There are many known examples of contact 3-manifolds that are not fillable by these various definitions: the original such result, that the so-called *overtwisted* contact manifolds are not weakly fillable, was proved by Gromov [Gro85] and Eliashberg [Eli89]. In contrast, Etnyre and Honda [EH02] showed that every contact 3-manifold admits a **symplectic cap**, meaning it is strongly cobordant to $\emptyset$ (though never exactly, due to Stokes' theorem).

There is an obvious obstruction to the relation $(M_-, \xi_-) \preccurlyeq (M_+, \xi_+)$ whenever (M_-, ξ_-) is strongly fillable but (M_+, ξ_+) is not, e.g. (M_-, ξ_-) cannot be the tight 3-sphere if (M_+, ξ_+) is overtwisted. Put another way, symplectic cobordisms imply filling obstructions, as (M, ξ) cannot be fillable if it is cobordant to anything overtwisted. The following question may be viewed as a test case for the existence of subtler obstructions to symplectic cobordisms.

Question 1. *Is every contact 3-manifold (M, ξ) that is not strongly fillable also symplectically cobordant to some overtwisted contact manifold $(M_{\mathrm{OT}}, \xi_{\mathrm{OT}})$?*

This question was open when the first version of the present article appeared, but it has since been answered in the negative: by an argument of Hutchings [Hut], $(M, \xi) \preccurlyeq (M_{\mathrm{OT}}, \xi_{\mathrm{OT}})$ implies that the contact invariant in the Embedded Contact Homology of (M, ξ) must vanish, and therefore (using [CGH]), so does the Ozsváth-Szabó contact invariant. A negative example for Question 1 is therefore furnished by any nonfillable contact manifold with nonvanishing Ozsváth-Szabó invariant; the first such examples were found by Lisca and Stipsicz in [LS04].

The answer to the corresponding question for exact cobordisms is much less subtle: by an argument originally due to Hofer [Hof93], $(M, \xi) \prec (M_{\mathrm{OT}}, \xi_{\mathrm{OT}})$ implies that every Reeb vector field on (M, ξ) admits a contractible periodic orbit, yet there are simple examples of contact manifolds without contractible orbits that are known to be non-fillable, e.g. all of the tight 3-tori other than the standard one. More generally, it has recently become clear that overtwistedness is only the first level in an infinite hierarchy of filling obstructions called *planar k-torsion* for integers $k \ge 0$, cf. [Wen]. A contact manifold is overtwisted if and only if it has planar 0-torsion, and there are many examples which are tight or have no Giroux torsion but have planar k-torsion for some $k \in \mathbb{N}$, and are thus not strongly fillable. The aforementioned argument of Hofer then generalizes to define an algebraic filling obstruction [LW11] that lives in Symplectic Field Theory and sometimes also gives obstructions to exact cobordisms from k-torsion to $(k-1)$-torsion. Our first main result says that no such obstructions exist for non-exact cobordisms, thus giving a large class of contact 3-manifolds for which the answer to Question 1 is yes.

Theorem 1. *Every closed contact 3-manifold with planar torsion admits a (non-exact) symplectic cobordism to an overtwisted contact manifold.*

This of course yields a new and comparatively low-tech proof of the fact, proved first in [Wen], that planar torsion obstructs strong fillings. It also generalizes a result proved by

David Gay in [Gay06], that any contact manifold with Giroux torsion at least 2 is cobordant to something overtwisted; as shown in [Wen], positive Giroux torsion implies planar 1-torsion (cf. §2.3). By a result of Etnyre and Honda [EH02], every connected overtwisted contact manifold admits a connected Stein cobordism to any other connected contact 3-manifold, and Gay [Gay06] showed that the word "connected" can be removed from this statement at the cost of dropping the Stein condition. We thus have the following consequence:

Corollary 1. *Every closed connected contact 3-manifold with planar torsion admits a connected strong symplectic cobordism to every other closed contact 3-manifold.*

It should be emphasized that due to the obstructions mentioned above, Corollary 1 is not true for *exact* cobordisms, not even if the positive boundary is required to be connected. In fact, there is no known example of an exact cobordism from anything tight to anything overtwisted, and many examples that are tight but non-fillable (e.g. the 3-tori with positive Giroux torsion) certainly do not admit such cobordisms.

There is also a version of Theorem 1 that implies the more general obstruction to weak fillings proved in [NW11]. Recall (see Definitions 3.1 and 3.3) that for a given closed 2-form Ω on a contact 3-manifold (M,ξ), we say that (M,ξ) has Ω-*separating* planar torsion if it contains a planar torsion domain in which a certain set of embedded 2-tori T all satisfy

$$\int_T \Omega = 0.$$

If this is true for all closed 2-forms Ω, then (M,ξ) is said to have *fully separating* planar torsion.

Theorem 2. *Suppose (M,ξ) is a closed contact 3-manifold with Ω-separating planar torsion for some closed 2-form Ω on M with $\Omega|_\xi > 0$. Then there exists a weak symplectic cobordism (W,ω) from (M,ξ) to an overtwisted contact manifold, with $\omega|_{TM} = \Omega$.*

Using a Darboux-type normal form near the boundary, weak symplectic cobordisms can be glued together along contactomorphic boundary components of opposite sign whenever the restrictions of the symplectic forms on the boundaries match (see Lemma 3.14). Thus if (M,ξ) has Ω-separating planar k-torsion and admits a weak filling (W,ω) with $[\omega|_{TM}] = [\Omega] \in H^2_{\mathrm{dR}}(M)$, then Theorem 2 yields a weak filling of an overtwisted contact manifold, and hence a contradiction due to the well known theorem of Gromov [Gro85] and Eliashberg [Eli90]. We thus obtain a much simplified proof of the following result, which was proved in [NW11] by a direct holomorphic curve argument and also follows from a computation of the twisted ECH contact invariant in [Wen].

Corollary ([NW11])**.** *If (M,ξ) has Ω-separating planar torsion for some closed 2-form Ω on M, then it does not admit any weak filling (W,ω) with $[\omega|_{TM}] = [\Omega] \in H^2_{\mathrm{dR}}(M)$. In particular, if (M,ξ) has fully separating planar torsion then it is not weakly fillable.*

We now state some related results that also apply to fillable contact manifolds. The aforementioned existence result of [EH02] for symplectic caps was generalized independently by Eliashberg [Eli04] and Etnyre [Etn04a] to weak cobordisms: they showed namely that for any (M,ξ) with a closed 2-form Ω that dominates ξ, there is a symplectic cap (W,ω) with $\partial W = -M$ and $\omega|_{TM} = \Omega$. Our next result concerns a large class of contact manifolds for which this cap may be assumed to have a certain very restrictive property.

Theorem 3. *Suppose (M,ξ) is a contact 3-manifold containing an Ω-separating partially planar domain $M_0 \subset M$ (see Definition 3.1) for some closed 2-form Ω on M with $\Omega|_\xi > 0$. Then (M,ξ) admits a symplectic cap (W,ω) such that $\omega|_{TM} = \Omega$ and there exists a symplectically embedded 2-sphere $S \subset W$ with vanishing self-intersection number.*

As the work of McDuff [McD90] makes clear, symplectic manifolds that contain symplectic spheres of square 0 are quite special, and for instance any closed symplectic manifold obtained by gluing the cap from Theorem 3 to a filling of (M,ξ) must be rational or ruled. An easy adaptation of the main result in [ABW10] also provides the following consequence, which was proved using much harder punctured holomorphic curve arguments in [Wen, NW11]:

Corollary 2. *Suppose (M,ξ) contains an Ω-separating partially planar domain for some closed 2-form Ω on M. If (W,ω) is a closed symplectic 4-manifold and M admits an embedding $\iota : M \hookrightarrow W$ such that $\iota^*\omega|_\xi > 0$ and $[\iota^*\omega] = [\Omega] \in H^2_{\mathrm{dR}}(M)$, then $\iota(M)$ separates W.*

Since planar torsion domains are also partially planar domains, this implies that planar torsion is actually an obstruction to contact type embeddings into closed symplectic manifolds, not just symplectic fillings.

Some examples of contact manifolds admitting non-separating embeddings arise from special types of symplectic fillings: we shall say that (M,ξ) is (strongly or weakly) **co-fillable** if there is a connected (strong or weak) filling (W,ω) whose boundary is the disjoint union of (M,ξ) with an arbitrary non-empty contact manifold. Put another way, (M,ξ) admits a connected **semi-filling** with disconnected boundary. Given such a filling, one can always attach a symplectic 1-handle to connect distinct boundary components and then cap off the boundary to realize (M,ξ) as a non-separating contact hypersurface. Various examples of contact manifolds that are or are not co-fillable have been known for many years:

- The tight 3-sphere (S^3,ξ_0) is not weakly co-fillable, by arguments due to Gromov [Gro85], Eliashberg [Eli90] and McDuff [McD91]. Etnyre [Etn04b] extended this result to all planar contact manifolds.
- McDuff [McD91] showed that for any Riemann surface Σ of genus at least 2, the unit cotangent bundle $ST^*\Sigma$ with its canonical contact structure is strongly co-fillable. Further examples were found by Geiges [Gei95] and Mitsumatsu [Mit95].
- Giroux [Gir94] showed that every tight contact structure on T^3 is weakly co-fillable. However, none of them are strongly co-fillable, due to a result of the author [Wen10].

All of the negative results just mentioned can be viewed as special cases of Corollary 2, and so can the closely related result in [ABW10], that partially planar contact manifolds never admit non-separating contact type embeddings. Observe that any contact manifold cobordant to one for which Corollary 2 holds also cannot be co-fillable: in particular this shows that not every contact 3-manifold is cobordant to (S^3,ξ_0). We are thus led to an analogue of Question 1 that also applies to fillable contact manifolds:

Question 2. *Does every closed and connected contact 3-manifold (M,ξ) that is not strongly co-fillable satisfy $(M,\xi) \preccurlyeq (S^3,\xi_0)$?*

To the author's knowledge, this question is open. The answer is again clearly no for exact cobordisms, as a variation on Hofer's argument from [Hof93] also shows that (M,ξ) must always admit contractible Reeb orbits if $(M,\xi) \prec (S^3,\xi_0)$. The following result provides some evidence for a positive answer in the non-exact case, though it is not quite as general as one might have hoped. (See also Remark 1.1 below for a candidate counterexample.)

Theorem 4. *Suppose (M,ξ) is a connected contact 3-manifold containing a partially planar domain which either has more than one irreducible subdomain or has nonempty binding. Then $(M,\xi) \preccurlyeq (S^3,\xi_0)$, hence (M,ξ) admits a connected strong symplectic cobordism to every connected strongly fillable contact 3-manifold.*

The conditions of Theorem 4 hold in particular for all planar contact manifolds, and in fact a stronger version can be stated since the fully separating condition is always satisfied. We will show in §2.6 that this implies Etnyre's planarity obstruction from [Etn04b].

Theorem 4′. *Suppose (M,ξ) is a connected planar contact 3-manifold. Then there exists a compact connected 4-manifold W with $\partial W = S^3 \sqcup (-M)$, with the property that for every closed 2-form Ω on M with $\Omega|_\xi > 0$, W admits a symplectic structure ω such that $\omega|_{TM} = \Omega$ and (W,ω) is a weak symplectic cobordism from (M,ξ) to (S^3,ξ_0).*

Remark 1.1. Let us describe a contact manifold that could conceivably furnish a negative answer to Question 2. Consider the standard contact 3-torus (T^3,ξ_1) (the definition of ξ_1 is recalled in (2.1) below), and divide it by the $\mathbb{Z}_2$-action induced by the contact involution

$$T^3 \to T^3 : (\eta,\phi,\theta) \mapsto (\eta + 1/2, -\phi, -\theta).$$

The quotient $T^3/\mathbb{Z}_2$ then inherits a contact structure ξ, which is supported by a summed open book with empty binding, one interface torus $T := \{2\eta \in \mathbb{Z}\}$ and fibration

$$\pi([\eta,\phi,\theta]) = \begin{cases} \phi & \text{for } 0 < \eta < 1/2, \\ -\phi & \text{for } 1/2 < \eta < 1. \end{cases}$$

Since the pages are cylinders, $(T^3/\mathbb{Z}_2,\xi)$ is a partially planar domain, so Corollary 2 implies that it is not strongly co-fillable. (Note that $(T^3/\mathbb{Z}_2,\xi)$ *is* Stein fillable, as it can be constructed from the Stein fillable torus (T^3,ξ_1) by a sequence of contact (-1)-surgeries along Legendrian curves in the pre-Lagrangian fibers $\{\eta = \text{const}\}$.) Theorem 4 however does not apply, as there is only one irreducible subdomain and no binding. It is not clear whether $(T^3/\mathbb{Z}_2,\xi) \preccurlyeq (S^3,\xi_0)$.

1.2. **The main theorems on handle attaching.** The cobordisms of the previous section are constructed by repeated application of two handle attaching constructions that we shall now describe. The handles we will work with take the form

$$-\Sigma \times \mathbb{D} \quad \text{and} \quad -\Sigma \times [-1,1] \times S^1,$$

where in each case Σ is a compact oriented surface with boundary, appearing with reversed orientation because we think of it as a "symplectic cap" for the page of an open book decomposition. In the first case, we shall attach $\partial\Sigma \times \mathbb{D}$ to the neighborhood of a transverse link, and in the second case, $\partial\Sigma \times [-1,1] \times S^1$ is attached to the neighborhood of a disjoint union of pre-Lagrangian tori. It is important however to understand that these constructions are not truly *local*, as the attaching requires neighborhoods that are in some sense sufficiently large. This condition on the neighborhoods is most easily stated in the language of (possibly blown up and summed) open books—that is not necessarily the only natural setting in which these operations make sense, but it is the first that comes to mind.

We have derived considerable inspiration from the symplectic capping technique introduced by Eliashberg in [Eli04]. The goal of Eliashberg's construction was somewhat different, namely to embed any weak symplectic filling into a closed symplectic manifold, but it can also be used to construct symplectic cobordisms between contact manifolds with supporting open books that are related to each other by capping off binding components. Indeed, Eliashberg's capping construction works as follows:

(1) Given (M,ξ) with a supporting open book $\pi : M \setminus B \to S^1$, attach 2-handles to $[0,1]\times M$ at $\{1\}\times M$ along each component of B via the page framing. This transforms M by a 0-surgery along each binding component, producing a new 3-manifold M' with a fibration $M' \to S^1$ whose fibers are the closed surfaces obtained by capping the original pages with disks. Any symplectic structure on $[0,1] \times M$ dominating ξ can then be extended over the handles so that the fibers of $M' \to S^1$ are symplectic.
(2) Cap off the boundary of the cobordism above by presenting M' as the boundary of a Lefschetz fibration over the disk with closed fibers.

The first step can be generalized by observing that if we choose to attach 2-handles along some but *not all* components of the binding, then the new manifold M' inherits an open book decomposition

$$\pi' : M' \setminus B' \to S^1$$

obtained from π by *capping off* the corresponding boundary components of the pages (cf. [Bal]), and we will show that the symplectic structure can always be arranged so as to produce a weak symplectic cobordism from (M,ξ) to (M',ξ'), where ξ' is supported by π'. Under some additional topological assumptions one can actually arrange the weak cobordism to be strong; this variation on Eliashberg's construction has already been worked out in detail by Gay and Stipsicz [GS12]. To generalize further, one can also imagine replacing the usual 2-handle $\mathbb{D}\times\mathbb{D}$ by $\Sigma\times\mathbb{D}$ for any compact orientable surface Σ. We shall carry out this generalization below, though the reader may prefer to pretend $\Sigma = \mathbb{D}$ on first reading, and this suffices for most of the applications we will discuss.

The key to our construction will be to combine the above brand of handle attachment with a "blown up" version, in which a round handle is attached to (M,ξ) along a 2-torus that can be thought of as a blown up binding circle. This is most naturally described in the language of *blown up summed open books*, a generalization of open book decompositions that was introduced in [Wen] and will be reviewed in more detail in §3.1. Rougly speaking, a blown up summed open book on a compact 3-manifold M, possibly with boundary, defines a fibration

$$\pi : M \setminus (B \cup \mathcal{I}) \to S^1,$$

where the *binding* $B \subset M \setminus \partial M$ is an oriented link and the *interface* $\mathcal{I} \subset M \setminus \partial M$ is a set of disjoint 2-tori, and the connected components of the fibers, which intersect ∂M transversely, are called *pages*. An ordinary open book is the special case where $\mathcal{I} = \partial M = \emptyset$, and in general we allow any or all of B, $\mathcal{I}$ and ∂M to be empty, so there may be closed pages. As with ordinary open books there is a natural notion of contact structures being *supported* by a blown up summed open book, in which case binding components become positively transverse links and interface and boundary components become pre-Lagrangian tori. Such a contact structure exists and is unique up to deformation unless the pages are closed.

Suppose (M,ξ) is a closed contact 3-manifold containing a compact 3-dimensional submanifold $M_0 \subset M$, possibly with boundary, which carries a blown up summed open book $\boldsymbol{\pi}$ that supports $\xi|_{M_0}$ and has nonempty binding. Pick a set of binding components

$$B_0 = \gamma_1 \cup \ldots \cup \gamma_N \subset B,$$

each of which comes with a natural framing determined by the pages adjacent to γ, called the **page framing**. For each $\gamma_j \subset B_0$, we identify a tubular neighborhood $\mathcal{N}(\gamma_j) \subset M$ of γ_j

with the oriented solid torus $S^1 \times \mathbb{D}$ via this framing so that $\gamma_j = S^1 \times \{0\}$ with the correct orientation and the fibration π takes the form

$$\pi(\theta, \rho, \phi) = \phi$$

on $\mathcal{N}(\gamma_j) \setminus \gamma_j$, where (ρ, ϕ) denote polar coordinates on the disk, normalized so that $\phi \in S^1 = \mathbb{R}/\mathbb{Z}$. Assign to $\partial\mathcal{N}(\gamma_j)$ its natural orientation as the boundary of $\mathcal{N}(\gamma_j)$ and denote by

$$\{\mu_j, \lambda_j\} \subset H_1(\partial\mathcal{N}(\gamma_j))$$

the distinguished positively oriented homology basis for which μ_j is a meridian and λ_j is the longitude determined by the page framing. Denote $\mathcal{N}(B_0) = \mathcal{N}(\gamma_1) \cup \ldots \cup \mathcal{N}(\gamma_N)$.

Now pick a compact, connected and oriented surface Σ with N boundary components

$$\partial\Sigma = \partial_1\Sigma \cup \ldots \cup \partial_N\Sigma$$

and choose an orientation preserving diffeomorphism of each $\partial_j\Sigma$ to S^1, thus defining a coordinate $s \in S^1$ for $\partial_j\Sigma$. Using this, we define new compact oriented manifolds

$$M' = (M \setminus \mathcal{N}(B_0)) \cup (-\Sigma \times S^1),$$
$$M_0' = (M_0 \setminus \mathcal{N}(B_0)) \cup (-\Sigma \times S^1)$$

by gluing in $\Sigma \times S^1$ via orientation reversing diffeomorphisms $\partial\Sigma_j \times S^1 \to \partial\mathcal{N}(\gamma_j)$ that take the form

$$(s,t) \mapsto (s, 1, t)$$

in the chosen coordinates. On the level of homology, the map $\partial\Sigma \times S^1 \to \partial\mathcal{N}(B_0)$ identifies $[\partial_j\Sigma \times \{*\}]$ with λ_j and $[\{z\} \times S^1]$ for $z \in \partial_j\Sigma$ with μ_j.

Remark 1.2. In the special case $\Sigma = \mathbb{D}$, the operation just defined is simply a Dehn surgery along a binding component $\gamma \subset B$ with framing 0 relative to the page framing.

The fibration $\pi : M \setminus (B \cup \mathcal{I}) \to S^1$ extends smoothly over $\Sigma \times S^1$ as the projection to the second factor, thus M_0' inherits from $\boldsymbol{\pi}$ a natural blown up summed open book $\boldsymbol{\pi}'$, with binding $B \setminus B_0$, interface $\mathcal{I}$ and pages that are obtained from the pages of $\boldsymbol{\pi}$ by attaching $-\Sigma$, gluing $\partial_j\Sigma$ to the boundary component adjacent to γ_j. We say that $\boldsymbol{\pi}'$ is obtained from $\boldsymbol{\pi}$ by **Σ-capping surgery along** B_0. If $\boldsymbol{\pi}'$ does not have closed pages, then it supports a contact structure ξ' on M_0' which can be assumed to match ξ outside the region of surgery, and thus extends to M'.

The Σ-capping surgery can also be defined by attaching a generalized version of a 4-dimensional 2-handle: define

$$\mathcal{H}_\Sigma = -\Sigma \times \mathbb{D},$$

with boundary

$$\partial\mathcal{H}_\Sigma = -\partial^-\mathcal{H}_\Sigma \cup \partial^+\mathcal{H}_\Sigma := -(\partial\Sigma \times \mathbb{D}) \cup (-\Sigma \times S^1).$$

The above identifications of the neighborhoods $\mathcal{N}(\gamma_j)$ with $S^1 \times \mathbb{D}$ yield an identification of $\mathcal{N}(B_0)$ with $\partial^-\mathcal{H}_\Sigma = \partial\Sigma \times \mathbb{D}$, which we use to attach $\mathcal{H}_\Sigma$ to the trivial cobordism $[0,1] \times M$ by gluing $\partial^-\mathcal{H}_\Sigma$ to $\mathcal{N}(B_0) \subset \{1\} \times M$, defining

$$W = ([0,1] \times M) \cup_{\mathcal{N}(B_0)} \mathcal{H}_\Sigma, \tag{1.1}$$

which after smoothing the corners has boundary

$$\partial W = M' \sqcup (-M).$$

We will refer to the oriented submanifolds

$$\mathcal{K}_\Sigma := ([0,1] \times B_0) \cup_{B_0} (-\Sigma \times \{0\}) \subset W$$

and

$$\mathcal{K}_\Sigma' := \{p\} \times \mathbb{D} \subset W$$

for an arbitrary interior point $p \in \Sigma$ as the **core** and **co-core** respectively. Note that $\partial\mathcal{K}_\Sigma = -B_0 \subset M$, $\partial\mathcal{K}_\Sigma' \subset M'$ and $\mathcal{K}_\Sigma \bullet \mathcal{K}_\Sigma' = 1$, where $\bullet$ denotes the algebraic count of intersections.

The following generalizes results in [Eli04] and [GS12].

Theorem 5. *Suppose ω is a symplectic form on $[0,1] \times M$ with $\omega|_\xi > 0$, and let W denote the handle cobordism defined in* (1.1)*, after smoothing corners. Then after a symplectic deformation of ω away from $\{0\} \times M$, ω can be extended symplectically over W so that it is positive on $\mathcal{K}_\Sigma$, $\mathcal{K}_\Sigma'$ and the pages of $\boldsymbol{\pi}'$. Moreover, if the latter pages are not closed, then ω also dominates a supported contact structure ξ' on M', thus defining a weak symplectic cobordism from (M, ξ) to (M', ξ').*

We will refer to the cobordism (W, ω) of Theorem 5 henceforward as a **Σ-capping cobordism**. In general it is a weak cobordism, but under certain conditions that depend only on the topology of the setup, it can also be made strong. Recall the standard fact, observed originally by Eliashberg [Eli91, Proposition 3.1] (see also [Eli04, Prop. 4.1]), that whenever (W, ω) has a boundary component M on which ω dominates a positive contact structure ξ and is exact, ω can be deformed in a collar neighborhood to make M strongly convex, with ξ as the induced contact structure. In §3.6 we will use routine Mayer-Vietoris arguments to characterize the situations in which this trick can be applied to the above construction.

Theorem 5′. *The symplectic cobordism (W, ω) constructed by Theorem 5 can be arranged so that the following holds. Choose a real 1-cycle h in $M \setminus \mathcal{N}(B_0)$ such that $[h] \in H_1(M; \mathbb{R})$ is Poincaré dual to the restriction of ω to $\{0\} \times M$. Then there is a number $c > 0$ such that*

$$\mathrm{PD}([\omega]) = [0,1] \times [h] + c\,[\mathcal{K}_\Sigma'] \in H_2(W, \partial W; \mathbb{R}),$$

where $\mathrm{PD} : H^2_{\mathrm{dR}}(W) \to H_2(W, \partial W; \mathbb{R})$ denotes the Poincaré-Lefschetz duality isomorphism. In particular, if $\{0\} \times M \subset (W, \omega)$ is (strongly) concave then the following conditions are equivalent:

(i) *ω is exact.*
(ii) $[\mathcal{K}_\Sigma'] = 0 \in H_2(W, \partial W; \mathbb{R})$.
(iii) $[\gamma_1] + \ldots + [\gamma_N]$ *is not torsion in* $H_1(M)$.

Further, assuming that $\{0\} \times M$ is concave, the following conditions are also equivalent:

(i) *(W, ω) can be arranged to be a strong symplectic cobordism from (M, ξ) to (M', ξ').*
(ii) $[\partial\mathcal{K}_\Sigma'] = 0 \in H_1(M'; \mathbb{R})$.
(iii) *$\lambda_1 + \ldots + \lambda_N$ is not torsion in $H_1(M \setminus B_0)$, where λ_j denote the longitudes on $\partial\mathcal{N}(\gamma_j)$ determined by the page framing.*

It should be emphasized that the above theorem assumes Σ is *connected*. The case where Σ is disconnected is equivalent to performing multiple surgery operations in succession, but the statement of Theorem 5′ would then become more complicated.

Remark 1.3. For the case $\Sigma = \mathbb{D}$, if $\gamma \subset B$ denotes the binding component where 0-surgery is performed, then Theorem 5′ means that ω will be exact on W if and only if γ is not torsion in $H_1(M)$, and (W, ω) can be made into a strong cobordism if and only if γ has no nullhomologous cover whose page framing matches its Seifert framing. An equivalent condition is assumed in [GS12], which only constructs strong cobordisms.

Remark 1.4. Though ω in the above construction is sometimes an exact symplectic form, (W,ω) is *never* an exact cobordism, i.e. it does not admit a global primitive that restricts to suitable contact forms on both boundary components. This follows immediately from the observation that the core $\mathcal{K}_\Sigma \subset W$ is a symplectic submanifold whose oriented boundary is a *negatively transverse* link in (M,ξ), hence if $\omega = d\lambda$ and $\lambda|_{TM}$ defines a contact form on (M,ξ) with the proper co-orientation, then

$$0 < \int_{\mathcal{K}_\Sigma} \omega = \int_{\partial \mathcal{K}_\Sigma} \lambda < 0,$$

a contradiction. A similar remark applies to the round handle cobordism considered in Theorems 6 and 6′ below. The non-exactness of (W,ω) is important because there are examples in which it is known that no exact cobordism from (M,ξ) to (M',ξ') exists (see §2.4).

To describe the blown up version of these results, we continue with the same setup as above and choose a set of interface tori,

$$\mathcal{I}_0 = T_1 \cup \ldots \cup T_N \subset \mathcal{I},$$

together with an orientation for each $T_j \subset \mathcal{I}_0$. There is then a distinguished positively oriented homology basis

$$\{\mu_j, \lambda_j\} \subset H_1(T_j),$$

where λ_j is represented by some oriented boundary component of a page adjacent to T_j, and μ_j is represented by a closed leaf of the characteristic foliation defined on T_j by ξ. Choose tubular neighborhoods $\mathcal{N}(T_j) \subset M$ of T_j and identify them with $S^1 \times [-1,1] \times S^1$ to define positively oriented coordinates (θ,ρ,ϕ) in which $\lambda_j = [S^1 \times \{*\}]$ and $\mu_j = [\{*\} \times S^1]$. We may then assume that for every $\theta_0 \in S^1$ the loop $\{(\theta_0, 0)\} \times S^1$ is Legendrian, and the fibration π takes the form

$$\pi(\theta,\rho,\phi) = \begin{cases} \phi & \text{for } \rho > 0, \\ -\phi & \text{for } \rho < 0. \end{cases}$$

Denote the two oriented boundary components of $\mathcal{N}(T_j)$ by

$$\partial \mathcal{N}(T_j) = \partial_+ \mathcal{N}(T_j) \sqcup \partial_- \mathcal{N}(T_j),$$

where we define the oriented tori $\partial_\pm \mathcal{N}(T_j) = \pm(S^1 \times \{\pm 1\} \times S^1)$ with corresponding homology bases $\{\mu_j^\pm, \lambda_j^\pm\} \subset H_1(\partial_\pm \mathcal{N}(T_j))$ such that

$$\lambda_j^\pm := \lambda_j \in H_1(\mathcal{N}(T_j)) \quad \text{and} \quad \mu_j^\pm := \pm \mu_j \in H_1(\mathcal{N}(T_j)).$$

Denote the union of all the neighborhoods $\mathcal{N}(T_j)$ by $\mathcal{N}(\mathcal{I}_0)$. Then writing two identical copies of Σ as $\Sigma_\pm$ and choosing a positively oriented coordinate $s \in S^1$ for each boundary component $\partial_j \Sigma_\pm$, we construct new compact oriented manifolds

$$M' = (M \setminus \mathcal{N}(\mathcal{I}_0)) \cup (-\Sigma_+ \times S^1) \cup (-\Sigma_- \times S^1),$$
$$M_0' = (M_0 \setminus \mathcal{N}(\mathcal{I}_0)) \cup (-\Sigma_+ \times S^1) \cup (-\Sigma_- \times S^1)$$

from M and M_0 respectively by gluing along orientation reversing diffeomorphisms $\partial_j \Sigma_\pm \times S^1 \to \partial_\pm \mathcal{N}(T_j)$ that take the form

$$(s,t) \mapsto (s, \pm 1, \pm t)$$

in the chosen coordinates. Thus in homology, $[\partial_j \Sigma_\pm \times \{*\}] \in H_1(\partial_j \Sigma_\pm \times S^1)$ is identified with $\lambda_j^\pm$ and $[\{*\} \times S^1] \in H_1(\partial_j \Sigma_\pm \times S^1)$ with $\mu_j^\pm$.

Once again the fibration $\pi : M \setminus (B \cup \mathcal{I}) \to S^1$ extends smoothly over the glued in region $(\Sigma_+ \sqcup \Sigma_-) \times S^1$ as the projection to S^1, so M_0' inherits from $\boldsymbol{\pi}$ a natural blown up summed open book $\boldsymbol{\pi}'$, with interface $\mathcal{I} \setminus \mathcal{I}_0$, binding B and fibers that are obtained from the fibers of $\boldsymbol{\pi}$ by attaching $-(\Sigma_+ \sqcup \Sigma_-)$ along the boundary components adjacent to $\mathcal{I}_0$. We say that $\boldsymbol{\pi}'$ is obtained by **Σ-decoupling surgery along $\mathcal{I}_0$**.

Remark 1.5. The choice of the term *decoupling* is easiest to justify in the special case $\Sigma = \mathbb{D}$: then the surgery cuts open M along T and glues in two solid tori that cap off the corresponding boundary components of the pages.

Even if M is connected, M' may in general be disconnected, and there is a (possibly empty) component

$$M'_{\text{flat}} \subset M',$$

defined as the union of all the closed pages of $\boldsymbol{\pi}'$. Denote $M'_{\text{convex}} := M' \setminus M'_{\text{flat}}$, so that

$$M' = M'_{\text{convex}} \sqcup M'_{\text{flat}}.$$

On M'_{convex} there is a contact structure ξ' which matches ξ away from the region of surgery and is supported by $\boldsymbol{\pi}'$ in $M'_{\text{convex}} \cap M_0'$.

The above surgery corresponds topologically to the attachment of a **round handle**: denote the annulus by

$$\mathbb{A} = [-1,1] \times S^1$$

and define

$$\widehat{\mathcal{H}}_\Sigma = -\Sigma \times \mathbb{A},$$

with boundary

$$\partial \widehat{\mathcal{H}}_\Sigma = -\partial^- \widehat{\mathcal{H}}_\Sigma \cup \partial^+ \widehat{\mathcal{H}}_\Sigma := -(\partial\Sigma \times \mathbb{A}) \cup (-\Sigma \times \partial \mathbb{A}),$$

where we identify the two connected components of $\partial^+ \widehat{\mathcal{H}}_\Sigma = -\Sigma \times \{-1,1\} \times S^1$ with $-\Sigma_\pm \times S^1$ via the orientation preserving maps

$$-\Sigma_\pm \times S^1 \to -\Sigma \times \{\pm 1\} \times S^1 : (p,\phi) \mapsto (p, \pm 1, \pm\phi). \tag{1.2}$$

Using the identifications of the neighborhoods $\mathcal{N}(T_j)$ with $S^1 \times [-1,1] \times S^1$ chosen above, we can identify

$$\partial^- \widehat{\mathcal{H}}_\Sigma = \partial\Sigma \times \mathbb{A} = \bigsqcup_{j=1}^N \partial_j \Sigma \times [-1,1] \times S^1$$

with $\mathcal{N}(\mathcal{I}_0)$ and use this to attach $\widehat{\mathcal{H}}_\Sigma$ to $[0,1] \times M$ by gluing $\partial^- \widehat{\mathcal{H}}_\Sigma$ to $\mathcal{N}(\mathcal{I}_0) \subset \{1\} \times M$, defining an oriented cobordism

$$W = ([0,1] \times M) \cup_{\mathcal{N}(\mathcal{I}_0)} \widehat{\mathcal{H}}_\Sigma \tag{1.3}$$

with boundary $\partial W = M' \sqcup (-M)$. Use the coordinates $(\theta,\rho,\phi) \in S^1 \times [-1,1] \times S^1$ on each $\mathcal{N}(T_j) \subset \mathcal{N}(\mathcal{I}_0)$ to define an oriented link $\widehat{B}_0$ as the union of all the loops

$$S^1 \times \{(0,0)\} \subset T_j \subset \mathcal{I}_0.$$

Then the core and co-core respectively can be defined as oriented submanifolds by

$$\widehat{\mathcal{K}}_\Sigma := ([0,1] \times \widehat{B}_0) \cup_{\widehat{B}_0} (-\Sigma \times \{(0,0)\}) \subset W$$

and

$$\widehat{\mathcal{K}}'_\Sigma := \{p\} \times \mathbb{A} \subset W$$

for an arbitrary interior point $p \in \Sigma$. We have $\partial\widehat{\mathcal{K}}_\Sigma = -\widehat{B}_0 \subset M$, $\partial\widehat{\mathcal{K}}'_\Sigma \subset M'$ and $\widehat{\mathcal{K}}_\Sigma \bullet \widehat{\mathcal{K}}'_\Sigma = 1$.

Theorem 6. *Suppose ω is a symplectic form on $[0,1] \times M$ which satisfies $\omega|_\xi > 0$ and*

$$\sum_{j=1}^N \int_{T_j} \omega = 0, \tag{1.4}$$

and W denotes the round handle cobordism of (1.3)*. Then after a symplectic deformation away from $\{0\} \times M$, ω can be extended symplectically over W so that it is positive on $\widehat{\mathcal{K}}_\Sigma$, $\widehat{\mathcal{K}}'_\Sigma$ and the pages of $\boldsymbol{\pi}'$, and ω dominates a supported contact structure ξ' on M'_{convex}. In particular, after capping M'_{flat} by attaching a Lefschetz fibration over the disk as in* [Eli04]*, this defines a weak symplectic cobordism from (M, ξ) to (M'_{convex}, ξ').*

We will refer to (W, ω) in this construction from now on as a **Σ-decoupling cobordism**.

Remark 1.6. The homological condition (1.4) is clearly not removable since the 2-cycles $\sum_{j=1}^N [\partial_\pm \mathcal{N}(T_j)]$ both become nullhomologous in M'. Note that here the chosen orientations of the tori T_j play a role, i.e. they cannot in general be chosen arbitrarily unless ω is exact. No such issue arose in Theorem 5 because ω is always exact on a neighborhood of a binding circle. This is the reason why the "Ω-separating" condition is needed for many of the results in §1.1, and there are easy examples to show that those theorems are not true without it (cf. Remark 2.3).

For the analogue of Theorem 5′ in this context, we shall restrict for simplicity to the case where $\int_{T_j} \omega$ vanishes for every $T_j \subset \mathcal{I}_0$. Note that in this case, the Poincaré dual of $\omega|_{TM}$ can be represented by a real 1-cycle in $M \setminus \mathcal{N}(\mathcal{I}_0)$.

Theorem 6′. *If $\int_{T_j} \omega = 0$ for each $T_1, \ldots, T_N \subset \mathcal{I}_0$, then the symplectic cobordism (W, ω) constructed by Theorem 6 can be arranged so that the following holds. Choose a real 1-cycle h in $M \setminus \mathcal{N}(\mathcal{I}_0)$ with $[h] \in H_1(M; \mathbb{R})$ Poincaré dual to the restriction of ω to $\{0\} \times M$. Then there is a number $c > 0$ with*

$$\mathrm{PD}([\omega]) = [0,1] \times [h] + c\,[\widehat{\mathcal{K}}'_\Sigma] \in H_2(W, \partial W; \mathbb{R}).$$

In particular, if $\{0\} \times M \subset (W, \omega)$ is (strongly) concave then the following conditions are equivalent:

(i) *ω is exact.*
(ii) $[\widehat{\mathcal{K}}'_\Sigma] = 0 \in H_2(W, \partial W; \mathbb{R})$.
(iii) *There are no integers $k, m_1, \ldots, m_N \in \mathbb{Z}$ with $k > 0$ and $\sum_{j=1}^N m_j = 0$ such that the homology class*

$$k(\lambda_1 + \ldots + \lambda_N) + \sum_{j=1}^N m_j \mu_j \in H_1(\mathcal{I}_0)$$

is trivial in $H_1(M)$.

Further, if $\{0\} \times M$ is concave and $M'_{flat} = \emptyset$, the following conditions are also equivalent:

(i) *(W, ω) can be arranged to be a strong symplectic cobordism from (M, ξ) to (M', ξ').*
(ii) $[\partial\widehat{\mathcal{K}}'_\Sigma] = 0 \in H_1(M'; \mathbb{R})$.

(iii) *There are no integers $k_\pm, m_1^\pm, \ldots, m_N^\pm \in \mathbb{Z}$ with $k_- + k_+ > 0$ and $\sum_j m_j^+ = \sum_j m_j^- = 0$ such that*

$$k_+ \sum_{j=1}^N \lambda_j^+ + k_- \sum_{j=1}^N \lambda_j^- + \sum_{j=1}^N m_j^+ \mu_j^+ + \sum_{j=1}^N m_j^- \mu_j^- = 0 \in H_1(M \setminus \mathcal{I}_0).$$

Finally, if M'_{flat} and M'_{convex} are both nonempty, assume the labels are chosen so that $\Sigma_+ \times S^1 \subset M'_{convex}$ and $\Sigma_- \times S^1 \subset M'_{flat}$, and consider the weak cobordism

$$(\overline{W}, \omega) = (W, \omega) \cup_{M'_{flat}} (X, \omega_X)$$

from (M, ξ) to (M'_{convex}, ξ') obtained by capping off M'_{flat} with a symplectic Lefschetz fibration $X \to \mathbb{D}$ as in [Eli04]*. The following conditions are then equivalent:*

(i) *(W, ω) can be arranged to be a strong symplectic cobordism from (M, ξ) to (M'_{convex}, ξ').*
(ii) $[\partial\widehat{\mathcal{K}}'_\Sigma \cap M'_{convex}] = 0 \in H_1(M'_{convex}; \mathbb{R})$.
(iii) *There are no integers $k, m_1, \ldots, m_N \in \mathbb{Z}$ with $k > 0$ and $\sum_j m_j = 0$ such that the homology class*

$$k \sum_{j=1}^N \lambda_j^+ + \sum_{j=1}^N m_j \mu_j^+$$

is trivial in $H_1(M \setminus \mathcal{I}_0)$.

We now discuss some applications of the capping and decoupling cobordisms to Embedded Contact Homology (cf. [Hut10]). Recall that for a closed contact 3-manifold (M, ξ) and homology class $h \in H_1(M)$, $\mathrm{ECH}_*(M, \xi; h)$ is the homology of a chain complex generated by sets of Reeb orbits with multiplicities whose homology classes add up to h, with a differential counting embedded index 1 holomorphic curves with positive and negative cylindrical ends in the symplectization $\mathbb{R} \times M$. Similarly, counting embedded index 2 holomorphic curves through a generic point in M yields the so-called U-map,

$$U : \mathrm{ECH}_*(M, \xi; h) \to \mathrm{ECH}_{*-2}(M, \xi; h).$$

The **ECH contact invariant**

$$c(M, \xi) \in \mathrm{ECH}_*(M, \xi; 0)$$

is the homology class represented by the "empty orbit set". It is equivalent via an isomorphism of Taubes [Tau10] to a corresponding invariant in Seiberg-Witten theory, and also to the Ozsváth-Szabó contact invariant [OS05] by recent work of Colin-Ghiggini-Honda [CGH] and independently Kutluhan-Lee-Taubes [KLT]. Like those invariants, its vanishing gives an obstruction to strong symplectic fillings, and a version with twisted coefficients also obstructs weak fillings.

Remark 1.7. Technically the definitions of $\mathrm{ECH}_*(M, \xi; h)$ and $c(M, \xi)$ depend not just on ξ but also on a choice of contact form and almost complex structure. However, Taubes' isomorphism to Seiberg-Witten Floer homology implies that they are actually independent of these choices, thus we are safe in writing $\mathrm{ECH}_*(M, \xi; h)$ without explicitly mentioning the extra data.

An argument due to Eliashberg[3] shows that $c(M,\xi)=0$ whenever (M,ξ) is overtwisted, and a much more general computation in [Wen] established the same result whenever (M,ξ) has planar k-torsion for any $k\ge 0$. The latter result can now be recovered as a consequence of Theorem 1, using a result recently announced by Hutchings [Hut] that $(M_-,\xi_-)\preccurlyeq(M_+,\xi_+)$ and $c(M_+,\xi_+)=0$ imply $c(M_-,\xi_-)=0$. This is highly non-obvious since the cobordisms we construct are never exact (see Remark 1.4), and non-exact cobordisms do not in general give rise to well-behaved maps on ECH in its standard form. The situation becomes slightly simpler however under stricter assumptions, e.g. Hutchings and Taubes have explained in [HT] how to construct such maps for the case $h=0$ whenever (W,ω) is a strong cobordism with an exact symplectic form, sometimes called a "weakly exact" cobordism:

Proposition 1.8 ([HT]). *Suppose (W,ω) is a strong symplectic cobordism from (M_-,ξ_-) to (M_+,ξ_+) such that ω is exact. Then there is a U-equivariant map*

$$\mathrm{ECH}_*(M_+,\xi_+;0)\to\mathrm{ECH}_*(M_-,\xi_-;0)$$

that takes $c(M_+,\xi_+)$ to $c(M_-,\xi_-)$.

Remark 1.9. For the example of a 2-handle cobordism constructed from an ordinary open book decomposition, the analogue of Proposition 1.8 in Heegaard Floer homology has been established by John Baldwin [Bal].

Let us now discuss a conjectural generalization of Proposition 1.8 which could remove all conditions on ω. Recall that for any closed 2-form Ω on M, one can define ECH with **twisted coefficients** in the group ring $\mathbb{Z}[H_2(M)/\ker\Omega]$, which we shall abbreviate by

$$\mathrm{ECH}(M,\xi;h,\Omega):=\mathrm{ECH}\big(M,\xi;h,\mathbb{Z}[H_2(M)/\ker\Omega]\big).$$

Here the differential keeps track of the homology classes in $H_2(M)/\ker\Omega$ of the holomorphic curves being counted, see [HS06]. The U-map can again be defined as a degree -2 map on $\mathrm{ECH}(M,\xi;h,\Omega)$, and the twisted contact invariant $c(M,\xi;\Omega)$ is again the homology class in $\mathrm{ECH}(M,\xi;0,\Omega)$ generated by the empty orbit set. The vanishing results in [Wen] give convincing evidence that a more general version of the map in Proposition 1.8 should exist, in particular with the following consequence:

Conjecture 1. *Suppose (W,ω) is a Σ-capping or Σ-decoupling cobordism from (M_-,ξ_-) to (M_+,ξ_+), and write $\Omega_\pm=\omega|_{TM\pm}$. Then:*

1. *If $c(M_+,\xi_+;\Omega_+)$ vanishes, then so does $c(M_-,\xi_-;\Omega_-)$.*
2. *If $c(M_+,\xi_+;\Omega_+)$ is in the image of the map U^k on* $\mathrm{ECH}(M_+,\xi_+;0,\Omega_+)$ *for some $k\in\mathbb{N}$, then $c(M_-,\xi_-;\Omega_-)$ is in the image of U^k on* $\mathrm{ECH}(M_-,\xi_-;0,\Omega_-)$.

The first part of the conjecture would reduce both the untwisted and twisted vanishing results in [Wen] to the fact, proved essentially by Eliashberg in the appendix of [Yau06], that the fully twisted contact invariant vanishes for every overtwisted contact manifold. The second part is related to another result proved in [Wen], namely the twisted ECH version of the planarity obstruction of Oszváth-Stipsicz-Szabó [OSS05] in Heegaard Floer homology: if (M,ξ) is planar, then $c(M,\xi;\Omega)$ is in the image of U^k for all k and all Ω. If the conjecture holds, then this fact follows from Theorem 4′ and the computation of $\mathrm{ECH}(S^3,\xi_0)$.

[3]In the appendix of [Yau06], Eliashberg sketches an argument to show that every overtwisted contact manifold has trivial contact homology, and this argument also implies the vanishing of the ECH contact invariant.

The obvious way to try to prove Conjecture 1 would be by constructing a U-equivariant map

$$\mathrm{ECH}(M_+,\xi_+;0,\Omega_+)\to\mathrm{ECH}(M_-,\xi_-;0,\Omega_-)$$

which takes $c(M_+,\xi_+;\Omega_+)$ to $c(M_-,\xi_-;\Omega_-)$. Due to the non-exactness of ω and a resulting lack of energy bounds, it seems unlikely that such a map would exist in general, but a more probable scenario is to obtain a map

$$\mathrm{ECH}(M_+,\xi_+;0,\Lambda_\omega)\to\mathrm{ECH}(M_-,\xi_-;0,\Lambda_\omega),$$

where Λ_ω is a Novikov completion of $\mathbb{Z}[H_2(W)/\ker\omega]$, and we take advantage of the natural inclusions

$$H_2(M_\pm)/\ker\Omega_\pm\hookrightarrow H_2(W)/\ker\omega$$

to define the ECH of $(M_\pm,\xi_\pm)$ with coefficients in Λ_ω. In cases where M_+ has connected components with closed leaves, one would expect this map to involve also the Periodic Floer Homology (cf. [HS05]) of the resulting mapping tori. Defining such a map would require a slightly more careful construction of the weak cobordism (W,ω), such that both boundary components inherit stable Hamiltonian structures which can be used to attach cylindrical ends and define reasonable moduli spaces of finite energy punctured holomorphic curves. This can always be done due to a construction in [NW11], which shows that suitable stable Hamiltonian structures exist for any desired cohomology class on the boundary. It is probably also useful to observe that for an intelligent choice of data, the holomorphic curves in (W,ω) with no positive ends can be enumerated precisely: we will show in Proposition 3.20 that all of them arise from the symplectic core of the handle.

2. Further applications, examples and discussion

We shall now give some concrete examples of capping and decoupling cobordisms and survey a few more applications, including new proofs of several known results and one or two new ones.

2.1. **The Gromov-Eliashberg theorem using holomorphic spheres.** In [Wen, NW11], holomorphic curve arguments were used to show that planar torsion is a filling obstruction, but Theorems 1 and 2 make these proofs much easier by using essentially "soft" methods to reduce them to the well-known result of Gromov-Eliashberg that overtwisted contact manifolds are not weakly fillable. This does not of course make everything elementary, as the Gromov-Eliashberg theorem still requires some technology—the original proof used a "Bishop family" of holomorphic disks with totally real boundary, and these days one can instead use punctured holomorphic curves, Seiberg-Witten theory or Heegaard Floer homology if preferred. While this technological overhead is probably not removable, we can use a decoupling cobordism to simplify the level of technology a tiny bit: namely we can reduce it to the following standard fact whose proof requires only *closed* holomorphic spheres, e.g. the methods used in [McD90].

Lemma 2.1. *If (W,ω) is a connected weak filling of a nonempty contact manifold (M,ξ), then it contains no embedded symplectic sphere with vanishing self-intersection.*

This lemma follows essentially from McDuff's results [McD90], but by today's standards it is also easy to prove on its own: if one chooses a compatible almost complex structure to make the boundary J-convex and the embedded symplectic sphere J-holomorphic, then vanishing self-intersection implies that the latter lives in a smooth 2-dimensional moduli space

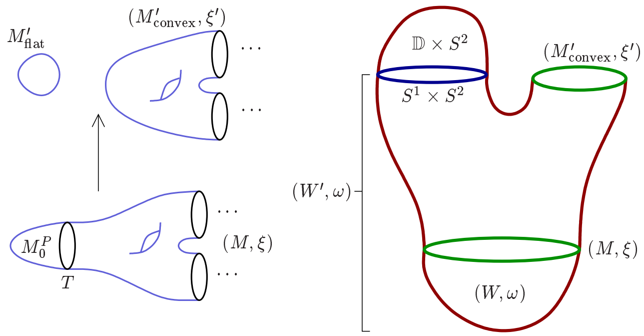

FIGURE 1. A proof of the Gromov-Eliashberg theorem using a $\mathbb{D}$-decoupling cobordism. The left shows the effect on the pages when a round handle $\mathbb{D} \times \mathbb{A}$ is attached to a planar 0-torsion domain. The right shows the resulting cobordism and consequent weak filling which furnishes a contradiction to Lemma 2.1.

of holomorphic spheres that foliate W (except at finitely many nodal singularities). This forces some leaf of the foliation to hit the boundary tangentially, thus contradicting J-convexity.

Corollary ([Gro85, Eli90]). *Every weakly fillable contact manifold is tight.*

Proof. A schematic diagram of the proof is shown in Figure 1. Suppose (W, ω) is a weak filling of (M, ξ) and the latter is overtwisted. Then (M, ξ) contains a planar 0-torsion domain[4] M_0, whose planar piece M_0^P is a solid torus with disk-like pages, attached along an interface torus $T = \partial M_0^P$ to another subdomain whose pages are not disks. Since $[T] = 0 \in H_2(M)$, $\int_T \omega = 0$ and we can attach a $\mathbb{D}$-decoupling cobordism along T, producing a larger symplectic manifold (W', ω) whose boundary has two connected components

$$\partial W' = M'_{\mathrm{flat}} \sqcup M'_{\mathrm{convex}},$$

of which the latter carries a contact structure ξ' dominated by ω. The component M'_{flat} has closed sphere-like pages, and is thus the trivial symplectic fibration $S^1 \times S^2 \to S^1$. After capping M'_{flat} by a symplectic fibration $\mathbb{D} \times S^2 \to \mathbb{D}$, we then obtain a weak filling of $(M'_{\mathrm{convex}}, \xi')$ containing a symplectic sphere with vanishing self-intersection, contradicting Lemma 2.1. □

Remark 2.2. A related argument appears in [Gay06], using the fact that overtwisted contact manifolds always have Giroux torsion; see also §2.3 below.

[4] The fact that overtwistedness implies planar 0-torsion relies on Eliashberg's classification of overtwisted contact structures [Eli89], quite a large result in itself. The original "Bishop disk" argument of Gromov and Eliashberg had the advantage of not requiring this.

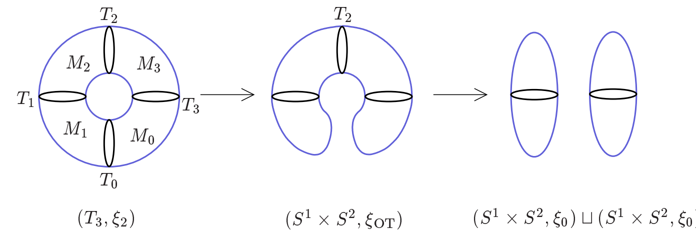

FIGURE 2. The torus (T^3, ξ_2) can be constructed out of four irreducible subdomains containing cylindrical pages with trivial monodromy. Attaching one $\mathbb{D}$-decoupling cobordism yields an overtwisted $S^1 \times S^2$, and one can then attach a second one to obtain a disjoint union of two copies of the tight $S^1 \times S^2$.

2.2. **Eliashberg's cobordisms from T^3 to $S^3 \sqcup \ldots \sqcup S^3$.** Let $T^3 = S^1 \times S^1 \times S^1$ with coordinates (η, ϕ, θ) and define for $n \in \mathbb{N}$ the contact structure

$$\xi_n = \ker \left[\cos(2\pi n \eta)\, d\theta + \sin(2\pi n \eta)\, d\phi\right]. \tag{2.1}$$

These contact structures are all tight, but Eliashberg showed in [Eli96] that they are not strongly fillable for $n \geq 2$, which follows from the fact that disjoint unions of multiple copies of (S^3, ξ_0) are not fillable, together with the following:

Theorem ([Eli96]). *For any $n \in \mathbb{N}$, (T^3, ξ_n) is symplectically cobordant to the disjoint union of n copies of the tight 3-sphere.*

Proof. The torus (T^3, ξ_n) admits a supporting summed open book decomposition with $2n$ irreducible subdomains M_j each having cylindrical pages and trivial monodromy, attached to each other along $2n$ interface tori $\mathcal{I} = \bigcup_j T_j$ such that $T_j = M_j \cap M_{j+1}$ for $j = \mathbb{Z}_{2n}$. Attaching round handles $\mathbb{D} \times \mathbb{A}$ along every second interface torus $T_0, T_2, \ldots, T_{2n-2}$ yields a weak symplectic cobordism to the disjoint union of n copies of the tight $S^1 \times S^2$ (Figure 2). The latter is also supported by an open book with cylindrical pages and trivial monodromy, so we can attach a 2-handle $\mathbb{D} \times \mathbb{D}$ along one binding component to create a weak cobordism to the tight S^3. The resulting weak cobordism from T^3 to $S^3 \sqcup \ldots \sqcup S^3$ can be deformed to a strong cobordism since the symplectic form is necessarily exact near $S^3 \sqcup \ldots \sqcup S^3$. □

Remark 2.3. Note that (T^3, ξ_n) is always *weakly* fillable [Gir94], and indeed, the above cobordism cannot be attached to any weak filling (W, ω) of (T^3, ξ_n) for which $\int_{T_j} \omega \neq 0$. This shows that the homological condition in Theorem 6 cannot be removed.

2.3. **Gay's cobordisms for Giroux torsion.** Recall that a contact manifold (M, ξ) is said to have **Giroux torsion** $\mathrm{GT}(M, \xi) = n \in \mathbb{N}$ if n is the largest integer for which (M, ξ) admits a contact embedding of $([0, 1] \times T^2, \xi_n)$, where ξ_n is given by (2.1); we write $\mathrm{GT}(M, \xi) = 0$ if there are no such embeddings and $\mathrm{GT}(M, \xi) = \infty$ if they exist for arbitrarily large n. Every contact manifold with positive Giroux torsion also has planar 1-torsion (see [Wen]), thus as a special case of Theorem 1, every (M, ξ) with $\mathrm{GT}(M, \xi) \geq 1$ is symplectically cobordant to something overtwisted; this was proved by David Gay in [Gay06] for $\mathrm{GT}(M, \xi) \geq 2$. A

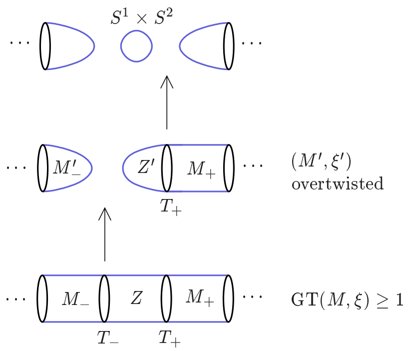

FIGURE 3. Attaching a $\mathbb{D}$-decoupling cobordism to (M, ξ) with Giroux torsion at least 1 yields an overtwisted contact manifold (M', ξ'). Attaching one more yields the disjoint union of a contact manifold with a trivial symplectic S^2-fibration over S^1.

concrete picture of this cobordism[5] is shown in Figure 3: namely, if $\mathrm{GT}(M, \xi) \geq 1$, then M contains a domain $[0, 1] \times T^2 \cong M_0 \subset M$ on which ξ is supported by a blown up summed open book with three irreducible subdomains

$$M_0 = M_- \cup_{T_-} Z \cup_{T_+} M_+$$

attached to each other in a chain along two interface tori $T_\pm = Z \cap M_\pm$. As is explained in [Wen], M_0 is literally the closure of some open neighborhood of the standard Giroux torsion domain $([0, 1] \times T^2, \xi_1)$ in M, and the middle segment Z can be identified with $[1/4, 3/4] \times T^2$ in $([0, 1] \times T^2, \xi_1)$: in particular it has cylindrical pages with trivial monodromy. Likewise M_+ and M_- have cylindrical pages but nontrivial monodromy in general—this detail will play no role in the following. Attaching a round handle $\mathbb{D} \times \mathbb{A}$ along T_- produces a weak symplectic cobordism to a new contact manifold M', containing the disconnected domain M'_0 shown in Figure 3: in particular M_- and Z are each transformed into subdomains M'_- and Z' with disk-like pages. Now $Z' \cup M_+ \subset M'$ contains an overtwisted disk; indeed, it is a planar 0-torsion domain. Observe that this construction can also be used to show that (M, ξ) is not *weakly* fillable if the torsion domain separates M, as then $\int_{T_-} \omega = 0$ for any symplectic form ω arising from a weak filling.

Gay's proof in [Gay06] that Giroux torsion obstructs strong filling did not directly use the above cobordism, but proved instead that (M, ξ) with $\mathrm{GT}(M, \xi) \geq 1$ admits a symplectic cobordism to some non-empty contact manifold such that the cobordism itself contains a symplectic sphere with vanishing self-intersection—Gay's argument then used gauge theory

[5] Both the cobordism in Figure 3 and the one that is constructed explicitly in [Gay06] for the case $\mathrm{GT}(M, \xi) \geq 2$ are *weak* cobordisms, not strong in general. As David Gay has pointed out to me, these can always be turned into strong cobordisms by attaching additional 2-handles to make the positive boundary an overtwisted rational homology sphere (see the proof of Theorem 1 in §3.7).

to derive a contradiction if (M, ξ) has a filling, but one can just as well use Lemma 2.1 above. A close relative of Gay's cobordism construction is easily obtained from the above picture: attaching round handles $\mathbb{D} \times \mathbb{A}$ along both T_- and T_+, the top of the cobordism contains a connected component with closed sphere-like pages (the top picture in Figure 3), which can be capped by $\mathbb{D} \times S^2$ to produce a cobordism that contains symplectic spheres of self-intersection number 0.

2.4. **Some new examples with $M_- \preccurlyeq M_+$ but $M_- \not\preccurlyeq M_+$.** Gromov's theorem [Gro85] on the non-existence of exact Lagrangians in $\mathbb{R}^{2n}$ provides perhaps the original example of a pair of contact manifolds that are strongly but not exactly cobordant: indeed, viewing (T^3, ξ_1) as the boundary of a Weinstein neighborhood of any Lagrangian torus in the standard strong filling of the tight 3-sphere (S^3, ξ_0), we obtain

$$(T^3, \xi_1) \preccurlyeq (S^3, \xi_0) \quad \text{but} \quad (T^3, \xi_1) \not\preccurlyeq (S^3, \xi_0).$$

The nonexistence of the exact cobordism here can also be proved by the argument of Hofer [Hof93] mentioned in the introduction, which implies that if (M, ξ) admits a Reeb vector field with no contractible periodic orbit, then $(M, \xi) \not\preccurlyeq (M', \xi')$ whenever either (M', ξ') is overtwisted or $M' \cong S^3$. Together with Theorem 1, this implies that for any $(M_{\mathrm{OT}}, \xi_{\mathrm{OT}})$ overtwisted and $n \geq 2$,

$$(T^3, \xi_n) \preccurlyeq (M_{\mathrm{OT}}, \xi_{\mathrm{OT}}) \quad \text{but} \quad (T^3, \xi_n) \not\preccurlyeq (M_{\mathrm{OT}}, \xi_{\mathrm{OT}}).$$

A subtler obstruction to exact cobordisms is defined in joint work of the author with Janko Latschev [LW11] via Symplectic Field Theory, leading to the following example. For any integer $k \geq 1$, suppose Σ is a closed, connected and oriented surface of genus $g \geq k$, and $\Gamma \subset \Sigma$ is a multicurve consisting of k disjoint embedded loops which divide Σ into exactly two connected components

$$\Sigma = \Sigma_+ \cup_\Gamma \Sigma_-,$$

such that Σ_+ has genus 0 and Σ_- has genus $g - k + 1 > 0$. By a construction due to Lutz [Lut77], the product

$$M_{k,g} := S^1 \times \Sigma$$

then admits a unique (up to isotopy) S^1-invariant contact structure $\xi_{k,g}$ such that the convex surfaces $\{*\} \times \Sigma$ have dividing set Γ. The contact manifold $(M_{k,g}, \xi_{k,g})$ then has planar $(k-1)$-torsion, as the two subsets $S^1 \times \Sigma_\pm$ can be regarded as the irreducible subdomains of a supporting summed open book with pages $\{*\} \times \Sigma_\pm$, so we view $S^1 \times \Sigma_+$ as the planar piece and $S^1 \times \Sigma_-$ as the padding (see Definition 3.3). In particular, $(M_{k,g}, \xi_{k,g})$ is overtwisted if and only if $k = 1$, and for $k \geq 2$ it has a Reeb vector field with no contractible periodic orbits. It turns out in fact that each increment of k contains an obstruction to exact fillings that is invisible in the non-exact case.

Theorem 7. *If $k > \ell \geq 1$ then for any $g \geq k$ and $g' \geq \ell$,*

$$(M_{k,g}, \xi_{k,g}) \preccurlyeq (M_{\ell,g'}, \xi_{\ell,g'}) \quad \textit{but} \quad (M_{k,g}, \xi_{k,g}) \not\preccurlyeq (M_{\ell,g'}, \xi_{\ell,g'})$$

Proof. The nonexistence of the exact cobordism is a result of [LW11]. The existence of the non-exact cobordism follows immediately from Corollary 1, but in certain cases one can construct it much more explicitly as in Figure 4. In particular, $(M_{k,g}, \xi_{k,g})$ is supported by a summed open book consisting of the two irreducible subdomains $S^1 \times \Sigma_\pm$ with pages $\{*\} \times \Sigma_\pm$ attached along k interface tori. Attaching $\mathbb{D} \times \mathbb{A}$ along one of the interface tori gives a weak $\mathbb{D}$-decoupling cobordism to $(M_{k-1,g-1}, \xi_{k-1,g-1})$. Theorem 6′ then implies that this can be

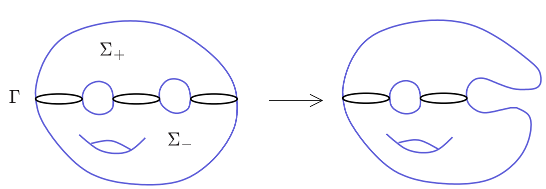

FIGURE 4. An explicit cobordism from $(M_{3,3}, \xi_{3,3})$ to $(M_{2,2}, \xi_{2,2})$ as in the proof of Theorem 7 can be realized as a $\mathbb{D}$-decoupling cobordism.

deformed to a strong cobordism, as the restriction of the symplectic form to $M_{k-1,g-1}$ is Poincaré dual to a multiple of the boundary of the co-core; the latter consists of two loops of the form $S^1 \times \{*\}$ with opposite orientations and is thus nullhomologous. $\square$

2.5. **Open books with reducible monodromy.** Any compact, connected and oriented surface Σ with boundary, together with a diffeomorphism $\varphi : \Sigma \to \Sigma$ fixing the boundary, determines a contact 3-manifold (M_φ, ξ_φ), namely the one supported by the open book decomposition with page Σ and monodromy φ. Recall that the mapping class of the monodromy map φ is said to be **reducible** if it has a representative that preserves some multicurve $\Gamma \subset \Sigma$ such that no component of $\Sigma \setminus \Gamma$ is a disk or an annulus. Consider the simple case in which φ preserves each individual connected component $\gamma \subset \Gamma$ and also preserves its orientation (note that this is always true for some iterate of φ). In this case we may assume after a suitable isotopy that φ is the identity on a neighborhood of $\partial\Sigma \cup \Gamma$, so that for some open annular neighborhood $\gamma \subset \mathcal{N}(\gamma) \subset \Sigma$ of each curve $\gamma \subset \Gamma$, M_φ contains a thickened torus region

$$S^1 \times \mathcal{N}(\gamma) \subset M_\varphi$$

on which the open book decomposition is the projection to the first factor. Let $\mathcal{N}(\Gamma) \subset \Sigma$ denote the union of all the neighborhoods $\mathcal{N}(\gamma)$ and define the possibly disconnected compact surface

$$\Sigma_\Gamma = \Sigma \setminus \mathcal{N}(\Gamma)$$

with boundary; then φ restricts to this surface as an orientation preserving diffeomorphism $\varphi_\Gamma : \Sigma_\Gamma \to \Sigma_\Gamma$ that preserves each connected component and equals the identity near $\partial\Sigma_\Gamma$. Denote the connected components of Σ_Γ by

$$\Sigma_\Gamma = \Sigma_\Gamma^1 \sqcup \ldots \sqcup \Sigma_\Gamma^N$$

and the corresponding restrictions of φ_Γ by

$$\varphi_\Gamma^j : \Sigma_\Gamma^j \to \Sigma_\Gamma^j$$

for $j = 1, \ldots, N$. Since each Σ_Γ^j necessarily has nonempty boundary, each gives rise to a connected contact manifold $(M_{\varphi_\Gamma^j}, \xi_{\varphi_\Gamma^j})$.

Theorem 8. *Given a reducible monodromy map $\varphi : \Sigma \to \Sigma$ as described above, there exists a weak symplectic cobordism (W, ω) from*

$$(M_{\varphi_\Gamma^1}, \xi_{\varphi_\Gamma^1}) \sqcup \ldots \sqcup (M_{\varphi_\Gamma^N}, \xi_{\varphi_\Gamma^N}) \quad \text{to} \quad (M_\varphi, \xi_\varphi),$$

which is (strongly) concave at the negative boundary and such that the restriction of ω to the positive boundary is Poincaré dual to a positive multiple of

$$\sum_{\gamma \subset \Gamma} [S^1 \times \{p_\gamma\}] \in H_1(M_\varphi; \mathbb{R}),$$

where the summation is over the connected components of Γ and $p_\gamma \subset \mathcal{N}(\gamma)$ denotes an arbitrarily chosen point.

Moreover, given any closed 2-form Ω on $M_{\varphi_\Gamma^1} \sqcup \ldots \sqcup M_{\varphi_\Gamma^N}$ that dominates the respective contact structures, one can also construct a weak cobordism between the contact manifolds above such that ω matches Ω at the negative boundary.

Proof. The cobordism is a stack of $\mathbb{A}$-capping cobordisms, constructed by attaching handles of the form $[-1,1] \times S^1 \times \mathbb{D}$ via Theorem 5 along all pairs of binding circles in $M_{\varphi_\Gamma^1} \sqcup \ldots \sqcup M_{\varphi_\Gamma^N}$ that correspond to the same curve in Γ. The co-core of each of these handles is a disk with boundary of the form $S^1 \times \{*\} \subset S^1 \times \mathcal{N}(\gamma) \subset M_\varphi$, thus the cohomology class of ω at the positive boundary follows immediately from Theorem 5′. $\square$

Corollary 3. *If the contact manifolds $(M_{\varphi_\Gamma^j}, \xi_{\varphi_\Gamma^j})$ for $j = 1, \ldots, N$ are all weakly fillable, then so is (M_φ, ξ_φ).*

Remark 2.4. John Baldwin [Bal] has observed that topologically, the cobordism of Theorem 8 can also be obtained by performing boundary connected sums on the pages and then using $\mathbb{D}$-capping cobordisms to remove extra boundary components; in [Bal] this is used to deduce a relation between the Ozsváth-Szabó contact invariants of (M_φ, ξ_φ) and the pieces $(M_{\varphi_\Gamma^1}, \xi_{\varphi_\Gamma^1}), \ldots, (M_{\varphi_\Gamma^N}, \xi_{\varphi_\Gamma^N})$. Additionally, Jeremy Van Horn-Morris and John Etnyre have pointed out to me that if one also assumes every component of $\Sigma \setminus \Gamma$ to intersect $\partial\Sigma$, then one can replace the weak cobordism of Theorem 8 with a Stein cobordism. This does not appear to be possible if any component of $\Sigma \setminus \Gamma$ has its full boundary in Γ.

2.6. **Etnyre's planarity obstruction.** Let us say that a connected contact 3-manifold (M, ξ) is **maximally cobordant to** S^3 if there exists a compact connected 4-manifold W with $\partial W = S^3 \sqcup (-M)$ such that for every closed 2-form Ω on M with $\Omega|_\xi > 0$, there is a symplectic form ω on W with $\omega|_{TM} = \Omega$ defining a weak symplectic cobordism from (M, ξ) to (S^3, ξ_0). Theorem 4′ says that every planar contact manifold is maximally cobordant to S^3. It turns out that this suffices to give an alternative proof of the planarity obstruction in [Etn04b, Theorem 4.1].

Theorem 9. *Suppose (M, ξ) is maximally cobordant to S^3. Then every connected weak semi-filling of (M, ξ) has connected boundary and negative-definite intersection form.*

Proof. Let W_1 be the compact 4-manifold with $\partial W_1 = S^3 \sqcup (-M)$ guaranteed by the assumption, and suppose (W_0, ω) is a weak filling of $(M, \xi) \sqcup (M', \xi')$, where (M', ξ') is some other contact manifold, possibly empty. If $W = W_0 \cup_M W_1$ is defined by gluing these two along M, then by assumption ω can be extended over W_1 so that (W, ω) becomes a weak filling of $(S^3, \xi_0) \sqcup (M', \xi')$, implying that M' must be empty since (S^3, ξ_0) is not weakly co-fillable. Now ω is exact near $\partial W = S^3$, so without loss of generality we may assume (W, ω) is a strong filling of (S^3, ξ_0).

We claim that the map induced on homology $H_2(W_0; \mathbb{Q}) \to H_2(W; \mathbb{Q})$ by the inclusion $\iota : W_0 \hookrightarrow W$ is injective. Indeed, if $A \in H_2(W_0; \mathbb{Q})$ satisfies $\int_A \omega \neq 0$, then obviously $\int_{\iota_* A} \omega$

is also nonzero and thus $\iota_* A \neq 0 \in H_2(W;\mathbb{Q})$. If $\int_A \omega = 0$ but $A \in H_2(W_0;\mathbb{Q})$ is nontrivial, we can pick any closed 2-form σ on W_0 with $\int_A \sigma \neq 0$ and replace ω by $\omega + \epsilon\sigma$ for any $\epsilon > 0$ sufficiently small so that $(W_0, \omega + \epsilon\sigma)$ remains a weak filling of (M, ξ). Then $\omega + \epsilon\sigma$ also extends over W_1, so that the above argument goes through again to prove that $\iota_* A$ is nontrivial.

Finally, we use the fact that the strong fillings of (S^3, ξ_0) have been classified: by a result of Gromov [Gro85] and Eliashberg [Eli90], W is necessarily diffeomorphic to a symplectic blow-up of the 4-ball, i.e.

$$W \cong B^4 \# \overline{\mathbb{C}P^2} \# \ldots \# \overline{\mathbb{C}P^2}.$$

Since the latter has a negative-definite intersection form and $\iota_* : H_2(W_0;\mathbb{Q}) \to H_2(W;\mathbb{Q})$ is injective, the result follows. $\square$

Our proof of Theorem $4'$ combined with Conjecture 1 would also reprove the algebraic planarity obstruction established in [Wen], which is the twisted ECH version of a Heegaard Floer theoretic result by due to Oszváth, Stipsicz and Szabó [OSS05]. Note that the condition of being maximally cobordant to S^3 does not require (M, ξ) to be fillable. It is also not clear whether there can exist non-planar contact manifolds that also satisfy this condition; the author is unaware of any known invariants that would be able to detect this distinction.

Question 3. *Is there a non-planar contact 3-manifold which is maximally cobordant to S^3?*

Note that if the assumption of Theorem 9 is relaxed to $(M, \xi) \preccurlyeq (S^3, \xi_0)$, then the result becomes false: a counterexample is furnished by the standard 3-torus (T^3, ξ_1), which admits a cobordism to (S^3, ξ_0) by Theorem 4 but also is strongly filled by T^*T^2, whose intersection form is indefinite. Assuming a strong filling (W_0, ω) of (M, ξ), the proof above fails precisely at the point where the inclusion $W_0 \hookrightarrow W$ is required to induce an *injective* map $H_2(W_0;\mathbb{Q}) \to H_2(W;\mathbb{Q})$. However, it still follows by the same argument that $H_2(W_0;\mathbb{Q})$ cannot contain any class with strictly positive square, hence we obtain the following weaker result with more general assumptions—it applies in particular to all the contact manifolds covered by Theorem 4.

Theorem 10. *Suppose (M, ξ) is a closed connected contact 3-manifold with $(M, \xi) \preccurlyeq (S^3, \xi_0)$. Then every strong semi-filling (W, ω) of (M, ξ) has connected boundary and $b_2^+(W) = 0$.*

2.7. **Some remarks on planar torsion.** The filling obstruction known as planar torsion was introduced in [Wen] with mainly holomorphic curves as motivation, as it provides the most general setting known so far in which the existence and uniqueness of certain embedded holomorphic curves leads to a vanishing result for the ECH contact invariant. In light of our cobordism construction, however, one can now provide an alternative motivation for the definition in purely symplectic topological terms. The first step is to understand what kinds of blown up summed open books automatically support overtwisted contact structures: using Eliashberg's classification theorem [Eli89] and Giroux's criterion (cf. [Gei08]), this naturally leads to the notion of planar 0-torsion. Then a more general blown up summed open book defines a planar k-torsion domain for some $k \geq 1$ if and only if it can be transformed into a planar 0-torsion domain by a sequence of $\mathbb{D}$-capping and $\mathbb{D}$-decoupling surgeries; this is the essence of Proposition 3.4 proved below. From this perspective, the definition of planar torsion and the crucial role played by blown up summed open books seem completely natural.

More generally, the partially planar domains are precisely the blown up summed open books for which a sequence of $\mathbb{D}$-capping and $\mathbb{D}$-decoupling cobordisms can be used to construct a symplectic cap that contains a symplectic sphere with square 0. As far as the author is aware, almost all existing uniqueness or classification results for symplectic fillings (e.g. [Wen10, Lis08, OO05]) apply to contact manifolds that admit caps of this type. However, it does not always suffice to construct an appropriate cap and then apply McDuff's results [McD90]: e.g. the classification of strong fillings of planar contact manifolds in terms of Lefschetz fibrations [Wen10, LVW] truly relies on *punctured* holomorphic curves, as there is no obvious way to produce a Lefschetz fibration with bounded fibers out of a family of holomorphic spheres in a cap.

Finally, we remark that while Theorems 1 and 2 substantially simplify the proof that planar torsion is a filling obstruction, they do not reproduce all of the results in [Wen]: in particular the technology of Embedded Contact Homology is not yet far enough along to deduce the vanishing of the contact invariant from a non-exact cobordism. Moreover, a proof using capping and decoupling cobordisms simplifies the technology needed but does not remove it, as a simplified version of the very same technology is required to prove the Gromov-Eliashberg theorem (cf. §2.1). From the author's own perspective, the idea for constructing symplectic cobordisms out of these types of handles would never have emerged without a holomorphic curve picture in the background (cf. Figure 7), and as we will discuss in §3.8, after one has constructed the symplectic structure, it is practically no extra effort to add a foliation by embedded J-holomorphic curves which reproduces the J-holomorphic blown up open books of [Wen] on both boundary components. The moral is that whether one prefers to prove non-fillability results by direct holomorphic curve arguments or by constructing cobordisms to reduce them to previously known results, it is essentially the same thing: neither proof would be possible without the other.

3. The details

The plan for proving the main results is as follows. We begin in §3.1 by reviewing the fundamental definitions involving blown up summed open books and planar torsion, culminating with the (more or less obvious) observation that one can always use capping or decoupling surgery to decrease the order of a planar torsion domain. In §3.2, we introduce a useful concrete model for a blown up summed open book and its supported contact structure. This is applied in §3.3 to write down a model of a weak Σ-decoupling cobordism, and minor modifications explained in §3.4 yield a similar model for the Σ-capping cobordism. This completes the cobordism construction for the case where the negative boundary is strongly concave (or more generally when the given symplectic form ω in Theorem 5 or 6 is exact), see Remark 3.12. For the general case, we need to show additionally that any given symplectic form on $[0,1] \times M$ satisfying the necessary cohomological condition can be deformed so as to attach smoothly to the model cobordisms we've constructed; this is shown in §3.5, thus completing the proofs of Theorems 5 and 6. We prove Theorems $5'$ and $6'$ in §3.6, answering the essentially cohomological question of when the weak cobordism can be made strong, and when its symplectic form is exact. With these ingredients all in place, the proofs of the main results from §1.1 are completed in §3.7. Finally, §3.8 gives a brief discussion of the existence and uniqueness of holomorphic curves in the cobordisms we've constructed.

3.1. **Review of summed open books and planar torsion.** The following notions were introduced in [Wen], and we refer to that paper for more precise definitions and further discussion.

Assume M is a compact oriented 3-manifold, possibly with boundary, the latter consisting of a union of 2-tori. A **blown up summed open book** $\boldsymbol{\pi}$ on M can be described via the following data.

(1) An oriented link $B \subset M \setminus \partial M$, called the **binding**.
(2) A disjoint union of 2-tori $\mathcal{I} \subset M \setminus \partial M$, called the **interface**.
(3) For each interface torus $T \subset \mathcal{I}$ a distinguished basis (λ, μ) of $H_1(T)$, where μ is defined only up to sign.
(4) For each boundary torus $T \subset \partial M$ a distinguished basis (λ, μ) of $H_1(T)$.
(5) A fibration
$$\pi : M \setminus (B \cup \mathcal{I}) \to S^1$$
whose restriction to ∂M is a submersion.

The distinguished homology classes $\lambda, \mu \in H_1(T)$ associated to each torus $T \subset \mathcal{I} \cup \partial M$ are called **longitudes** and **meridians** respectively, and the oriented connected components of the fibers $\pi^{-1}(\text{const})$ are called **pages**. We assume moreover that the fibration π can be expressed in the following normal forms near the components of $B \cup \mathcal{I} \cup \partial M$. As in an ordinary open book decomposition, each binding circle $\gamma \subset B$ has a neighborhood admitting coordinates $(\theta, \rho, \phi) \in S^1 \times \mathbb{D}$, where (ρ, ϕ) are polar coordinates on the disk (normalized so that $\phi \in S^1 = \mathbb{R}/\mathbb{Z}$), such that $\gamma = \{\rho = 0\}$ and

$$\pi(\theta, \rho, \phi) = \phi. \tag{3.1}$$

Near an interface torus $T \subset \mathcal{I}$, we can find a neighborhood with coordinates $(\theta, \rho, \phi) \in S^1 \times [-1, 1] \times S^1$ such that $T = \{\rho = 0\} = S^1 \times \{0\} \times S^1$ with (λ, μ) matching the natural basis of $H_1(S^1 \times \{0\} \times S^1)$, and

$$\pi(\theta, \rho, \phi) = \begin{cases} \phi & \text{for } \rho > 0, \\ -\phi & \text{for } \rho < 0. \end{cases} \tag{3.2}$$

A neighborhood of a boundary torus $T \subset \partial M$ similarly admits coordinates $(\theta, \rho, \phi) \in S^1 \times [0, 1] \times S^1$ with $T = S^1 \times \{0\} \times S^1$ and

$$\pi(\theta, \rho, \phi) = \phi. \tag{3.3}$$

Observe that unlike the normal form (3.1), the map (3.3) is well defined at $\rho = 0$, since there are no polar coordinates and hence no coordinate singularity. The above conditions imply that the closure of each page is a smoothly *immersed* surface, whose boundary components are each embedded submanifolds of B, $\mathcal{I}$ or ∂M, and in the last two cases homologous to the distinguished longitude λ. The "generic" page has an embedded closure, but in isolated cases there may be pairs of boundary components that are identical as oriented 1-dimensional submanifolds in $\mathcal{I}$.

In general, any or all of B, $\mathcal{I}$ and ∂M may be empty, and M may also be disconnected. If $B \cup \mathcal{I} \cup \partial M = \emptyset$ we have simply a fibration $\pi : M \to S^1$ whose fibers are closed oriented surfaces. If $\mathcal{I} \cup \partial M = \emptyset$ but $B \neq \emptyset$ and M is connected, we have an ordinary open book.

We say that $\boldsymbol{\pi}$ is **irreducible** if the fibers $\pi^{-1}(\text{const})$ are connected, i.e. there is only one S^1-parametrized family of pages. More generally, any blown up summed open book can be presented uniquely as a union of **irreducible subdomains**

$$M = M_1 \cup \ldots \cup M_N,$$

which each inherit irreducible blown up summed open books and are attached together along boundary tori (which become interface tori in M).

The notion of a contact structure supported by an open book generalizes in a natural way: we say that a contact structure ξ on M is **supported** by $\boldsymbol{\pi}$ if it is the kernel of a **Giroux form**, a contact form whose Reeb vector field is everywhere positively transverse to the pages and positively tangent to their boundaries, and which induces a characteristic foliation on $\mathcal{I} \cup \partial M$ with closed leaves parallel to the distinguished meridians. A Giroux form exists and is unique up to homotopy through Giroux forms on any connected manifold with a blown up summed open book, except in the case where the pages are closed, i.e. $B \cup \mathcal{I} \cup \partial M = \emptyset$. The binding is then a positively transverse link, and the interface and boundary are disjoint unions of pre-Lagrangian tori.

Definition 3.1. An irreducible blown up summed open book is called **planar** if its pages have genus zero. An arbitrary blown up summed open book is then called **partially planar** if its interior contains a planar irreducible subdomain, which we call a **planar piece**. A **partially planar domain** is a contact 3-manifold (M, ξ), possibly with boundary, together with a supporting blown up summed open book that is partially planar. For a given closed 2-form Ω on M, and a partially planar domain (M, ξ) with planar piece $M^P \subset M$, we say that (M, ξ) is Ω**-separating** if $\int_T \Omega = 0$ for all interface tori T of M that lie in M^P, and **fully separating** if this is true for all Ω.

Definition 3.2. A blown up summed open book is called **symmetric** if it has empty boundary, all its pages are diffeomorphic and it contains exactly two irreducible subdomains

$$M = M_+ \cup M_-,$$

each of which has empty binding and interface.

The simplest example of a symmetric summed open book is the one whose pages are disks: this supports the tight contact structure on $S^1 \times S^2$ (cf. Figure 2, right).

Definition 3.3. For any integer $k \geq 0$, an Ω-separating partially planar domain (M, ξ) with planar piece $M^P \subset M$ is called an Ω**-separating planar k-torsion domain** if it satisfies the following conditions:

- (M, ξ) is not symmetric.
- $\partial M^P \neq \emptyset$.
- The pages in M^P have $k + 1$ boundary components.

The (necessarily nonempty) subdomain $\overline{M \setminus M^P}$ is then called the **padding**.

We say that a contact manifold (M, ξ) with closed 2-form Ω has Ω**-separating planar k-torsion** if it contains an Ω-separating planar k-torsion domain. If this is true for all closed 2-forms Ω on M, then we say (M, ξ) has **fully separating** planar k-torsion.

It was shown in [Wen] that a contact manifold is overtwisted if and only if it has planar 0-torsion, which is always fully separating since the interface then intersects the planar piece only at its boundary, a single nullhomologous torus. The proofs of Theorems 1 and 2 thus rest on the following easy consequence of the preceeding definitions.

Proposition 3.4. *If M is a planar k-torsion domain for some $k \geq 1$, then it contains a binding circle γ or interface torus T in its planar piece such that the following is true. Let M' denote the manifold with corresponding blown up summed open book obtained from M by $\mathbb{D}$-capping surgery along γ or $\mathbb{D}$-decoupling surgery along T respetively. Then some connected component of M' is a planar ℓ-torsion domain for some $\ell \in \{k-2, k-1\}$.*

Proof. By assumption, M contains a planar piece M^P with nonempty boundary, and if $T_0 \subset \partial M^P$ denotes a boundary component, then the pages in M^P have exactly one boundary component adjacent to T_0. The pages in M^P have $k+1$ boundary components, and without loss of generality we may assume no other irreducible subdomain in the interior of M has planar pages with fewer boundary components than this. Since $k \ge 1$, these pages have at least one boundary component adjacent to some binding circle γ or interface torus T distinct from T_0. Performing $\mathbb{D}$-capping sugery to remove γ or $\mathbb{D}$-decoupling surgery to remove T produces a new manifold M_1 containing a planar irreducible subdomain M_1^P whose pages have ℓ boundary components where ℓ is either k or $k-1$; the latter can only result from a decoupling surgery along $T \subset M^P \setminus \partial M^P$. Thus M_1 is a planar $(\ell-1)$-torsion domain unless it is symmetric. The latter would mean $\partial M_1 = \emptyset$, hence also $\partial M = \emptyset$, and $\overline{M_1 \setminus M_1^P}$ is also irreducible and has planar pages with ℓ boundary components. This cannot arise from capping surgery along a binding circle or decoupling surgery along a torus in the interior of M^P, as we assumed all planar pages in the interior of M outside of M^P to have at least $k+1 \ge \ell+1$ boundary components. The only remaining possibility would be decoupling surgery along $T \subset \partial M^P$, but then symmetry of M_1 would imply that M must also have been symmetric, hence a contradiction. □

3.2. **A model for a blown up summed open book.** Assume (M_0, ξ) is a compact contact 3-manifold, possibly with boundary, supported by a blown up summed open book $\boldsymbol{\pi}$ with binding B, interface $\mathcal{I}$ and fibration

$$\pi : M_0 \setminus (B \cup \mathcal{I}) \to S^1.$$

We assume that each connected component of M_0 contains at least one component of $B \cup \mathcal{I} \cup \partial M_0$, so that $\boldsymbol{\pi}$ will support a contact structure everywhere. It will be useful to identify this with the following generalization of the notion of an *abstract* open book (cf. [Etn06]). The closure of a fiber $\overline{\pi^{-1}(\text{const})} \subset M_0$ is the image of some compact oriented surface S with boundary under an immersion

$$\iota : S \looparrowright M_0,$$

which is an embedding on the interior. The monodromy of the fibration then determines (up to isotopy) a diffeomorphism $\psi : S \to S$ which preserves connected components and is the identity in a neighborhood of the boundary, and we define the mapping torus

$$S_\psi = (S \times \mathbb{R})/\sim$$

with $(z, t+1) \sim (\psi(z), t)$ for all $t \in \mathbb{R}$, $z \in S$. Denote by

$$\phi : S_\psi \to \mathbb{R}/\mathbb{Z} = S^1$$

the natural fibration.

Let us label the connected components of ∂S by

$$\partial S = \partial_1 S \cup \ldots \cup \partial_n S,$$

and for each $i = 1, \ldots, n$ choose an open collar neighborhood $\mathcal{U}^i \subset S$ of $\partial_i S$ on which ψ is the identity. Denote the union of all these neighborhoods by $\mathcal{U} \subset S$. Now for each $i = 1, \ldots, n$, choose positively oriented coordinates

$$(\theta, \rho) : \mathcal{U}^i \to S^1 \times [r, 1)$$

for some $r \in (0,1)$. These neighborhoods give rise to corresponding collar neighborhoods of ∂S_ψ,

$$\mathcal{U}^i_\psi = \mathcal{U}^i \times S^1 \subset S_\psi,$$

which can be identified with $S^1 \times [r,1) \times S^1$ via the coordinates (θ, ρ, ϕ). The index set $I := \{1, \ldots, n\}$ comes with an obvious partition

$$I = I_B \cup I_{\mathcal{I}} \cup I_\partial,$$

where

$$\begin{aligned} I_B &= \{i \in I \mid \iota(\partial_i S) \subset B\}, \\ I_{\mathcal{I}} &= \{i \in I \mid \iota(\partial_i S) \subset \mathcal{I}\}, \\ I_\partial &= \{i \in I \mid \iota(\partial_i S) \subset \partial M_0\}. \end{aligned}$$

There is also a free $\mathbb{Z}_2$-action on $I_{\mathcal{I}}$ defined via an involution

$$\sigma : I_{\mathcal{I}} \to I_{\mathcal{I}}$$

such that $j = \sigma(i)$ if and only if $\iota(\partial_i S)$ and $\iota(\partial_j S)$ lie in the same connected component of $\mathcal{I}$. Now define for each $i \in I$ the domain

$$\mathcal{N}_i = \begin{cases} S^1 \times \mathbb{D} & \text{if } i \in I_B, \\ S^1 \times [-1,1] \times S^1 & \text{if } i \in I_{\mathcal{I}}, \\ S^1 \times [0,1] \times S^1 & \text{if } i \in I_\partial, \end{cases}$$

and denote by (θ, ρ, ϕ) the natural coordinates on $\mathcal{N}_i$, where for $i \in I_B$ we view (ρ, ϕ) as polar coordinates on the disk with the angle normalized to take values in $S^1 = \mathbb{R}/\mathbb{Z}$. Denote the subsets $\{\rho = 0\}$ by

$$B^{\text{abs}} = \bigsqcup_{i \in I_B} S^1 \times \{0\} \subset \bigsqcup_{i \in I_B} \mathcal{N}_i, \qquad \mathcal{I}^{\text{abs}} = \bigsqcup_{i \in I_{\mathcal{I}}} S^1 \times \{0\} \times S^1 \subset \bigsqcup_{i \in I_{\mathcal{I}}} \mathcal{N}_i.$$

The chosen coordinates on the neighborhoods $\mathcal{U}^i_\psi$ then determine a gluing map

$$\Phi : \bigcup_{i \in I} \mathcal{U}^i_\psi \to \bigsqcup_{i \in I} \mathcal{N}_i$$

which takes $\mathcal{U}^i_\psi$ to $\mathcal{N}_i$, and we use this to define a new compact and oriented manifold, possibly with boundary,

$$M_0^{\text{abs}} = S_\psi \cup_\Phi \left(\bigsqcup_{i \in I} \mathcal{N}_i \right) \Big/ \sim,$$

where the equivalence relation identifies $(\theta, \rho, \phi) \in \mathcal{N}_i$ for $i \in I_{\mathcal{I}}$ with $(\theta, -\rho, -\phi) \in \mathcal{N}_{\sigma(i)}$. This naturally contains B^{abs} and $\mathcal{I}^{\text{abs}}$ as submanifolds, and the fibration $\phi : S_\psi \to S^1$ can be extended over $M_0^{\text{abs}} \setminus (B^{\text{abs}} \cup \mathcal{I}^{\text{abs}})$ so that it matches the canonical ϕ-coordinate on $\mathcal{N}_i$ wherever $\rho > 0$. Now M_0 can be identified with M_0^{abs} via a diffeomorphism that maps B to B^{abs} and $\mathcal{I}$ to $\mathcal{I}^{\text{abs}}$, and transforms the fibration $\pi : M_0 \setminus (B \cup \mathcal{I}) \to S^1$ to ϕ.

A supported contact structure on M_0^{abs} can be defined as follows. First, define a smooth 1-form of the form

$$\lambda_0 = \begin{cases} d\phi & \text{on } S_\psi, \\ f_i(\rho)\, d\theta + g_i(\rho)\, d\phi & \text{on } \mathcal{N}_i,\ i \in I, \end{cases}$$

where $f_i, g_i : [0,1] \to \mathbb{R}$ are smooth functions chosen to have the following properties:

(1) As ρ moves from 0 to 1, $\rho \mapsto (f_i(\rho), g_i(\rho)) \in \mathbb{R}^2 \setminus \{0\}$ defines a path through the first quadrant from $(1,0)$ to $(0,1)$.
(2) λ_0 is contact on $\{0 \le \rho < r\} \subset \mathcal{N}_i$.
(3) $f_i(\rho) = 0$ for $\rho \in [r,1]$.
(4) $g_i(\rho) = 1$ for $\rho \in [r',1]$, for some positive number $r' < r$.
(5) $g_i'(\rho) > 0$ for $\rho \in (0,r')$.

Remark 3.5. The contact condition is satisfied if and only if $f_i g_i' - f_i' g_i \neq 0$, except at B^{abs}, where the coordinate singularity changes the condition to $g_i''(0) \neq 0$. One consequence is that $f_i'(\rho) < 0$ for $\rho \in [r', r)$, hence $f_i(r') > 0$. The assumption that λ_0 is a *smooth* 1-form imposes some additional conditions, namely for $i \in I_B$, $(\rho,\phi) \mapsto f_i(\rho)$ and $(\rho,\phi) \mapsto g_i(\rho)/\rho^2$ must define smooth functions at the origin in $\mathbb{R}^2$ (in polar coordinates), and for $i \in I_{\mathcal{I}}$, f_i and g_i can be extended smoothly over $[-1,1]$ such that

$$f_i(\rho) = f_{\sigma(i)}(-\rho), \qquad g_i(\rho) = -g_{\sigma(i)}(-\rho).$$

In particular this implies $(f_i(\rho), g_i(\rho)) = (0,-1)$ for $\rho \in [-1,-r]$. We will assume these conditions are always satisfied without further comment.

The co-oriented distribution

$$\xi_0 := \ker \lambda_0$$

is a confoliation on M_0^{abs}, which is integrable on the mapping torus S_ψ and outside of this is a positive contact structure. To perturb it to a global contact structure, choose a 1-form α on S which satisfies $d\alpha > 0$ and takes the form

$$\alpha = (2-\rho)\, d\theta \tag{3.4}$$

on $\mathcal{U}^i$. By a simple interpolation trick (cf. [Etn06]), α can be used to construct a 1-form α_ψ on S_ψ that satisfies

$$d\alpha_\psi|_{\xi_0} > 0 \quad \text{ and } \quad \alpha_\psi = (2-\rho)\, d\theta \text{ on } \mathcal{U}^i_\psi.$$

Choosing $\epsilon > 0$ sufficiently small, we can bring $\ker(d\phi + \epsilon\alpha_\psi)$ sufficiently C^0-close to ξ_0 on S_ψ so that $d\alpha_\psi|_{\ker(d\phi+\epsilon\alpha_\psi)} > 0$. Then a contact form that equals λ_0 near $\partial M_0^{\mathrm{abs}}$ can be defined by

$$\lambda_\epsilon = \begin{cases} d\phi + \epsilon\, \alpha_\psi & \text{on } S_\psi, \\ f_{i,\epsilon}(\rho)\, d\theta + d\phi & \text{on } \{\rho \in [r', r]\} \subset \mathcal{N}_i, \\ \lambda_0 & \text{on } \{\rho \le r'\} \subset \mathcal{N}_i, \end{cases} \tag{3.5}$$

where the fact that $f_i(r') > 0$ allows us for $\epsilon > 0$ sufficiently small to choose smooth functions $f_{i,\epsilon} : [0,1] \to \mathbb{R}$ satisfying

- $f_{i,\epsilon}(\rho) = f_i(\rho)$ for $\rho \in [0, r']$,
- $f_{i,\epsilon}' < 0$ for $\rho \in [r', r]$, and
- $f_{i,\epsilon}(\rho) = \epsilon(2-\rho)$ for $\rho \in [r,1]$.

Note that for $i \in I_{\mathcal{I}}$, $f_{i,\epsilon}$ also extends naturally over $[-1,1]$ with $f_{i,\epsilon}(\rho) = f_{\sigma(i),\epsilon}(-\rho)$. All contact forms that one can construct in this way are homotopic to each other through families of contact forms, so the resulting contact structure

$$\xi_\epsilon := \ker \lambda_\epsilon$$

is uniquely determined up to isotopy. Moreover, it is easy to check that the Reeb vector field determined by λ_ϵ is everywhere positively transverse to the pages: in particular, λ_ϵ is a

Giroux form for the blown up summed open book we've constructed on M_0^{abs}, thus $(M_0^{\mathrm{abs}}, \xi_\epsilon)$ is contactomorphic to (M_0, ξ).

3.3. **A model decoupling cobordism.**

Assume now that the manifold M_0 from the previous section is embedded into a closed contact 3-manifold (M,ξ) such that ξ is an extension of the contact structure that was given on M_0. Without loss of generality, we can identify (M_0,ξ) with the abstract model $(M_0^{\mathrm{abs}}, \xi_\epsilon)$, and assume in particular that λ_0 and λ_ϵ are 1-forms on M which restrict on M_0 to the models constructed above, and on a neighborhood of $\overline{M \setminus M_0}$ define matching contact forms whose kernel is ξ.

Our goal in this section is to construct a weak symplectic cobordism that realizes a Σ-decoupling surgery along some set of oriented interface tori

$$\mathcal{I}_0 = T_1 \cup \ldots \cup T_N \subset \mathcal{I}.$$

The chosen orientation of each T_j splits a tubular neighborhood $\mathcal{N}(T_j) \subset M$ of T_j naturally into positive and negative parts

$$\mathcal{N}(T_j) = \mathcal{N}_-(T_j) \cup \mathcal{N}_+(T_j)$$

whose intersection is T_j. To simplify notation in the following, let us assume these neighborhoods are chosen and the page boundary components $\partial S = \partial_1 S \cup \ldots \cup \partial_n S$ are ordered so that for each $j = 1, \ldots, N$,

$$\mathcal{N}(T_j) = \mathcal{N}_j = S^1 \times [-1,1] \times S^1 \quad \text{ and } \quad \mathcal{N}_+(T_j) = S^1 \times [0,1] \times S^1.$$

We will fix on $\mathcal{N}(T_j)$ the standard coordinates (θ, ρ, ϕ) of $\mathcal{N}_j$, and assume all the functions chosen to define λ_0 and λ_ϵ are the same for all of these neighborhoods, so we can write

$$f = f_j, \ g = g_j, \ f_\epsilon = f_{j,\epsilon}$$

for $j = 1, \ldots, N$.

For Theorems 5 and 6, the cobordism we construct will need to be attached to a trivial cobordism of the form $([0,1] \times M, \omega)$, which will be impossible if our model symplectic form does not match ω at least cohomologically at $\{1\} \times M$. In order to realize the right cohomology class in the model, we choose a closed 2-form Ω_0 on M representing an arbitrary cohomology class for which the condition (1.4) is satisfied. Since we only care about Ω_0 up to cohomology, we are free to add an exact 2-form and thus assume Ω_0 satisfies

$$\Omega_0 = c_j\, d\phi \wedge d\theta \quad \text{ on } \mathcal{N}(T_j)$$

for each $j = 1, \ldots, N$, where $c_j \in \mathbb{R}$ are constants satisfying

$$\sum_{j=1}^{N} c_j = 0. \tag{3.6}$$

Since $\lambda_0 \wedge d\lambda_\epsilon > 0$ everywhere on M, we can define an exact symplectic form on the trivial cobordism $[0,1] \times M$ as follows: fix any smooth, strictly increasing function $\varphi : [0,1] \to \mathbb{R}$ with $\varphi(0) = 0$ and $|\varphi(t)|$ uniformly small, and set

$$\omega_0 = d\left(\varphi(t)\lambda_0 + \lambda_\epsilon\right). \tag{3.7}$$

If $\|\varphi\|_{L^\infty}$ is sufficiently small then ω_0 is symplectic and restricts positively to both ξ and the pages of $\boldsymbol{\pi}$, everywhere on $[0,1] \times M$. Now if $C > 0$ is a sufficiently large constant, then the 2-form

$$\omega_C := C\omega_0 + \Omega_0 \tag{3.8}$$

also has these properties. In the following we shall always assume C is arbitrarily large whenever convenient. Note that for the case of Theorems 5 and 6 where the given ω on $[0,1] \times M$ is exact, we may assume without loss of generality that $\Omega_0 \equiv 0$, see Remark 3.12.

To construct a cobordism corresponding to the round handle attachment, we shall first "dig a hole" in the trivial cobordism $[0,1] \times M$ near each of the tori $\{1\} \times T_j$. In order to find nice coordinates near the boundary of the hole, it will be useful to consider the vector field X_θ on $[0,1] \times \mathcal{N}(T_j)$ defined by the condition

$$\omega_0(X_\theta, \cdot) = -d\theta.$$

Lemma 3.6. *The vector field X_θ is locally Hamiltonian with respect to ω_C and takes the form*

$$X_\theta = A(t,\rho)\,\partial_t + B(t,\rho)\,\partial_\rho \tag{3.9}$$

for some smooth functions $A, B : [0,1] \times [1,1] \to \mathbb{R}$ with the following properties:

1. *For $\pm\rho \in [r,1]$, $A(t,\rho) = 0$ and $B(t,\rho) = \pm\frac{1}{\epsilon}$.*
2. *For $\pm\rho \in [r',r]$, $A(t,\rho) = 0$ and $\pm B(t,\rho) > 0$.*
3. *For $\rho \in (-r',r')$, $A(t,\rho) < 0$.*

Proof. By a direct computation, X_θ takes the form (3.9) with A and B satisfying the linear system

$$\begin{pmatrix} -\varphi'(t) f(\rho) & -\left[\varphi(t) f'(\rho) + f_\epsilon'(\rho)\right] \\ \varphi'(t) g(\rho) & \left[\varphi(t) + 1\right] g'(\rho) \end{pmatrix} \begin{pmatrix} A(t,\rho) \\ B(t,\rho) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The determinant $\Delta(t,\rho)$ of this matrix is always negative since the contact condition requires $f(\rho)g'(\rho) - f'(\rho)g(\rho) > 0$ for $|\rho| < r$, and for $\pm\rho \in [r,1]$ we have $g(\rho) = \pm 1$, $\pm f'(\rho) \le 0$ and $\pm f_\epsilon'(\rho) < 0$. The general solution for A and B can thus be written as

$$\begin{pmatrix} A(t,\rho) \\ B(t,\rho) \end{pmatrix} = \frac{1}{\Delta(t,\rho)} \begin{pmatrix} \left[\varphi(t) + 1\right] g'(\rho) \\ -\varphi'(t) g(\rho) \end{pmatrix}.$$

The stated conditions on $A(t,\rho)$ and $B(t,\rho)$ then follow immediately from the conditions we've placed on f, g, f_ϵ and φ.

In light of (3.9), X_θ is in the kernel of $d\phi \wedge d\theta$, and we conclude easily that it is locally Hamiltonian since

$$\mathcal{L}_{X_\theta} \omega_C = d\iota_{X_\theta}(C\omega_0 + c_j\, d\phi \wedge d\theta) = d\left(-C\, d\theta\right) = 0.$$

□

Due to the lemma, we can choose a smoothly embedded curve

$$[-1,1] \to [1/2,1] \times [-1,1] : \tau \mapsto (t(\tau), \rho(\tau))$$

that is everywhere transverse to the vector field (3.9) and also satisfies $(t(0), \rho(0)) = (1/2, 0)$ and

$$(t(\tau), \rho(\tau)) = (\pm\tau, \pm 1)$$

near $\tau = \pm 1$ (see Figure 5). Writing the annulus as $\mathbb{A} = [-1,1] \times S^1$, use the curve just chosen to define an embedding

$$\Psi : S^1 \times \mathbb{A} \hookrightarrow [0,1] \times \mathcal{N}(T_j) : (\theta, \tau, \phi) \mapsto (t(\tau), \theta, \rho(\tau), \phi),$$

which traces out a smooth hypersurface $H_{T_j} \subset [0,1] \times \mathcal{N}(T_j)$ that meets $\{1\} \times M$ transversely at the pair of tori $\{1\} \times \partial\mathcal{N}(T_j)$. Denote by

$$\mathcal{U}_{T_j} \subset [0,1] \times M$$

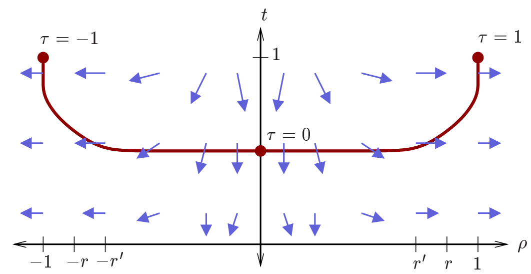

FIGURE 5. The path $(t(\tau), \rho(\tau))$ transverse to the vector field of (3.9).

the interior of the component of $([0,1] \times M) \setminus H_{T_j}$ that contains $\{1\} \times T_j$ (see Figure 6). Observe that by construction, $\mathcal{U}_{T_j}$ lies entirely within $[1/2,1] \times \mathcal{N}(T_j)$, and the locally Hamiltonian vector field X_θ points transversely outward at $\partial\overline{\mathcal{U}}_T = H_{T_j}$. Thus for sufficiently small $\delta > 0$, we can use the flow $\varphi^t_{X_\theta}$ of X_θ to parametrize a neighborhood of H_{T_j} in $\overline{\mathcal{U}}_T$ by an embedding

$$\begin{aligned} \widetilde{\Psi} : (1-\delta, 1] \times S^1 \times \mathbb{A} &\hookrightarrow [0,1] \times M \\ (\sigma, \theta, \tau, \phi) &\mapsto \varphi^{\sigma-1}_{X_\theta}(\Psi(\theta,\tau,\phi)). \end{aligned}$$

Lemma 3.7. *We have*

$$\widetilde{\Psi}^*\omega_0 = -d\left(\sigma\, d\theta\right) + d\eta, \tag{3.10}$$

where η is an S^1-invariant 1-form on $S^1 \times \mathbb{A}$ that satisfies

$$\eta = \pm\left[\varphi(\pm\tau) + 1\right]\, d\phi \quad \text{near } \{\tau = \pm 1\} = \partial(S^1 \times \mathbb{A}),$$

and $d\theta \wedge d\eta > 0$ everywhere.

Proof. In $[0,1] \times \mathcal{N}(T)$ we can write $\omega_0 = d\Lambda$, where

$$\Lambda := \varphi(t)\,\lambda_0 + \lambda_\epsilon - \epsilon\, d\theta.$$

Then defining $\eta := \Psi^*\Lambda$ on $S^1 \times \mathbb{A}$, we have $d\eta = \Psi^*\omega_0$ and can write η explicitly near $\tau = \pm 1$ by plugging in $t = \pm\tau$, $\rho = \pm 1$, $f(\rho) = 0$, $g(\rho) = \pm 1$ and $f_\epsilon(\rho) = \epsilon(2 - \rho) = \epsilon$, hence

$$\begin{aligned} \eta &= \Psi^*\left(\varphi(t)\,\lambda_0 + \lambda_\epsilon - \epsilon\, d\theta\right) = \varphi(\pm\tau)(\pm d\phi) + \epsilon\, d\theta \pm d\phi - \epsilon\, d\theta \\ &= \pm\left[\varphi(\pm\tau) + 1\right]\, d\phi \end{aligned}$$

as desired. Since Λ is invariant under the S^1-action by translation of θ, η is also S^1-invariant. The claim $d\theta \wedge d\eta > 0$ is a consequence of the fact that H_{T_j} is transverse to the vector field X_θ, which is ω_0-dual to $-d\theta$: indeed, ignoring combinatorial factors we find

$$\begin{aligned} d\theta \wedge d\eta(\partial_\theta, \partial_\tau, \partial_\phi) &= -\Psi^*(\iota_{X_\theta}\omega_0) \wedge \Psi^*\omega_0(\partial_\theta, \partial_\tau, \partial_\phi) \\ &\propto -\omega_0 \wedge \omega_0(X_\theta, \Psi_*\partial_\theta, \Psi_*\partial_\tau, \Psi_*\partial_\phi) \neq 0. \end{aligned}$$

It follows that $d\theta \wedge d\eta$ is positive since this is obviously true near $\tau = \pm 1$. The formula (3.10) now follows from the fact that $-d\theta = \iota_{X_\theta}\omega_0$ and X_θ has a symplectic flow. □

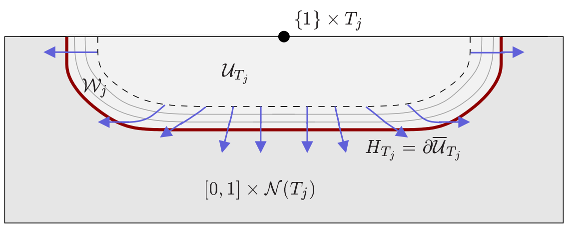

FIGURE 6. Digging a hole in $[0,1] \times \mathcal{N}(T_j)$ near $\{1\} \times T_j$.

Remark 3.8. The embedding $\widetilde{\Psi}$ reverses orientations. This will be convenient in the following since the handle $\widehat{\mathcal{H}}_\Sigma = -\Sigma \times \mathbb{A}$ also comes with a reversed orientation.

Since $\widetilde{\Psi}$ acts trivially on the coordinates ϕ and θ, the lemma also yields a formula for the pullback of ω_C, namely

$$\widetilde{\Psi}^*\omega_C = C\left[-d\left(\sigma\, d\theta\right) + d\eta\right] + c_j\, d\phi \wedge d\theta. \tag{3.11}$$

For each $j = 1, \dots, N$, denote by $\mathcal{W}_j \subset [0,1] \times \mathcal{N}(T_j)$ the image of the map $\widetilde{\Psi}$ as constructed above (Figure 6): $\mathcal{W}_j$ thus inherits negatively oriented coordinates $(\sigma, \theta, \tau, \phi) \in (1-\delta, 1] \times S^1 \times [-1,1] \times S^1$ in which ω_C has the form given in (3.11).

We are now ready to write down a smooth model of the round handle attachment. As in §1, assume Σ is a compact, connected and oriented surface with N boundary components

$$\partial\Sigma = \partial_1\Sigma \cup \ldots \cup \partial_N\Sigma.$$

Near each component $\partial_j\Sigma$, identify a collar neighborhood $\mathcal{V}_j \subset \Sigma$ with $(1-\delta, 1] \times S^1$ and denote the resulting oriented coordinates by (σ, θ). Then denote the union of all the subsets $\mathcal{U}_{T_j}$ by $\mathcal{U}_{\mathcal{I}_0}$ and define the cobordism

$$W = \left(\left([0,1] \times M\right) \setminus \mathcal{U}_{\mathcal{I}_0}\right) \cup \left(-\Sigma \times \mathbb{A}\right)$$

by removing $\mathcal{U}_{\mathcal{I}_0}$ from $[0,1] \times M$ and replacing it by the handle $-\Sigma \times \mathbb{A}$, gluing $\mathcal{V}_j \times \mathbb{A}$ to $\mathcal{W}_j$ via the natural identification of the coordinates $(\sigma, \theta, \tau, \phi)$. This yields a smooth 4-manifold with two boundary components

$$\partial W = M' \sqcup (-M),$$

where we identify M with $\{0\} \times M$ and write

$$M' = \left(\left(\{1\} \times M\right) \setminus \mathcal{N}(\mathcal{I}_0)\right) \cup \left(-\Sigma_+ \times S^1\right) \cup \left(-\Sigma_- \times S^1\right),$$

using the identification $\Sigma \times \partial\mathbb{A} = (\Sigma_+ \times S^1) \sqcup (\Sigma_- \times S^1)$ defined in (1.2). The oriented surfaces $-(\Sigma_+ \sqcup \Sigma_-) \times \{\phi\}$ now glue together smoothly with the fibers $\pi^{-1}(\phi)$ in $M \setminus \mathcal{N}(\mathcal{I}_0)$ to form the pages of the natural blown up summed open book $\boldsymbol{\pi}'$ on M' obtained from $\boldsymbol{\pi}$ by Σ-decoupling surgery along $\mathcal{I}_0$. It remains to define a suitable symplectic form on $\Sigma \times \mathbb{A}$ that matches (3.11) near $\partial\Sigma \times \mathbb{A}$ and is positive on these pages.

Lemma 3.9. *There exists a symplectic form on $-\Sigma \times \mathbb{A}$ that matches (3.10) near $\partial\Sigma \times \mathbb{A}$ and is positive on the oriented surfaces $\{p\} \times \mathbb{A}$ for any $p \in \Sigma \setminus (\mathcal{V}_1 \cup \ldots \cup \mathcal{V}_N)$ and $-\Sigma \times \{(\tau, \phi)\}$*

for any $(\tau, \phi) \in \mathbb{A}$, and makes $T(\Sigma \times \{*\})$ and $T(\{*\} \times \mathbb{A})$ into symplectically orthogonal symplectic subspaces everywhere along $\Sigma \times \partial\mathbb{A}$.

Proof. We will use a standard deformation trick to simplify (3.10) on each of the regions $\mathcal{V}_j \times \mathbb{A}$ so that it can be extended as a split symplectic form. Choose a 1-form η_0 on $\mathbb{A}$ with $d\eta_0 > 0$ and lift it in the obvious way to $S^1 \times \mathbb{A}$. Since

$$\int_{\{*\} \times \partial\mathbb{A}} \eta = 2\left[\varphi(1) + 1\right] > 0$$

and η has no $d\theta$-term near $S^1 \times \partial\mathbb{A}$, we can also arrange for η_0 to match η on a neighborhood of $S^1 \times \partial\mathbb{A}$. Next choose a smooth cutoff function $\tilde{\beta} : (1-\delta, 1] \to [0,1]$ that satisfies $\tilde{\beta}(\sigma) = 0$ near $\sigma = 1 - \delta$ and $\tilde{\beta}(\sigma) = 1$ near $\sigma = 1$, and use this to define a smooth function $\beta : \Sigma \to [0,1]$ which satisfies

$$\beta(\sigma, \theta) = \tilde{\beta}(\sigma) \text{ on } \mathcal{V}_j, \qquad \beta \equiv 0 \text{ on } \Sigma \setminus (\mathcal{V}_1 \cup \ldots \cup \mathcal{V}_N).$$

We observe that the expression

$$\beta\eta + (1 - \beta)\,\eta_0$$

now gives a well-defined 1-form on $\Sigma \times \mathbb{A}$ by lifting η_0 from $\mathbb{A}$ to $\Sigma \times \mathbb{A}$ and η from $S^1 \times \mathbb{A}$ to $(1-\delta, 1] \times S^1 \times \mathbb{A} = \mathcal{V}_j \times \mathbb{A}$ in the obvious ways.

Choose also a smooth function $\psi : (1-\delta, 1] \to [1, \infty)$ satisfying $\psi' > 0$ and $\psi'(\sigma) = 1$ near $\sigma = 1$, and a 1-form μ on Σ such that

$$\mu = \psi(\sigma)\, d\theta \text{ in } \mathcal{V}_j, \qquad d\mu > 0 \text{ everywhere.}$$

A suitable symplectic form on $\Sigma \times \mathbb{A}$ can then be defined by

$$\omega_0' = -d\mu + d\big(\beta\eta + (1-\beta)\,\eta_0\big). \tag{3.12}$$

By construction, ω_0' matches (3.10) near $\partial\Sigma \times \mathbb{A}$, while near $\Sigma \times \partial\mathbb{A}$ and outside of the regions $\mathcal{V}_j \times \mathbb{A}$ it takes the split form

$$-d\mu + d\eta_0,$$

which is symplectic and makes each of $T(\Sigma \times \{*\})$ and $T(\{*\} \times \mathbb{A})$ into symplectic subspaces which are symplectically orthogonal to each other. To test whether ω_0' is symplectic on $\mathcal{V}_j \times \mathbb{A}$, we compute

$$\begin{aligned}\frac{1}{2}\omega_0' \wedge \omega_0' &= \psi'\, d\theta \wedge d\sigma \wedge \left[\beta\, d\eta + (1-\beta)\, d\eta_0\right] \\ &\quad + \beta\beta'\, d\sigma \wedge (\eta - \eta_0) \wedge \left[\beta\, d\eta + (1-\beta)\, d\eta_0\right].\end{aligned}$$

The first term is always nonzero since $d\theta \wedge d\eta$ and $d\theta \wedge d\eta_0$ are both positive. The whole expression is therefore nonzero whenever either $\beta'(\sigma) = 0$ or $\psi'(\sigma)$ is sufficiently large, and we are free to choose ψ so that it increases fast in the region where β is not constant. This choice also ensures $\omega_0'(\partial_\theta, \partial_\sigma) > 0$ everywhere on $\mathcal{V}_j \times \mathbb{A}$. ☐

To find a symplectic extension of (3.11) over $\Sigma \times \mathbb{A}$, choose now a closed 1-form κ on Σ which takes the form

$$\kappa = c_j\, d\theta$$

near each boundary component $\partial_j\Sigma$; this is possible due to the homological condition (3.6). Then if ω_0' denotes the extension of $\widetilde{\Psi}^*\omega_0$ given by Lemma 3.9, we extend (3.11) as

$$\omega_C' := C\omega_0' + d\phi \wedge \kappa.$$

Whenever C is sufficiently large, Lemma 3.9 implies that this form is also symplectic and restricts positively to the surfaces $-\Sigma \times \{(\tau,\phi)\}$ and $\{p\} \times \mathbb{A}$ if $p \in \Sigma$ lies outside a neighborhood of the boundary. This implies that it is positive on the pages of $\boldsymbol{\pi}'$, as well as on the core $\widehat{\mathcal{K}}_\Sigma = ([0,1/2] \times \widehat{B}_0) \cup (-\Sigma \times \{(0,0)\}) \subset W$ and co-core $\widehat{\mathcal{K}}'_\Sigma = \{p\} \times \mathbb{A} \subset W$ (cf. §1.2).

To summarize: we have constructed a smooth cobordism W with symplectic form ω'_C that matches ω_C near $M = \{0\} \times M$ and is positive on the core and co-core and on the pages of the induced blown up summed open book at the other boundary component M'. An appropriate confoliation 1-form λ'_0 can now be defined on M' by

$$\lambda'_0 = \begin{cases} \lambda_0 & \text{on } M \setminus \mathcal{N}(\mathcal{I}_0), \\ d\phi & \text{on } \Sigma_\pm \times S^1, \end{cases} \tag{3.13}$$

where we use ϕ to denote the natural S^1-coordinate on $\Sigma_\pm \times S^1$. The distribution $\xi'_0 := \ker \lambda'_0$ is then tangent to the pages on the glued in region, hence $\omega'_C|_{\xi'_0} > 0$. It follows that on any connected component of M' that does not contain closed pages, ξ'_0 has a perturbation to a contact structure ξ' that is supported by $\boldsymbol{\pi}'$ and dominated by ω'_C.

Remark 3.10. It will be useful later to observe that $\int_{\widehat{\mathcal{K}}_\Sigma} \omega'_C$ is not only positive but can be assumed to be arbitrarily large. In fact it *must* in general be large due to the deformation trick used in the proof of Lemma 3.9.

Remark 3.11. If the constants c_j all vanish, i.e. $\Omega_0 = 0$ on $\mathcal{N}(\mathcal{I}_0)$, then one can choose the 1-form κ in the above construction to be identically zero. This has the useful consequence that for any $\tau \in [-1,1]$ and any closed embedded loop $\ell \subset \Sigma$ outside a neighborhood of $\partial\Sigma$, the torus $\ell \times \{\tau\} \times S^1 \subset \Sigma \times \mathbb{A}$ is Lagrangian. More generally, if $\ell \subset \Sigma$ is any properly embedded compact 1-dimensional submanifold transverse to $\partial\Sigma$, then

$$\int_{\ell \times \{\tau\} \times S^1} \omega'_C = 0.$$

Indeed, with $\kappa = 0$ it is equivalent to show that the integral of ω'_0 vanishes, and using (3.12) we find

$$\int_{\ell \times \{\tau\} \times S^1} \omega'_0 = \int_{\partial\ell \times \{\tau\} \times S^1} \eta$$

since μ vanishes on the S^1-factor in $\partial\ell \times \{\tau\} \times S^1$. Since η is S^1-invariant on $S^1 \times \mathbb{A}$, this integral doesn't depend on the position of any point in $\partial\ell \subset \partial\Sigma$ but only on the algebraic count of these points, which is zero, thus

$$\int_{\partial\ell \times \{\tau\} \times S^1} \eta = \#(\partial\ell) \int_{\{(\theta,\tau)\} \times S^1} \eta = 0.$$

Remark 3.12. The reader who is only interested in *strong* cobordisms, or more generally the case where the negative boundary of the cobordism is (strongly) concave, may assume throughout this section that $\Omega_0 \equiv 0$. In this case, the symplectic form we have defined on W is exact near $M \subset \partial W$ and has a primitive there which restricts to a constant multiple of the contact form λ_ϵ, so this boundary component is concave. The contents of this and the next section therefore suffice to complete the proofs of Theorems 6 and 5 respectively if the given ω on $[0,1] \times M$ is exact: indeed, by [Eli91, Proposition 3.1], ω can then be deformed to make it (strongly) convex at the positive boundary, so after a further deformation to match the contact forms, the Liouville flow can be used to attach it smoothly to our model as long

as the constant $C > 0$ is chosen sufficiently large. The case where ω is not exact requires the additional deformation argument of §3.5 below.

3.4. **Modifications for the capping cobordism.** The above construction works essentially the same way for the handle $\Sigma \times \mathbb{D}$, so we will be content to briefly summarize the differences. Here we pick binding components

$$B_0 = \gamma_1 \cup \ldots \cup \gamma_N \subset B$$

and denote the corresponding solid torus neighborhoods by $\mathcal{N}(\gamma_j) = S^1 \times \mathbb{D}$ with coordinates (θ,ρ,ϕ), viewing (ρ,ϕ) as polar coordinates, and denote the union of these neighborhoods by $\mathcal{N}(B_0)$. The model symplectic form ω_C on the trivial cobordism $[0,1] \times M$ is again defined via (3.7) and (3.8), with the difference that since every closed 2-form on $\mathcal{N}(\gamma_j)$ is exact, we can assume (after adding an exact 2-form) that Ω_0 *vanishes* on all of these neighborhoods. The role of H_{T_j} is now played by a hypersurface

$$H_{\gamma_j} \subset [0,1] \times \mathcal{N}(\gamma_j)$$

parametrized by an embedding

$$\Psi : S^1 \times \mathbb{D} \to [0,1] \times M,$$

thus defining a similar set of coordinates $(\theta,\tau,\phi) \in S^1 \times \mathbb{D}$ on H_{γ_j}, where (τ,ϕ) are now *polar* coordinates on $\mathbb{D}$. We can again arrange H_{γ_j} to be transverse to the vector field X_θ, defined exactly as before, and then use its flow to parametrize a neighborhood of H_{γ_j} in the region $\overline{\mathcal{U}}_{\gamma_j}$ that it bounds via a map

$$\widetilde{\Psi} : (1-\delta,1] \times S^1 \times \mathbb{D} \hookrightarrow [0,1] \times \mathcal{N}(\gamma_j) : (\sigma,\theta,\tau,\phi) \mapsto \varphi^{\sigma-1}_{X_\theta}(\Psi(\theta,\phi,\tau))$$

for which $\widetilde{\Psi}^*\omega_0$ again takes the form $-d(\sigma\, d\theta) + d\eta$ for some 1-form η on $S^1 \times \mathbb{D}$ that satisfies

$$\eta = [\varphi(\tau) + 1]\ d\phi$$

near $S^1 \times \partial\mathbb{D}$ and $d\theta \wedge d\eta > 0$ everywhere. Denote the image of $\widetilde{\Psi}$ corresponding to each γ_j by $\mathcal{W}_j$, with negatively oriented coordinates $(\sigma,\theta,\tau,\phi) \in (1-\delta,1] \times S^1 \times \mathbb{D}$. Writing the union of the regions $\mathcal{U}_{\gamma_j}$ as $\mathcal{U}_{B_0}$, the smooth cobordism is then defined by

$$W = (([0,1] \times M) \setminus \mathcal{U}_{B_0}) \cup (-\Sigma \times \mathbb{D}),$$

where $\Sigma \times \mathbb{D}$ is glued in by identifying $\mathcal{V}_j \times \mathbb{D}$ with $\mathcal{W}_j$ so that the coordinates match. This has boundary $\partial W = M' \sqcup (-M)$, where $M = \{0\} \times M$ and

$$M' = ((\{1\} \times M) \setminus \mathcal{N}(B_0)) \cup (-\Sigma \times S^1),$$

hence the glued in region $\Sigma \times S^1$ carries the coordinates (σ,θ,ϕ) near its boundary. Choosing a 1-form η_0 on $\mathbb{D}$ that matches η near $\partial\mathbb{D}$ and satisfies $d\eta_0 > 0$, the interpolation trick (3.12) can again be used to deform ω_0 in a collar neighborhood of $\partial\Sigma \times \mathbb{D}$ so that it admits a symplectic extension over the rest of $\Sigma \times \mathbb{D}$ in the form $\omega'_0 = -d\mu + d\eta_0$. The resulting form $\omega'_C = C\omega'_0 + \Omega_0$ is symplectic everywhere on W and is also positive on the pages of $\boldsymbol{\pi}'$ at M' if C is sufficiently large, as well as on the core

$$\mathcal{K}_\Sigma = ([0,1/2] \times B_0) \cup (-\Sigma \times \{0\}) \subset W \tag{3.14}$$

and the co-core

$$\mathcal{K}'_\Sigma = \{p\} \times \mathbb{D} \subset W$$

for an appropriate choice of $p \in \Sigma$. The confoliation 1-form extends smoothly over $\Sigma \times S^1$ as $\lambda_0' = d\phi$, so that ω_C' is also positive on $\xi_0' := \ker \lambda_0'$ and thus dominates any contact form obtained as a small perturbation.

3.5. **Symplectic deformation in a collar neighborhood.** To apply the constructions of the previous sections in proving Theorems 5 and 6 when the given symplectic form ω on $[0,1] \times M$ dominating ξ is non-exact, we must show that ω can be deformed away from $\{0\} \times M$ to reproduce the model

$$\omega_C = C\, d\left(\varphi(t)\lambda_0 + \lambda\right) + \Omega_0, \tag{3.15}$$

where $\lambda := \lambda_\epsilon$ and λ_0 are 1-forms as described at the beginning of §3.3, $\varphi : [0,1] \to \mathbb{R}$ is a smooth function with $\varphi' > 0$, $\varphi(0) = 0$ and $\|\varphi\|_{L^\infty}$ small, Ω_0 is some closed 2-form on M in the appropriate cohomology class, and $C > 0$ is a constant that we can assume to be as large as necessary. The following application of a standard Moser deformation argument (cf. [NW11, Lemma 2.3]) will be useful.

Lemma 3.13. *Suppose (W, ω) is a symplectic 4-manifold, M is a closed oriented 3-manifold with an embedding $\Phi : M \hookrightarrow W$ and λ is a 1-form on M that satisfies $\lambda \wedge \Phi^*\omega > 0$. Then for sufficiently small $\epsilon > 0$, Φ extends to an embedding*

$$\widetilde{\Phi} : (-\epsilon, \epsilon) \times M \hookrightarrow W$$

such that $\widetilde{\Phi}(0, \cdot) = \Phi$ and $\widetilde{\Phi}^*\omega = d(t\lambda) + \Phi^*\omega$.

Observe that if $M \subset W$ is an oriented hypersurface in a symplectic 4-manifold (W, ω) with a positive contact structure ξ, then ω dominates ξ if and only if it satisfies

$$\lambda \wedge \omega|_{TM} > 0$$

for every contact form λ on (M, ξ). Using the obvious variants of Lemma 3.13 when the hypersurface is a positive or negative boundary component of W, we obtain the following useful consequence:

Lemma 3.14. *Suppose (M, ξ) is a closed contact 3-manifold and $((-1, 0] \times M, \omega_-)$ and $([0, 1) \times M, \omega_+)$ are two symplectic manifolds such that the restrictions of ω_- and ω_+ to $\{0\} \times M$ define the same 2-form Ω on M, with $\Omega|_\xi > 0$. Then for any small $\epsilon > 0$, $(-1, 1) \times M$ admits a symplectic form which matches ω_+ on $[\epsilon, 1) \times M$ and ω_- on $(-1, -\epsilon] \times M$.*

Proposition 3.15. *Suppose (M, ξ) is any closed contact 3-manifold with contact form λ, λ_0 is a 1-form on M satisfying $\lambda_0 \wedge d\lambda > 0$, ω is a symplectic form on $[0, 1] \times M$ with $\omega|_\xi > 0$, and Ω_0 is a closed 2-form on M with $[\Omega_0] = [\omega|_{TM}] \in H^2_{\mathrm{dR}}(M)$. Then for any $\delta \in (0, 1)$ and sufficiently large $C > 0$, there exists a symplectic form ω' on $[0, 1] \times M$ that matches ω on a neighborhood of $\{0\} \times M$ and takes the form (3.15) on $[\delta, 1] \times M$.*

Proof. By Lemma 3.13 we can assume without loss of generality that ω has the form

$$\omega = d(t\lambda) + \Omega,$$

near $\{0\} \times M$, where Ω is the closed 2-form on M defined as the restriction of ω to $\{0\} \times M$.

The proof now proceeds in two steps, of which the first is to put the symplectic structure ω_C of (3.15) into a slightly simpler form via a coordinate change near $\{0\} \times M$. Define the 1-form

$$\Lambda_0 = \varphi(t)\, \lambda_0 + \lambda$$

on $[0, 1] \times M$ and write $\omega_0 := d\Lambda_0$, so $\omega_C = C\omega_0 + \Omega_0$. Let V denote the vector field that is ω_C-dual to $C\Lambda_0$, i.e. $\omega_C(V, \cdot) = C\Lambda_0$. For C sufficiently large, V is then a small perturbation of the vector field that is ω_0-dual to Λ_0, which is a Liouville (with respect to ω_0) vector field positively transverse to $\{0\} \times M$ since $\Lambda_0|_{\{0\}\times M} = \lambda$ is contact. Hence we may assume V is also positively transverse to $\{0\} \times M$ and use its flow φ_V^t to define an embedding

$$\psi : [0, \epsilon) \times M \hookrightarrow [0, 1] \times M : (t, m) \mapsto \varphi_V^t(m)$$

for $\epsilon > 0$ sufficiently small. If X_λ denotes the Reeb vector field determined by λ, along $\{0\} \times M$ we then have

$$\iota_{\partial_t}(\psi^*\omega_C) = C\lambda$$

and

$$\psi^*\omega_C|_{T(\{0\}\times M)} = C\, d\lambda + \Omega_0.$$

Hence $\psi^*\omega_C$ matches the symplectic form $d(t\,C\lambda) + C\, d\lambda + \Omega_0$ pointwise at $\{0\} \times M$, and another Moser deformation argument thus allows us to isotop the embedding ψ so that $\psi^*\omega_C$ takes this form on some neighborhood of $\{0\} \times M$. Equivalently, this means ω_C admits a deformation to a new symplectic form ω_C' which takes the form

$$\omega_C' = d(tC\lambda) + C\, d\lambda + \Omega_0 \tag{3.16}$$

on an arbitrarily small neighborhood of $\{0\} \times M$ and matches the original ω_C outside a slightly larger neighborhood.

For step two, we show that the given ω can be deformed outside a small neighborhood of $\{0\} \times M$ to a new symplectic form ω' that matches (3.16) outside a slightly larger neighborhood. Indeed, choose a constant $C' > 0$ large enough so that

$$(C'\, d\lambda + \Omega_0)|_\xi > 0,$$

and since Ω and Ω_0 are cohomologous by assumption, choose a 1-form η on M such that $C'\, d\lambda + \Omega_0 - \Omega = d\eta$. For some $\delta > 0$ small, choose a cutoff function $\beta(t)$ that equals 0 near $t = 0$ and 1 near $t = \delta$, and define

$$\omega' = d\left(f(t)\, \lambda\right) + \Omega + d\left(\beta(t)\, \eta\right),$$

with $f : [0, \delta] \to [0, \infty)$ a smooth function satisfying

- $f(t) = t$ near $t = 0$,
- $f' > 0$,
- $f(\delta) + C' = C(\delta + 1)$.

If f is chosen to increase sufficiently fast, then ω' is symplectic, and this can always be arranged if $C > 0$ is made sufficiently large. This depends in particular on the fact that the 2-forms Ω and $C\, d\lambda + \Omega_0$ are both positive on ξ. The restrictions of ω' and ω_C' to the hypersurface $\{\delta\} \times M$ now match, thus the two can be glued together smoothly by Lemma 3.14. ☐

Combining Proposition 3.15 with the cobordism constructions of §3.3 and §3.4 completes the proofs of Theorems 5 and 6.

3.6. **Cohomology.** We now prove Theorems $5'$ and $6'$ by characterizing the situations in which ω can be made exact on W or on M'_{convex}.

Assume first that (W, ω) is a Σ-capping cobordism $([0,1] \times M) \cup \mathcal{H}_\Sigma$, with $\mathcal{H}_\Sigma = -\Sigma \times \mathbb{D}$ attached along a neighborhood $\mathcal{N}(B_0)$ of $B_0 = \gamma_1 \cup \ldots \cup \gamma_N$. Write $\partial W = M' \sqcup (-M)$ and

$$\Omega := \omega|_{TM}, \qquad \Omega' := \omega|_{TM'}.$$

Due to §3.5, we may assume without loss of generality that Ω has the form

$$\Omega = C\, d\lambda + \Omega_0 \tag{3.17}$$

where $C > 0$ is arbitrarily large, λ is the usual contact form on M and Ω_0 vanishes on $\mathcal{N}(B_0)$. By Remark 3.10, we can also assume in the following that $\int_{\mathcal{K}_\Sigma} \omega$ is arbitrarily large.

The decomposition of W into $[0,1] \times M$ and $\mathcal{H}_\Sigma$, which intersect at $\mathcal{N}(B_0) \subset \{1\} \times M$, gives rise to the Mayer-Vietoris sequence,

$$\ldots \to H_2(\mathcal{N}(B_0)) \to H_2(M) \oplus H_2(\mathcal{H}_\Sigma) \to H_2(W) \to H_1(\mathcal{N}(B_0)) \to H_1(M) \oplus H_1(\mathcal{H}_\Sigma) \to \ldots$$

in which $H_2(\mathcal{N}(B_0)) = H_2(\mathcal{H}_\Sigma) = 0$, $H_1(\mathcal{H}_\Sigma) = H_1(\Sigma)$ and $H_1(\mathcal{N}(B_0)) = H_1(B_0) = \mathbb{Z}^N$. Thus there is an isomorphism

$$\begin{aligned} H_2(W) \cong \operatorname{im}\big(H_2(M) \to & H_2(W)\big) \\ & \oplus \ker\big(H_1(\mathcal{N}(B_0)) \to H_1(M) \oplus H_1(\mathcal{H}_\Sigma)\big), \end{aligned} \tag{3.18}$$

in which the first summand is an isomorphic copy of $H_2(M)$. Denote by $\iota^M : \mathcal{N}(B_0) \hookrightarrow M$ and $\iota^\Sigma : \mathcal{N}(B_0) \hookrightarrow \mathcal{H}_\Sigma$ the natural inclusions. Then $\iota^\Sigma_*([\gamma_j]) = [\partial_j \Sigma] \in H_1(\Sigma) = H_1(\mathcal{H}_\Sigma)$, so since Σ is connected, $\ker \iota^\Sigma_*$ is isomorphic to $\mathbb{Z}$ and is generated by $[\gamma_1] + \ldots + [\gamma_N]$. It follows that the second summand in (3.18) consists of all integer multiples of $[\gamma_1] + \ldots + [\gamma_N]$ which are also in $\ker \iota^M_*$, i.e. it is isomorphic to $\mathbb{Z}$ if $[\gamma_1] + \ldots + [\gamma_N]$ is torsion in $H_1(M)$, and is otherwise trivial. In the former case, let $k_0 \in \mathbb{N}$ be the smallest number for which $k_0([\gamma_1] + \ldots + [\gamma_N]) = 0 \in H_1(M)$, and construct a cycle $A_{k_0} \in H_2(W)$ in the form

$$A_{k_0} = C_M + k_0[\mathcal{K}_\Sigma], \tag{3.19}$$

where C_M is any 2-chain in $\{0\} \times M$ with $\partial C_M = k_0([\gamma_1] + \ldots + [\gamma_N])$ and $\mathcal{K}_\Sigma \subset W$ is the core (3.14). The isomorphism (3.18) implies that everything in $H_2(W)$ is an element of $H_2(M)$ plus an integer multiple of (3.19).

Let h denote a real 1-cycle in $M \setminus \mathcal{N}(B_0)$ such that $[h] = \mathrm{PD}([\Omega]) \in H_1(M; \mathbb{R})$; note that this is always possible since Ω is necessarily exact on $\mathcal{N}(B_0)$. The product $[0,1] \times h$ then represents a relative homology class in $H_2(W, \partial W; \mathbb{R})$.

Proposition 3.16. *There is a number $c > 0$ such that* $\mathrm{PD}([\omega]) = [0,1] \times [h] + c[\mathcal{K}'_\Sigma] \in H_2(W, \partial W; \mathbb{R})$.

Proof. It suffices to show that for every $A \in H_2(W)$, the evaluation of ω on A matches the intersection product

$$\int_A \omega = A \bullet \big([0,1] \times [h] + c[\mathcal{K}'_\Sigma]\big). \tag{3.20}$$

For any $A \in \operatorname{im}(H_2(M) \to H_2(W))$ this is immediately clear since

$$\int_A \omega = \int_A \Omega = A \bullet [h],$$

where the latter is the intersection product in M, and A does not intersect anything in the handle. By (3.18), either the image of $H_2(M) \to H_2(W)$ is the entirety of $H_2(W)$ or there is one more generator $A_{k_0} = C_M + k_0[\mathcal{K}_\Sigma]$. For the latter we have

$$\int_{A_{k_0}} \omega = \int_{C_M} \Omega + k_0 \int_{\mathcal{K}_\Sigma} \omega$$

and

$$A_{k_0} \bullet \big([0,1] \times [h] + c[\mathcal{K}'_\Sigma]\big) = C_M \bullet [h] + k_0 c,$$

so (3.20) is satisfied if and only if

$$c = \int_{\mathcal{K}_\Sigma} \omega + \frac{1}{k_0} \left(\int_{C_M} \Omega - C_M \bullet [h] \right).$$

This is positive without loss of generality since $\int_{\mathcal{K}_\Sigma} \omega$ was assumed to be arbitrarily large. □

The above argument also shows that if $\{0\} \times M \subset (W, \omega)$ is concave, then ω can never be exact if $[\gamma_1] + \ldots + [\gamma_N] \in H_1(M)$ is torsion, even without assuming $\int_{\mathcal{K}_\Sigma} \omega$ to be arbitrarily large. Indeed, in this case we have $\Omega = d\lambda$ for a contact form λ on (M, ξ), and $[h] = 0$, hence

$$\int_{A_{k_0}} \omega = \int_{C_M} d\lambda + k_0 \int_{\mathcal{K}_\Sigma} \omega = k_0 \sum_{j=1}^N \int_{\gamma_j} \lambda + k_0 \int_{\mathcal{K}_\Sigma} \omega > 0,$$

and

$$c = \int_{\mathcal{K}_\Sigma} \omega + \sum_{j=1}^N \int_{\gamma_j} \lambda > 0.$$

On the other hand if $[\gamma_1] + \ldots + [\gamma_N] \in H_1(M)$ is not torsion, then $H_2(M)$ generates everything in $H_2(W)$, so $\int_A \omega$ always vanishes since Ω is exact. This proves the first half of Theorem $5'$.

We also conclude from the above that if $\{0\} \times M \subset (W, \omega)$ is concave, then there is a constant $c > 0$ such that

$$\mathrm{PD}([\Omega']) = c[\partial \mathcal{K}'_\Sigma] \in H_1(M'; \mathbb{R}),$$

so the second half of the theorem is proved by showing that $[\partial \mathcal{K}'_\Sigma] = 0 \in H_1(M'; \mathbb{R})$ if and only if the stated homological condition on $\gamma_1, \ldots, \gamma_N$ is satisfied. Writing $M' = (M \setminus \mathcal{N}(B_0)) \cup (-\Sigma \times S^1)$, we obtain the Mayer-Vietoris sequence

$$\ldots \to H_2(M') \to H_1(\partial \mathcal{N}(B_0)) \to H_1(M \setminus B_0) \oplus H_1(\Sigma \times S^1) \to \ldots,$$

where $H_1(\partial \mathcal{N}(B_0)) \cong \mathbb{Z}^{2N}$, with each component $\partial \mathcal{N}(\gamma_j)$ carrying the two distinguished generators μ_j, λ_j defined in §1. Denote the inclusions $\iota^M : \partial \mathcal{N}(B_0) \to M \setminus B_0$ and $\iota^\Sigma : \partial \mathcal{N}(B_0) \to \Sigma \times S^1$. Then $\iota^\Sigma_* \lambda_j = [\partial_j \Sigma \times \{*\}] \in H_1(\Sigma \times S^1)$ and $\iota^\Sigma_* \mu_j = [\{*\} \times S^1] \in H_1(\Sigma \times S^1)$, so $\ker \iota^\Sigma_*$ consists of all classes of the form

$$k \sum_{j=1}^N \lambda_j + \sum_{j=1}^N m_j \mu_j$$

with $k, m_1, \ldots, m_N \in \mathbb{Z}$ and $\sum_j m_j = 0$. Now, $[\partial \mathcal{K}'_\Sigma]$ is represented by the cycle $\{*\} \times S^1 \subset \Sigma \times S^1 \subset M'$, and it vanishes in $H_1(M'; \mathbb{R})$ if and only if

$$A \bullet [\{*\} \times S^1] = 0$$

for every $A \in H_2(M')$. This is true if and only if the image of the map $H_2(M') \to H_1(\partial \mathcal{N}(B_0))$ in the above sequence contains only cycles of the form $\sum_j m_j \mu_j$. In light of the above description of $\ker \iota_*^\Sigma$, this is true if and only if

$$k(\lambda_1 + \ldots + \lambda_N) \notin \ker \iota_*^M$$

for all $k \neq 0$. This completes the proof of Theorem 5′.

The proof of Theorem 6′ proceeds similarly: Assume $W = ([0,1]\times M) \cup \widehat{\mathcal{H}}_\Sigma$ is a Σ-decoupling cobordism, with $\widehat{\mathcal{H}}_\Sigma = -\Sigma \times \mathbb{A}$ attached along a neighborhood $\mathcal{N}(\mathcal{I}_0)$ of $\mathcal{I}_0 = T_1 \cup \ldots \cup T_N$, write $\partial W = M' \sqcup (-M)$, $\Omega := \omega|_{TM}$ and $\Omega' := \omega|_{TM'}$. We again assume that $\int_{\widehat{\mathcal{K}}_\Sigma} \omega$ is arbitrarily large, and that Ω takes the form of (3.17), and we also impose the extra condition

$$\int_{T_j} \Omega = 0 \quad \text{for every component } T_j \subset \mathcal{I}_0.$$

In this case we can find a real 1-cycle h in $M \setminus \mathcal{N}(\mathcal{I}_0)$ that represents $\mathrm{PD}([\Omega]) \in H_1(M;\mathbb{R})$. Without changing the cohomology class or the symplectic properties of ω, we can then also assume that Ω_0 is supported in a tubular neighborhood of the cycle h.

Recall from §1 that each oriented torus $T_j \subset \mathcal{I}_0$ comes with a distinguished homology basis $\{\mu_j, \lambda_j\} \subset H_1(T_j)$, where λ_j is a boundary component of a page and μ_j is represented by a Legendrian loop in T_j. This also gives rise to bases $\{\mu_j^\pm, \lambda_j^\pm\}$ of $H_1(\partial_\pm \mathcal{N}(T_j))$, where the orientation of μ_j^- is reversed compared with μ_j. For $W = ([0,1] \times M) \cup \widehat{\mathcal{H}}_\Sigma$ we have the Mayer-Vietoris sequence

$$\ldots \to H_2(M) \oplus H_2(\widehat{\mathcal{H}}_\Sigma) \to H_2(W) \to H_1(\mathcal{N}(\mathcal{I}_0)) \to H_1(M) \oplus H_1(\widehat{\mathcal{H}}_\Sigma) \to \ldots$$

and resulting isomorphism

$$\begin{aligned} H_2(W) \cong \operatorname{im}\big(H_2(M) \oplus H_2(\widehat{\mathcal{H}}_\Sigma) \to & H_2(W)\big) \\ & \oplus \ker\big(H_1(\mathcal{N}(\mathcal{I}_0)) \to H_1(M) \oplus H_1(\widehat{\mathcal{H}}_\Sigma)\big). \end{aligned} \tag{3.21}$$

Denote the generator of $H_1(\mathbb{A}) = \mathbb{Z}$ by $[S^1]$, which can also naturally be regarded as a primitive class in $H_1(\widehat{\mathcal{H}}_\Sigma) = H_1(\Sigma) \oplus H_1(\mathbb{A})$. Then writing the inclusions as $\iota^M : \mathcal{N}(\mathcal{I}_0) \hookrightarrow M$ and $\iota^\Sigma : \mathcal{N}(\mathcal{I}_0) \hookrightarrow \widehat{\mathcal{H}}_\Sigma$, we have $\iota_*^\Sigma(\lambda_j) = [\partial_j \Sigma \times \{*\}]$ and $\iota_*^\Sigma(\mu_j) = [S^1]$, hence $\ker \iota_*^\Sigma$ consists of all classes of the form

$$k \sum_{j=1}^N \lambda_j + \sum_{j=1}^N m_j \mu_j$$

for $k, m_1, \ldots, m_N \in \mathbb{Z}$ with $\sum_j m_j = 0$. For any $\Gamma \in H_1(\mathcal{I}_0)$ of this form which is also nullhomologous in M, we can form a cycle $A_\Gamma \in H_2(W)$ as follows. First choose a 2-chain C_M in $\{0\} \times M$ with $\partial C_M = \Gamma$. Choose also a 1-chain ℓ in Σ with boundary in $\partial\Sigma$ such that the 2-chain $\ell \times \{*\} \times S^1$ in $\widehat{\mathcal{H}}_\Sigma$ has boundary

$$\partial(\ell \times \{*\} \times S^1) = -\sum_{j=1}^N m_j \mu_j,$$

which is always possible since $\sum_j m_j = 0$. We can represent ℓ by a properly immersed submanifold in Σ so that by Remark 3.11, $\int_{\ell \times \{*\} \times S^1} \omega = 0$. Now extend $\ell \times \{*\} \times S^1$ to a 2-chain in W with boundary in $\{0\} \times M$ by attaching trivial cylinders over the appropriate covers of Legendrian representatives of μ_j. Since these cylinders are Lagrangian, this construction yields an immersed submanifold $L_\ell \subset W$ which satisfies

$$\int_{L_\ell} \omega = 0 \tag{3.22}$$

and $\partial L_\ell \subset \mathcal{I}_0 \subset \{0\} \times M$, with $[\partial L_\ell] = -\sum_j m_j \mu_j \in H_1(\mathcal{I}_0)$. We define $A_\Gamma \in H_2(W)$ by

$$A_\Gamma = C_M + [L_\ell] + k[\widehat{\mathcal{K}}_\Sigma]. \tag{3.23}$$

Proposition 3.17. *There is a number $c > 0$ such that* $\mathrm{PD}([\omega]) = [0,1] \times [h] + c[\widehat{\mathcal{K}}'_\Sigma] \in H_2(W, \partial W; \mathbb{R})$.

Proof. The goal is again to prove

$$\int_A \omega = A \bullet \Big([0,1] \times [h] + c[\widehat{\mathcal{K}}'_\Sigma]\Big) \tag{3.24}$$

for every $A \in H_2(W)$, and it is again immediate if $A \in \operatorname{im}(H_2(M) \to H_2(W))$. It is also clear for $A \in \operatorname{im}(H_2(\widehat{\mathcal{H}}_\Sigma) \to H_2(W))$, as $H_2(\widehat{\mathcal{H}}_\Sigma)$ is generated by classes of the form $\ell' \times [S^1]$ for $\ell' \in H_1(\Sigma)$, hence both sides of (3.24) vanish (see Remark 3.11).

The rest of $H_2(W)$ is generated by classes of the form A_Γ defined in (3.23), for which

$$\int_{A_\Gamma} \omega = \int_{C_M} \Omega + k \int_{\widehat{\mathcal{K}}_\Sigma} \omega$$

in light of (3.22). Similarly, L_ℓ does not intersect either $[0,1] \times h$ or $\widehat{\mathcal{K}}'_\Sigma$, thus

$$A_\gamma \bullet \Big([0,1] \times [h] + c[\widehat{\mathcal{K}}'_\Sigma]\Big) = C_M \bullet [h] + kc,$$

and (3.24) is thus satisfied if and only if

$$c = \int_{\widehat{\mathcal{K}}_\Sigma} \omega + \frac{1}{k}\left(\int_{C_M} \Omega - C_M \bullet [h]\right),$$

which is positive if $\int_{\widehat{\mathcal{K}}_\Sigma} \omega$ is made sufficiently large. To see that this formula for c doesn't depend on any choices, observe that if Ω is exact, then $h = 0$ and $\Omega = C\, d\lambda$, so

$$\int_{C_M} \Omega - C_M \bullet [h] = C \int_{\partial C_M} \lambda$$

is proportional to k, as the integral of λ vanishes on all the meridians μ_j. When Ω is not exact but equals $C\, d\lambda + \Omega_0$ with Ω_0 supported in a tubular neighborhood of h, we can find a real homology class $B \in H_2(M;\mathbb{R})$ with $B \bullet [h] = C_M \bullet [h]$ and thus define a real 2-chain

$$C'_M := C_M - B$$

with $\partial C'_M = \partial C_M$ and $C'_M \bullet [h] = 0$. Then up to relative homology, C'_M can be represented by a real linear combination of immersed surfaces that have no geometric intersection with h, hence $\int_{C'_M} \Omega_0 = 0$. Now since $\int_B \Omega = B \bullet [h]$,

$$\int_{C_M} \Omega - C_M \bullet [h] = \int_{C'_M} \Omega = C \int_{C'_M} d\lambda = C \int_{\partial C_M} \lambda,$$

and this is again proportional to k. □

If $\{0\} \times M \subset (W, \omega)$ is concave, then writing $h = 0$ and $\Omega = d\lambda$ gives

$$\int_{A_\Gamma} \omega = \int_\Gamma \lambda + k \int_{\widehat{\mathcal{K}}_\Sigma} \omega$$

for any cycle $\Gamma = k(\lambda_1 + \ldots + \lambda_N) + \sum_j m_j \mu_j \in H_1(\mathcal{I}_0)$ with $\sum_j m_j = 0$ that is nullhomologous in M. Since $\int_\Gamma \lambda$ is also positively proportional to k, this proves that ω is exact if and only if there is no such nullhomologous cycle Γ with $k > 0$. Moreover, $\mathrm{PD}([\Omega']) = c[\partial\widehat{\mathcal{K}}'_\Sigma] \in H_1(M'; \mathbb{R})$ for some $c > 0$, so it remains to characterize the situations where this homology class vanishes. Write $M' = (M \setminus \mathcal{N}(\mathcal{I}_0)) \cup (-\Sigma_+ \times S^1) \cup (-\Sigma_- \times S^1)$ and consider the resulting Mayer-Vietoris sequence

$$\ldots \to H_2(M') \to H_1(\partial\mathcal{N}(\mathcal{I}_0)) \to H_1(M \setminus \mathcal{I}_0) \oplus H_1((\Sigma_+ \sqcup \Sigma_-) \times S^1) \to \ldots,$$

where $H_1(\partial\mathcal{N}(\mathcal{I}_0))$ is freely generated by the $4N$ cycles $\mu_j^\pm, \lambda_j^\pm$. Denote the inclusions $\iota^M : \partial\mathcal{N}(\mathcal{I}_0) \to M \setminus \mathcal{I}_0$ and $\iota^\Sigma : \partial\mathcal{N}(\mathcal{I}_0) \to (\Sigma_+ \sqcup \Sigma_-) \times S^1$, where the latter maps $\partial_\pm\mathcal{N}(T_j)$ into $\Sigma_\pm \times S^1$. Then $\iota_*^\Sigma \lambda_j^\pm = [\partial_j \Sigma \times \{*\}] \in H_1(\Sigma_\pm \times S^1)$ and $\iota_*^\Sigma \mu_j^\pm = [\{*\} \times S^1] \in H_1(\Sigma_\pm \times S^1)$, thus $\ker \iota_*^\Sigma$ consists of all classes of the form

$$k_+ \sum_{j=1}^N \lambda_j^+ + k_- \sum_{j=1}^N \lambda_j^- + \sum_{j=1}^N m_j^+ \mu_j^+ + \sum_{j=1}^N m_j^- \mu_j^-, \tag{3.25}$$

with $k_\pm, m_1^\pm, \ldots, m_N^\pm \in \mathbb{Z}$ satisfying $\sum_j m_j^+ = \sum_j m_j^- = 0$. The co-core $\widehat{\mathcal{K}}'_\Sigma$ has two boundary components, one generating each of the cycles $\{*\} \times S^1 \subset \Sigma_\pm \times S^1 \subset M'$, which we will denote by $[S^1]_\pm \in H_1(M')$. Thus $[\partial\widehat{\mathcal{K}}'_\Sigma]$ vanishes in $H_1(M'; \mathbb{R})$ if and only if

$$A \bullet ([S^1]_+ + [S^1]_-) = 0$$

for every $A \in H_2(M')$. This is true if and only if the image of the map $H_2(M') \to H_1(\partial\mathcal{N}(\mathcal{I}_0))$ in the above sequence contains only cycles of the form (3.25) with $k_+ + k_- = 0$, meaning that cycles of this form with $k_+ + k_- \neq 0$ are never trivial in $H_1(M \setminus \mathcal{I}_0)$.

We've now characterized the cases in which Ω' is globally exact on M'; of course this never happens if $M'_{\text{flat}} \neq \emptyset$ since the latter then contains closed pages on which Ω' is positive. If both M'_{convex} and M'_{flat} are nonempty, then the interesting question is when Ω' will be exact on M'_{convex}, which is the case if and only if

$$[\partial\widehat{\mathcal{K}}'_\Sigma \cap M'_{\text{convex}}] = 0 \in H_1(M'_{\text{convex}}; \mathbb{R}).$$

Assuming the labels are chosen so that $\Sigma_+ \times S^1 \subset M'_{\text{convex}}$ and $\Sigma_- \times S^1 \subset M'_{\text{flat}}$, $[\partial\widehat{\mathcal{K}}'_\Sigma \cap M'_{\text{convex}}]$ is now represented by $\{*\} \times S^1 \subset \Sigma_+ \times S^1$, and a repeat of the above argument shows that this cycle vanishes if and only if $H_1(\partial\mathcal{N}(\mathcal{I}_0))$ contains no cycle of the form (3.25) which vanishes in $H_1(M \setminus \mathcal{I}_0)$ and has $k_+ \neq 0$. We observe however that in this situation, $\mathcal{I}_0$ must separate M so that each $\partial_+\mathcal{N}(T_j)$ lies in a different connected component of $M \setminus \mathcal{I}_0$ from each $\partial_-\mathcal{N}(T_i)$, hence a cycle of the form (3.25) vanishes in $H_1(M \setminus \mathcal{I}_0)$ if and only if both the plus and minus parts vanish. Our condition is thus reduced to the nonexistence of a cycle

$$k \sum_{j=1}^N \lambda_j^+ + \sum_{j=1}^N m_j \mu_j^+$$

with $k \neq 0$ that vanishes in $H_1(M \setminus \mathcal{I}_0)$.

3.7. Proofs of the results from §1.1.

Proofs of Theorems 1 and 2. To prove Theorem 2, suppose (M, ξ) contains an Ω-separating planar k-torsion domain M_0 for some closed 2-form Ω with $\Omega|_\xi > 0$ and an integer $k \geq 1$. Then $\int_T \Omega = 0$ for every interface torus T in M_0 that lies in the planar piece, so we are free to remove any such torus by attaching a $\mathbb{D}$-decoupling cobordism whose symplectic structure matches Ω at M. By Proposition 3.4, one can find a binding component γ or interface torus T such that if (W, ω) with $\partial W = M' \sqcup (-M)$ denotes the result of attaching the corresponding $\mathbb{D}$-capping or $\mathbb{D}$-decoupling cobordism respectively, then M' contains a planar torsion domain of order either $k-1$ or $k-2$. Writing $\Omega' := \omega|_{TM'}$, the latter is also Ω'-separating since near each of the remaining interface tori, which lie outside the region of surgery, ω is still cohomologous to the original Ω. The process can therefore be repeated until the manifold at the top has planar 0-torsion, meaning it is overtwisted.

Theorem 1 is essentially the special case of Theorem 2 for which we assume Ω is exact to start with, except that the above argument actually gives a *weak* symplectic cobordism (W, ω) from (M, ξ) to some overtwisted $(M_{\text{OT}}, \xi_{\text{OT}})$, where we can assume $M \subset (W, \omega)$ is concave and $M_{\text{OT}} \subset (W, \omega)$ is not necessarily convex, but $\omega|_{\xi_{\text{OT}}} > 0$. This can now be turned into a strong cobordism by the following trick which was suggested to me by David Gay: first, observe that if M_{OT} is a rational homology sphere, then ω is exact near M_{OT} and can thus be deformed to make M_{OT} convex using the argument of Eliashberg in [Eli91, Proposition 3.1]. Otherwise, take any knot $K \subset M_{\text{OT}}$ that is nontrivial in $H_1(M_{\text{OT}}; \mathbb{Q})$, and isotop it if necessary so that it is disjoint from some overtwisted disk. Then after a C^0-small perturbation to make K Legendrian, one can attach a symplectic 2-handle along K so that the new positive boundary becomes an overtwisted contact manifold $(M'_{\text{OT}}, \xi'_{\text{OT}})$ with $\dim H_1(M'_{\text{OT}}; \mathbb{Q}) = \dim H_1(M_{\text{OT}}; \mathbb{Q}) - 1$, see Lemma 3.18 below. Repeating this process enough times, the positive boundary eventually becomes an overtwisted rational homology sphere, so that the weak cobordism can be deformed to a strong one. □

The final step in the above proof is justified by the following simple homological lemma:

Lemma 3.18. *Suppose M is a closed oriented 3-manifold, $K \subset M$ is a knot with $[K] \neq 0 \in H_1(M; \mathbb{Q})$ and M' is the result of performing Dehn surgery along K with any framing. Then $\dim H_1(M'; \mathbb{Q}) = \dim H_1(M; \mathbb{Q}) - 1$.*

Proof. As preparation, suppose K is any knot in a closed oriented 3-manifold M, denote a neighborhood of K in M by N_K and let (μ, λ) denote any basis of $H_1(\partial N_K)$ such that μ is a meridian. If $\iota : \partial N_K \hookrightarrow M \setminus K$ denotes the inclusion, we claim that

$$\dim H_1(M; \mathbb{Q}) = \begin{cases} \dim H_1(M \setminus K; \mathbb{Q}) & \text{if } \iota_*\mu = 0 \in H_1(M \setminus K; \mathbb{Q}), \\ \dim H_1(M \setminus K; \mathbb{Q}) - 1 & \text{if } \iota_*\mu \neq 0 \in H_1(M \setminus K; \mathbb{Q}). \end{cases}$$

This follows from the Mayer-Vietoris sequence for $M = N_K \cup (M \setminus K)$:

$$\ldots H_2(M; \mathbb{Q}) \to H_1(\partial N_K; \mathbb{Q}) \xrightarrow{\Phi} H_1(N_K; \mathbb{Q}) \oplus H_1(M \setminus K; \mathbb{Q}) \to H_1(M; \mathbb{Q}) \to H_0(\partial N_K; \mathbb{Q}) \ldots$$

Since any 1-cycle in M can be disjoined from N_K, the map $H_1(M; \mathbb{Q}) \to H_0(\partial N_K; \mathbb{Q})$ in this sequence is trivial, thus exactness implies

$$H_1(M; \mathbb{Q}) \cong \left(H_1(N_K; \mathbb{Q}) \oplus H_1(M \setminus K; \mathbb{Q})\right) \big/ \operatorname{im} \Phi.$$

The map $\Phi : H_1(\partial N_K;\mathbb{Q}) \to H_1(N_K;\mathbb{Q}) \oplus H_1(M \setminus K;\mathbb{Q})$ is nontrivial since λ maps to the generator of $H_1(N_K;\mathbb{Q}) = \mathbb{Q}$. Since μ maps to 0 in $H_1(N_K;\mathbb{Q})$, $\operatorname{im}\Phi$ is thus 1-dimensional if and only if $\iota_*\mu = 0 \in H_1(M \setminus K;\mathbb{Q})$, and is otherwise 2-dimensional. This proves the claim.

Now assume $[K] \neq 0 \in H_1(M;\mathbb{Q})$, and we are given a framing such that λ is the preferred longitude. This implies immediately that $\iota_*\lambda \neq 0 \in H_1(M \setminus K;\mathbb{Q})$. Likewise, $\iota_*\mu = 0 \in H_1(M \setminus K;\mathbb{Q})$: to see this, note that by the nondegeneracy of the intersection form, there exists a 2-cycle C in M such that $[C] \bullet [K] = 1$, hence the restriction of C to the complement of N_K defines a chain whose boundary is μ; alternatively, one can also derive this from the exact sequence above by considering the image of $[C]$ under $H_2(M;\mathbb{Q}) \to H_1(\partial N_K;\mathbb{Q})$. We therefore have $\dim H_1(M;\mathbb{Q}) = \dim H_1(M \setminus K;\mathbb{Q})$ by the claim above. If M' is now defined by gluing another solid torus into $M \setminus N_K$ such that λ becomes the meridian, then the claim is again applicable and implies $\dim H_1(M';\mathbb{Q}) = \dim H_1(M \setminus K;\mathbb{Q}) - 1$. □

Proof of Theorem 3. Suppose (M,ξ) contains an Ω-separating partially planar domain $M_0 \subset M$ with planar piece $M_0^P \subset M_0$, where Ω is a closed 2-form on M satisfying $\Omega|_\xi > 0$. Then for every binding circle or interface torus in M_0^P, we can attach $\mathbb{D}$-capping or $\mathbb{D}$-decoupling cobordisms to produce a symplectic manifold (W,ω) with $\partial W = M' \sqcup (-M)$, $\omega|_{TM} = \Omega$ and

$$M' = M'_{\text{flat}} \sqcup M'_{\text{convex}},$$

where M'_{flat} contains a component that is a symplectic S^2-fibration over S^1, and M'_{convex} carries a contact structure ξ' with $\omega|_{\xi'} > 0$. The desired cap is then obtained from (W,ω) after capping $M'_{\text{flat}} \sqcup M'_{\text{convex}}$ via [Eli04] or [Etn04a]. □

Proof of Theorem 4. Note that since $H^2_{\text{dR}}(S^3) = 0$, any weak cobordism from (M,ξ) to (S^3,ξ_0) that is concave at M can be deformed to a strong cobordism, so it suffices to prove that (S^3,ξ_0) can be obtained from (M,ξ) by a finite sequence of (generally weak) capping and decoupling cobordisms.

Suppose $M_0 \subset M$ is a partially planar domain. If it is also a planar torsion domain then the result already follows from Corollary 1, thus assume not. If M_0 has only one irreducible subdomain with nonempty binding, we can remove binding components by $\mathbb{D}$-capping cobordisms and interface tori by $\mathbb{D}$-decoupling cobordisms until the planar piece has exactly one binding component left and no interface or boundary, which means it is the tight S^3. The desired cobordism can then be obtained by capping any additional components that may remain at the end of this process.

If M_0 has more than one irreducible subdomain but does not have planar torsion, then it must be symmetric (see Definition 3.2). This means that $M = M_0$, the binding and boundary are empty and the interface tori divide M into exactly two irreducible subdomains that have diffeomorphic planar pages. Then we can remove interface tori by $\mathbb{D}$-decoupling cobordisms until exactly one remains, and the resulting contact manifold is the tight $S^1 \times S^2$. The latter is cobordant to S^3 by a $\mathbb{D}$-capping cobordism that removes one binding component from the supporting open book with cylindrical pages and trivial monodromy.

Theorem 4′ follows essentially by the same argument since every planar open book is also a fully twisted partially planar domain. We only need to add that the topological construction of the cobordism by attaching 2-handles along binding components does not depend on the choice of 2-form Ω on M, which after the deformation carried out in §3.5, always looks the same on a large tubular neighborhood of the binding. □

3.8. **Holomorphic curves.** For applications to Embedded Contact Homology and Symplectic Field Theory among other things, it may be quite helpful to observe that the cobordism (W,ω) generally admits not only a symplectic structure but also a foliation by J-holomorphic curves. We don't plan to pursue this here in full detail, but we shall give a sketch of the general picture. For simplicity, we consider only the Σ-decoupling cobordism along $\mathcal{I}_0 = T_1 \cup \ldots \cup T_N$ in the case where the negative boundary component (M,ξ) is concave, so we can arrange ω near $\{0\} \times M$ to have the form

$$\omega = d\left(\varphi(t)\lambda_0 + \lambda\right) \tag{3.26}$$

as in §3.3, where λ_0 and λ are the confoliation 1-form and contact form respectively that were constructed in §3.2. These have the following convenient properties:

- $\lambda_0 \wedge d\lambda > 0$
- $\ker d\lambda \subset \ker d\lambda_0$

Together with the obvious fact that $d\lambda$ is closed, these properties mean that the pair $(\lambda_0, d\lambda)$ is a **stable Hamiltonian structure**, to which we associate the co-oriented distribution $\xi_0 = \ker\lambda_0$ and positively transverse vector field X_0 on M such that

$$\lambda_0(X_0) \equiv 1, \qquad d\lambda(X_0,\cdot) \equiv 0.$$

Similarly, writing $\Omega' := \omega|_{TM'}$ and recalling the confoliation 1-form λ_0' defined on M' by (3.13), the pair (λ_0', Ω') forms a stable Hamiltonian structure on M', and we define the corresponding distribution $\xi_0' = \ker\lambda_0'$ and vector field X_0' on M' such that

$$\lambda_0'(X_0') \equiv 1, \qquad \Omega'(X_0',\cdot) \equiv 0.$$

In fact, on components of M' where the pages are not closed, one can show with a little more effort that λ_0' admits a perturbation to a contact form λ' on M' such that $\xi' := \ker\lambda'$ is dominated by Ω', where

$$\Omega' = F\, d\lambda'$$

for some smooth function $F : M' \to (0,\infty)$ that satisfies $dF \wedge d\lambda' \equiv 0$ and is constant outside a neighborhood of the boundary of the co-core, $\partial\widehat{\mathcal{K}}_\Sigma' \subset M'$, hence X_0' is colinear with the Reeb vector field determined by λ'. There is now a collar neighborhood $(-\epsilon, 0] \times M' \subset W$ of M' on which ω takes the form $d(t\lambda_0') + \Omega'$, thus we can attach positive and negative cylindrical ends to define the **completion** of (W,ω),

$$W^\infty := W \cup ((-\infty,0] \times M) \cup \left([0,\infty) \times M'\right)$$

and extend ω symplectically so that it takes the form $d(\varphi(t)\,\lambda_0) + d\lambda$ on $(-\infty,0] \times M$ and $d(\psi(t)\,\lambda_0') + \Omega'$ on $[0,\infty) \times M'$ for suitable choices of functions φ and ψ. Equivalently, (W^∞,ω) can be constructed directly from the symplectization of (M,ξ) as follows: extend the function $\varphi(t)$ to $\mathbb{R}$ so that (3.26) defines a symplectic form on $\mathbb{R} \times M$, and extend the "hole" $\mathcal{U}_{\mathcal{I}_0}$ defined in §3.3 to a hole in $\mathbb{R} \times M$ by including the interior of the region $(1,\infty) \times \mathcal{N}(\mathcal{I}_0)$; let

$$\mathcal{U}^\infty_{\mathcal{I}_0} \subset \mathbb{R} \times M$$

denote the extended hole. Then (W^∞,ω) can be obtained by removing $\mathcal{U}^\infty_{\mathcal{I}_0}$ from $(\mathbb{R} \times M, d(\varphi(t)\lambda_0 + \lambda))$ and replacing it with the completion of the round handle,

$$\widehat{\mathcal{H}}^\infty_\Sigma := \Sigma \times (-\infty,\infty) \times S^1,$$

extending the symplectic form over $\widehat{\mathcal{H}}^\infty_\Sigma$ by an adaptation of the argument in §3.3.

An almost complex structure J on (W^∞, ω) is now **admissible** if it is ω-compatible on W and is compatible with the stable Hamiltonian structures on both cylindrical ends, meaning it is $\mathbb{R}$-invariant, restricts to a complex structure on the respective distributions ξ_0 and ξ_0' defining the correct orientations, and maps the unit vector ∂_t in the $\mathbb{R}$-direction to the vector field X_0 or X_0'.

It was shown in [Wen] that for a suitable choice of almost complex structure J_0 on $\mathbb{R} \times M$ compatible with $(\lambda_0, d\lambda)$, the pages of the blown up summed open book $\boldsymbol{\pi}$ in M_0 admit lifts to embedded J_0-holomorphic curves in $\mathbb{R} \times M$ which match the fibers of the mapping torus S_ψ outside of the neighborhoods $\mathcal{N}_i$ of $B \cup \mathcal{I} \cup \partial M_0$, have positive cylindrical ends approaching closed orbits of X_0 in $B \cup \mathcal{I} \cup \partial M_0$ and satisfy a suitable finite energy condition. We can now define an admissible almost complex structure on (W^∞, ω) which matches J_0 outside of $\mathcal{U}^\infty_{\mathcal{I}_0}$ and is ω-compatible on $\widehat{\mathcal{H}}^\infty_\Sigma$. The J_0-holomorphic curves in $(\mathbb{R} \times M) \setminus \mathcal{U}^\infty_{\mathcal{I}_0}$ can be extended into $\widehat{\mathcal{H}}^\infty_\Sigma$ as symplectic surfaces that are diffeomorphic to Σ and foliate $\widehat{\mathcal{H}}^\infty_\Sigma$, thus we can extend J_0 into the handle so that it is ω-compatible and these surfaces become J_0-holomorphic. In doing this, we can also make the natural completion of the core

$$\widehat{\mathcal{K}}_{\Sigma,\infty} := \widehat{\mathcal{K}}_\Sigma \cup_{\widehat{B}_0} ((-\infty, 0] \times \widehat{B}_0)$$

J_0-holomorphic, as well as its translations under the S^1-action by translating the local ϕ-coordinates, and the completion of the co-core

$$\widehat{\mathcal{K}}'_{\Sigma,\infty} := \widehat{\mathcal{K}}'_\Sigma \cup_{\partial \widehat{\mathcal{K}}'_\Sigma} ([1, \infty) \times \partial \widehat{\mathcal{K}}'_\Sigma).$$

The result is a foliation of W^∞ (or at least the region outside of $\mathbb{R} \times (M \setminus M_0)$) by finite energy J_0-holomorphic curves. We summarize this construction as follows (see Figure 7).

Proposition 3.19. *One can choose an admissible almost complex structure J_0 on the completion (W^∞, ω) of a Σ-decoupling cobordism (W, ω), such that there exists a foliation $\mathcal{F}$ by embedded J_0-holomorphic curves with the following properties:*

(1) *In each cylindrical end, the leaves of $\mathcal{F}$ match the holomorphic lifts of the pages of $\boldsymbol{\pi}$ and $\boldsymbol{\pi}'$ constructed in* [Wen].
(2) *The completed core $\widehat{\mathcal{K}}_{\Sigma,\infty}$ and all its S^1-translations are leaves of $\mathcal{F}$.*
(3) *The trivial cylinders over periodic orbits in B, ∂M_0 and $\mathcal{I} \setminus \mathcal{I}_0$ are all leaves of $\mathcal{F}$.*
(4) *All other leaves of $\mathcal{F}$ have only positive cylindrical ends asymptotic to orbits in $B \cup (\mathcal{I} \setminus \mathcal{I}_0) \cup \partial M$, and they are homotopic in the moduli space to the holomorphic lifts of the pages of $\boldsymbol{\pi}'$ in $[1, \infty) \times M'$.*
(5) *The completed co-core $\widehat{\mathcal{K}}'_{\Sigma,\infty}$ is also J_0-holomorphic and intersects the leaves of $\mathcal{F}$ transversely.*

In considering the behavior of holomorphic curve invariants under symplectic cobordisms, a special role is typically played by curves that have no positive ends—such curves can only exist in non-exact cobordisms. One useful application of the foliation constructed above is that we can now characterize all such curves precisely:

Proposition 3.20. *Suppose $u : \dot{\Sigma} \to W^\infty$ is a finite energy J_0-holomorphic curve that is connected, somewhere injective and has no positive ends. Then u is a leaf of $\mathcal{F}$, specifically it is an S^1-translation of the core $\widehat{\mathcal{K}}_{\Sigma,\infty}$.*

Proof. There are no curves without positive ends outside the region of surgery since here the symplectic form is exact, thus we may assume u intersects both the handle and its complement.

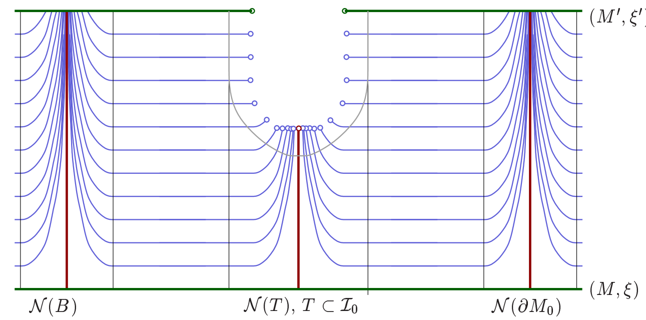

FIGURE 7. The J_0-holomorphic foliation described by Proposition 3.19 for a Σ-decoupling cobordism, not including the cylindrical ends. The picture shows the t and ρ coordinates near various binding, interface and boundary components, including an interface torus T where a handle $\Sigma \times \mathbb{A}$ has been attached. The circles at the ends of leaves in this region represent capping by Σ. The core $\widehat{\mathcal{K}}_{\Sigma,\infty}$ is shown as the vertical leaf directly in the center, which emerges from $-\infty$ and is capped off in the handle.

If u is a leaf of $\mathcal{F}$ then it must be an S^1-translation of the core, as all other leaves have positive ends. If it is not a leaf of $\mathcal{F}$ then it has a positive intersection with some leaf v, and without loss of generality we may suppose that v has only positive ends. Then v is homotopic in the moduli space to a holomorphic lift of a page of $\boldsymbol{\pi}'$, which we may assume lives in the region $[c, \infty) \times M'$ for an arbitrarily large number $c > 0$ and thus cannot intersect u. This is a contradiction, due to positivity of intersections. □

To apply these constructions to ECH, or to Symplectic Field Theory for that matter, one must perturb ξ_0 and ξ_0' to contact structures and perturb J_0 with them. The J_0-holomorphic leaves of $\mathcal{F}$ will not generally behave well under this perturbation: a leaf with only positive ends for instance, if it has genus g, will have Fredholm index $2 - 2g$ and thus disappears under any generic perturbation of the data unless $g = 0$. Proposition 3.20, however, should still hold under a sufficiently small perturbation, because for any sequence J_k of perturbed almost complex structures converging to J_0, a sequence of J_k-holomorphic curves should converge in the sense of [BEH+03] to a J_0-holomorphic building, and Proposition 3.20 determines what this building can look like. This is a variation on the uniqueness argument used in [Wen] to prove vanishing of the ECH contact invariant: higher genus holomorphic curves do not generically exist, but they remain useful for proving that no other curves can exist either.

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