NON-EXACT SYMPLECTIC COHORDISMS BETWEEN CONTACT 3-MANIFOLDS

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Abstract. We show that the pre-order defined on the category of contact 3-manifolds by arbitrary symplectic cobordisms is considerably less rigid than its counterparts for exact or Stein cobordisms; in particular, we exhibit large new classes of contact 3-manifolds which are symplectically cobordant to something overtwisted, or to the tight 3-sphere, or which admit symplectic fillings containing symplectically embedded spheres with vanishing self-intersection. These constructions imply new and simplified proofs of several recent results involving flexibility, planarity and non-separating contact type embeddings. The cobordisms are built from symplectic handles of the form $\Sigma \times \mathbb{S}^1$ and $\Sigma \times [-1,1] \times \mathbb{S}^1$, which have symplectic cores and can be attached to contact 3-manifolds along sufficiently large neighborhoods of transverse links and pre-Lagrangian tori. We also sketch a construction of holomorphic foliations in these cobordisms and formulate a conjecture regarding maps induced on Embedded Contact Homology with twisted coefficients.

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2010 Mathematics Subject Classification. Primary 57R17; Secondary 53D10, 57K20, 53D42.
Research supported by an Alexander von Humboldt Research Fellowship.
and [14] discusses the construction of a holomorphic foliation in the cobordisms, which we expect should have interesting applications in Embedded Contact Homology and for Symplectic Field Theory.

Acknowledgments. The writing of this article benefited substantially from discussions with John Etnyre, David Gay, Michael Hutchings, Klaus Niederkrüger, Andras Stipsicz, Cliff Taubes and Jeremy Van Horn-Morris. Several of these conversations took place at the March 2010 MSRI Workshop Symplectic and Contact Topology and Dynamics: Puzzles and Horizons. I would also like to thank Patrick Massot for suggesting the use of the term “co-fillable,” as well as Paolo Ghigi and an anonymous referee, both of whom read the original version quite carefully and made several suggestions for improving the exposition.

1. Some background and sample results. In topology, an oriented cobordism from one closed oriented manifold $M_+$ to another $M_-$ is a compact oriented manifold $W$ such that $\partial W = M_+ \cup (-M_-)$. If $W$ has dimension $2n$ and also carries a symplectic structure $\omega$, then it is natural to consider the case where $(W,\omega)$ is symplectically convex at $M_+$, and concave at $M_-$. This means that there exists a vector field $Y$ near $\partial W$ which points transversely outward at $M_+$, and inward at $M_-$, and is a Liouville vector field, i.e., $\Lambda Y = \omega$. This case has a symplectic form, meaning it satisfies $\Lambda \wedge (\alpha F) > 0$. The induced contact structure on $M_+$ is the co-oriented symplectic cobordism from $(M_+,\omega)$ to $(M_-,\xi)$, and when such a cobordism exists, we say that $(M_+,\omega)$ is strongly symplectically cobordant to $(M_-,\xi)$. If $W$ also extends to a globally convex primitive of $\omega$, then $Y$ extends to a global Liouville vector field on $W$, when we call $(W,\omega)$ an exact symplectic cobordism from $(M_+,\omega)$ to $(M_-,\xi)$. Whenever $(M_+,\omega)$ and $(M_-,\xi)$ are both connected, we shall abbreviate the existence of a connected symplectic cobordism from $(M_+,\omega)$ to $(M_-,\xi)$ by writing $(M_+,\omega) \sim (M_-,\xi)$ for the general case, and $(M_+,\omega) \prec (M_-,\xi)$ for the exact case 1.

When $\dim W = 4$, it is also interesting to consider a much weaker notion: assuming that $\omega$ is exact near $\partial W$, we call $(W,\omega)$ a weak symplectic cobordism from $(M_+,\omega)$ to $(M_-,\xi)$ if $\xi$ are any two positive co-oriented (and hence also oriented) contact structures such that $\alpha_{\xi} > 0$. Then we say that $\omega$ dominates the contact structures on both boundary components. In order to distinguish strong symplectic cobordisms from this weaker notion, we will sometimes refer to convex/concave boundary components of strong cobordisms as strongly convex/concave.

1 Though contact structures need not be co-extendable in general, all contact structures considered in this paper will be, and we shall regard the co-orientation always as an essential part of the data, though it will usually be suppressed in the notation.

2 The reason for single not connected cobordisms is that technically, every closed contact 3-manifold $(M_+,\xi)$ is symplectically cobordant to the standard contact 3-sphere $(S^3,\xi_s)$, namely via the disjoint union of a symplectic cap for $(M_+\times \{0\},\omega)$ with a symplectic filling of $(S^3,\xi_s)$. We will see however that if the cobordism is required to be connected, then the question of when $(M_+,\xi) \sim (S^3,\xi_s)$ becomes an interesting one, cf. Thm. 1.8.1.
David Gay in [Gay00], that any contact manifold with Giroux torsion at least 2 is cobordant to something overtwisted; as shown in [Ném01], positive Giroux torsion implies planar 1-torsion (cf. [BG]). By a result of Eliyahu andHonda [EH04], every connected overtwisted contact manifold admits a connected Stein cobordism to any other connected contact 3-manifold, and Gay [Gay00] showed that the word “connected” can be removed from this statement at the cost of dropping the Stein condition. We thus have the following consequence:

**Corollary 1.** Every closed connected contact 3-manifold with planar torsion admits a connected strong symplectic cobordism to every other closed contact 3-manifold.

It should be emphasized that due to the obstructions mentioned above, Corollary 1 is not true for exact cobordisms, not even if the positive boundary is required to be connected. In fact, there is no known example of an exact cobordism from anything tight to anything overtwisted, and many examples that are tight but non-fillable (e.g. the 3-torus with positive Giroux torsion) certainly do not admit such cobordisms.

There is also a version of Theorem 1 that implies the mere general obstruction to weak fillings proved in [NW11]. Recall (see Definitions 5.1 and 5.3) that for a given closed 2-form $\Omega$ on a contact 3-manifold $(M,\xi)$, we say that $(M,\xi)$ has $\Omega$-separating planar torsion if it contains a planar torsion domain in which a certain set of embedded 2-tori $T$ all satisfy

$$\int_T \Omega = 0.$$

If this is true for all closed 2-forms $\Omega$, then $(M,\xi)$ is said to have fully separating planar torsion.

**Theorem 2.** Suppose $(M,\xi)$ is a closed contact 3-manifold with $\Omega$-separating planar torsion for some closed 2-form $\Omega$ on $M$ with $\int_M \Omega > 0$. Then there exists a weak symplectic cobordism $(W,\omega)$ from $(M,\xi)$ to an overtwisted contact manifold, with $\partial \omega = \Omega$.

Using a Darboux-type normal form near the boundary, weak symplectic cobordisms can be glued together along contactomorphic boundary components of opposite sign whenever the restrictions of the symplectic forms on the boundaries match (see Lemma 6.1). Thus if $(M,\xi)$ has $\Omega$-separating planar torsion and admits a weak filling $(W,\omega)$ with $\partial \omega = \Omega = [\Omega]$ in $H^{2n}(M)$, then Theorem 2 yields a weak filling of an overtwisted contact manifold, and hence a contradiction due to the well known theorem of Gromov [Grom99] and Eliashberg [Eli99]. We thus obtain a much simplified proof of the following result, which was proved in [NW11] by a direct holomorphic curve argument and also follows from a computation of the twisted ECH contact invariant in [Ném11].

**Corollary (NW11).** If $(M,\xi)$ has $\Omega$-separating planar torsion for some closed 2-form $\Omega$ on $M$, then it does not admit any weak filling $(W,\omega)$ with $\partial \omega = \Omega = [\Omega]$ in $H^{2n}(M)$. In particular, if $(M,\xi)$ has fully separating planar torsion then it is not weakly fillable.

We now state some related results that also apply to fillable contact manifolds. The aforementioned existence result of [EH04] for symplectic caps was generalized independently by Eliashberg [Eli04] and Eliyahu and Honda [EH04] to weak cobordisms: they showed namely that for any $(M,\xi)$ with a closed 2-form $\Omega$ that dominates $\xi$, there is a symplectic cap $(W,\omega)$ with $\partial W = -\hat{M}$ and $\partial \omega = \Omega$. Our next result concerns a large class of contact manifolds for which this cap may be assumed to have a certain very restrictive property.

**Theorem 3.** Suppose $(M,\xi)$ is a contact 3-manifold containing an $\Omega$-separating partially planar domain $M_0 \subset M$ (see Definition 7.1 for some closed 2-form $\Omega$ on $M$ with $\int_M \Omega > 0$). Then $(M,\xi)$ admits a symplectic cap $(W,\omega)$ such that $\omega|_{M_0} = \Omega$ and there exists a symplectically embedded 2-sphere $S \subset W$ with vanishing self-intersection number.

As the work of McDuff [McDuf90] makes clear, symplectic manifolds that contain symplectic spheres of square (are quite special, and for instance any closed symplectic manifold obtained by giving the cap from Theorem 3 a filling of $(M,\xi)$ must be rational or ruled. An easy adaptation of the main result in [McDuf90] also provides the following consequence, which was proved using much harder punctured holomorphic curve arguments in [AusKWH11].

**Corollary 2.** Suppose $(M,\xi)$ contains an $\Omega$-separating partially planar domain for some closed 2-form $\Omega$ on $M$. If $(W,\omega)$ is a closed symplectic 4-manifold and $M$ admits an embedding $i : M \to W$ such that $i^*\omega > 0$ and $[\iota^*\omega] = [\Omega] \in H^2(M)$, then $(M,\xi)$ separates $W$.

Since planar torsion domains are also partially planar domains, this implies that planar torsion is actually an obstruction to contact type embeddings into closed symplectic manifolds, not just symplectic fillings.

Some examples of contact manifolds admitting non-separating embeddings arise from special types of symplectic filling: we shall say that $(M,\xi)$ is (strongly or weakly) co-fillable if there is a connected (strong or weak) filling $(W,\omega)$ whose boundary is the disjoint union of $(M,\xi)$ with an arbitrary non-empty contact manifold. Put another way, $(M,\xi)$ admits a connected semi-filling with disconnected boundary. Given such a filling, one can always attach a symplectic $1$-handle to connect distinct boundary components and then cap off the boundary to realize $(M,\xi)$ as a non-separating contact hypersurface. Various examples of contact manifolds that are are not co-fillable have been known for many years:

- The tight 3-sphere $(S^3,\xi_0)$ is not weakly co-fillable, by arguments due to Gromov [Grom99], Eliashberg [Eli99] and McDuff [McDuf90].
- Entry [Eli04] extended this result to all planar contact manifolds.
- McDuff [McDuf90] showed that for any Riemann surface $S$ of genus at least 2, the unit cotangent bundle $TS^*S$ with its canonical contact structure is strongly co-fillable.
- Further examples were found by Geiges [Gei10] and Mitsuhashi [Mis10].
- Giron [Giron02] showed that every tight contact structure on $T^3$ is weakly co-fillable.

However, none of them are strongly co-fillable, due to a result of the author [Wend11].

All of the negative results just mentioned can be viewed as special cases of Corollary 2 and so can the closely related result in [ABW10] that partially planar contact manifolds never admit non-separating contact type embeddings. Observe that any contact manifold cobordant to one for which Corollary 2 holds also cannot be co-fillable: in particular this shows that not every contact 3-manifold is cobordant to $(S^3,\xi_0)$. We can thus led to an analogue of Question 1 that also applies to fillable contact manifolds: The following theorem gives the answer. The question is again clearly no for exact cobordisms, as a variation on Hofer’s argument from [Hofer95] also shows that $(M,\xi)$ must always admit contractible Reeb orbits if $(M,\xi) \prec (S^3,\xi_0)$. The following result provides some evidence for a positive answer in the non-exact case, though it is not quite as general as one might have hoped. (See also Remmert [Remm75] below for a candidate counterexample.)
Theorem 4. Suppose $(M, \xi)$ is a connected contact 3-manifold containing a partially ordered set of \( \geq 3 \) compact connected 4-manifolds whose dimension is greater than or equal to 3. Then $(M, \xi)$ is a contact 3-manifold containing a partially ordered set of \( \geq 3 \) compact connected 4-manifolds.

In this paper, we show that if $(M, \xi)$ is a connected contact 3-manifold containing a partially ordered set of \( \geq 3 \) compact connected 4-manifolds, then $(M, \xi)$ is a contact 3-manifold containing a partially ordered set of \( \geq 3 \) compact connected 4-manifolds.

The proof of Theorem 4 follows from the fact that the contact structure is uniquely determined by the partially ordered set of\( \geq 3 \) compact connected 4-manifolds.

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with the oriented solid torus $S^1 \times \mathbb{D}$ via this framing so that $\gamma_j = S^1 \times \{0\}$ with the correct orientation and the fibration $\pi$ takes the form
\[ \pi(\theta, \rho, \phi) = \phi \]
on $N(\gamma_j) \setminus \gamma_j$, where $(\rho, \phi)$ denote polar coordinates on the disk, normalized so that $\phi \in S^1 = \mathbb{R}/\mathbb{Z}$. Assign to $\partial N(\gamma_j)$ its natural orientation as the boundary of $N(\gamma_j)$ and denote by
\[ \{\mu_j, \lambda_j\} \subset H_1(\partial N(\gamma_j)) \]
the distinguished positively oriented homology basis for which $\mu_j$ is a meridian and $\lambda_j$ is the longitude determined by the page framing. Denote $N(B_0) = N(\gamma_0) \cup \ldots \cup N(\gamma_N)$.

Now pick a compact, connected and oriented surface $\Sigma$ with $N$ boundary components
\[ \partial \Sigma = \partial_+ \Sigma \cup \ldots \cup \partial_{N-1} \Sigma \]
and choose an orientation preserving diffeomorphism of each $\partial_j \Sigma$ to $S^1$, thus defining a coordinate $s \in S^1$ for $\partial_j \Sigma$. Using this, we define new compact oriented manifolds
\[ M' = (M \setminus N(B_0)) \cup (\Sigma \times S^1), \]
\[ M' = (M \setminus N(B_0)) \cup (\Sigma \times S^1) \]
by gluing in $\Sigma \times S^1$ via orientation reversing diffeomorphisms $\partial \Sigma \times S^1 \to \partial N(\gamma_j)$ that take the form
\[ (s, t) \mapsto (s, 1, t) \]
in the chosen coordinates. On the level of homology, the map $\partial \Sigma \times S^1 \to \partial N(\gamma_j)$ identifies $\partial \Sigma \times \{1\}$ with $\lambda_j$ and $\partial \Sigma \times \{0\}$ for $z \in \partial \Sigma$ with $\mu_j$.

Remark 2.1. In the special case $\Sigma = S^1$, the operation just defined is simply a Dehn surgery along a bounding component $\gamma \subset B$ with framing 0 relative to the page framing.

The fibration $\pi : M \setminus (B \cup \Sigma) \to S^1$ extends smoothly over $\Sigma \times S^1$ as the projection to the second factor, thus $M'_{\Sigma}$ inherits from $\pi$ a natural blown up summed open book $\mathcal{F}_\Sigma$, with binding $B_0 \setminus B_0$, interface $\Sigma$ and pages that are obtained from the pages of $\pi$ by attaching $\Sigma$, giving $\partial \Sigma$ to the boundary component adjacent to $\gamma_j$. We say that $\pi'$ is obtained from $\pi$ by $\Sigma$-capping surgery along $B_0$. If $\pi'$ does not have closed pages, then it supports a contact structure $\xi'$ on $M'_\Sigma$ which can be assumed to match $\xi$ outside of the region of surgery, and thus extends to $M'$.

The $\Sigma$-capping surgery can also be defined by attaching a generalized version of a 4-dimensional 2-handle; define
\[ N_\Sigma := \Sigma \times \mathbb{D}, \]
with boundary
\[ \partial N_\Sigma = \partial^+ N_\Sigma \cup \partial^+ N_\Sigma := \partial (\Sigma \times \mathbb{D}) \cup (\Sigma \times S^1). \]
The above identifications of the neighborhoods $N(\gamma_j)$ with $S^1 \times \mathbb{D}$ yield an identification of $N(B_0)$ with $\partial^+ N_\Sigma = \Sigma \times \mathbb{D}$, which we use to attach $N_\Sigma$ to the trivial cobordism $[0, 1] \times M$ by gluing $\partial^+ N_\Sigma$ to $N(B_0) \subset \{1\} \times M$, defining
\[ W = ([0, 1] \times M) \cup N(B_0) \cup N_\Sigma, \]
which after smoothing the corners has boundary
\[ \partial W = M' \cup (-M). \]

We will refer to the oriented submanifolds
\[ K_\Sigma := ([0, 1] \times B_0) \cup (-\Sigma \times \{0\}) \subset W \]
and
\[ K_\Sigma := \{\rho\} \times \mathbb{D} \subset W \]
for an arbitrary interior point $p \in \Sigma$ as the core and co-core respectively. Note that $\partial K_\Sigma = (-B_0 \cup M_\Sigma, \Sigma)$, $\partial K_\Sigma \subset M_\Sigma$ and $K_\Sigma \cdot K_\Sigma = 1$, where $\cdot$ denotes the algebraic count of intersections.

The following generalizes results in [37BD] and [36BD].

Theorem 5. Suppose $\omega$ is a symplectic form on $[0, 1] \times M$ with $\omega|_{\{1\}} > 0$, and let $W$ denote the handle cobordism defined in (36), after smoothing corners. Then after a symplectic deformation of $\omega$ away from $\{0\} \times M$, $\omega$ can be extended symplectically over $W$ so that it is positive on $K_\Sigma$, $K_\Sigma$, and the pages of $\pi'$. Moreover, if the latter pages are not closed, then $\omega$ also dominates a supported contact structure $\xi'$ on $M'$, thus defining a weak symplectic cobordism from $(M, \xi)$ to $(M', \xi')$.

We will refer to the cobordism $(W, \omega)$ of Theorem 5 henceforward as a $\Sigma$-capping cobordism. In general it is a weak cobordism, but under certain conditions that depend only on the topology of the setup, it can also be made strong. Recall the standard fact, observed originally by Eliashberg [36BD Proposition 3.1] (see also [36BD Prop. 4.1]), that whenever $(W, \omega)$ has a boundary component $M$ on which $\omega$ dominates a positive contact structure $\xi$ and is exact, $\omega$ can be deformed in a collar neighborhood to make $M$ strongly convex, with $\xi$ as the induced contact structure. In 36BD we will use routine Mayer-Vietoris arguments to characterize the situations in which this trick can be applied to the above construction.

Theorem 5'. The symplectic cobordism $(W, \omega)$ constructed by Theorem 4 can be arranged so that the following holds. Choose a real 1-cycle $h \in M \setminus N(B_0)$ such that $[h] \in H_1(M; \mathbb{R})$ is Poincaré dual to the restriction of $\omega$ to $[0] \times M$. Then there is a number $c > 0$ such that
\[ PD([\omega]) = [0, 1] \times [0, c] \subset H_2(W, \partial W; \mathbb{R}), \]
where $PD : H_2(W) \to H_2(W, \partial W; \mathbb{R})$ denotes the Poincaré-Lefschetz duality isomorphism. In particular, if $[0] \times M \subset (W, \omega)$ is (strongly) concave then the following conditions are equivalent:

(i) $\omega$ is exact,
(ii) $[\int_\partial \omega] = c \in H_2(W, \partial W; \mathbb{R}),$ [\int_\partial \omega] = c \in H_2(W, \partial W; \mathbb{R},
(iii) $[\gamma_1] + \ldots + [\gamma_N]$ is not torsion in $H_1(M; \mathbb{Z}).$

Further, assuming that $[0] \times M \subset (W, \omega)$ is concave, the following conditions are also equivalent:

(i) $(W, \omega)$ can be arranged to be a strong symplectic cobordism from $(M, \xi)$ to $(M'; \xi'),$
(ii) $[\int_\partial \omega] = c \in H_2(M; \mathbb{R}),$
(iii) $\lambda_1 + \ldots + \lambda_N$ is not torsion in $H_1(M \setminus B_0)$, where $\lambda_j$ denote the longitudes on $\partial N(\gamma_j)$ determined by the page framing.

It should be emphasized that the above theorem assumes $\Sigma$ is connected. The case where $\Sigma$ is disconnected is equivalent to performing multiple surgery operations in succession, but the statement of Theorem 5 would then become more complicated.

Remark 2.3. For the case $\Sigma = S^1$, if $\gamma \subset B$ denotes the binding component where 0-surgery is performed, then Theorem 5 means that $\omega$ will be exact on $W$ if and only if $\gamma$ is not torsion in $H_1(M)$, and $(W, \omega)$ can be made into a strong cobordism if and only if $\gamma$ has no nonhomologous
cover whose page framing matches its Seifert framing. An equivalent condition is assumed in \cite{GS12}, which only constructs strong cobordisms.

**Remark 1.4.** Though \( \omega \) in the above construction is sometimes an exact symplectic form, \( [W, \omega] \) is never an exact cobordism, i.e. it does not admit a global primitive that restricts to suitable contact forms on both boundary components. This follows immediately from the observation that the core \( K \subset W \) is a symplectic submanifold whose oriented boundary is a **negatively transverse** link in \( (M, \xi) \), hence if \( \omega = d\lambda \) and \( \lambda |_{\partial M} \) defines a contact form on \( (M, \xi) \) with the proper co-orientation, then

\[
0 < \int_{\partial \Sigma_0} \omega = \int_{\partial \Sigma_0} \lambda < 0,
\]

a contradiction. A similar remark applies to the round handle cobordism considered in Theorem \( \text{B} \) and \( \text{C} \) below. The nonexistence of \( [W, \omega] \) is important because there are examples in which it is known that no exact cobordism from \( (M, \xi) \) to \( (M', \xi') \) exists (see \( \text{2.2} \)).

To describe the blown up version of these results, we continue with the same setup as above and choose a set of interface tori,

\[
\mathcal{X}_0 = T_1 \cup \ldots \cup T_N \subset \Sigma,
\]

together with an orientation for each \( T_j \subset \mathcal{X}_0 \). There is then a distinguished positively oriented homology basis

\[
\{ \mu_j, \lambda_j \} \subset H_*(T_j),
\]

where \( \lambda_j \) is represented by some oriented boundary component of a page adjacent to \( T_j \), and \( \mu_j \) is represented by a closed leaf of the characteristic foliation defined on \( T_j \) by \( \xi \). Choose tubular neighborhoods \( \mathcal{N}(T_j) \subset M \) of \( T_j \) and identify them with \( S^2 \times [-1, 1] \times S^1 \) to define positively oriented coordinates \((\theta, \rho, \phi)\) in which \( \lambda_j = [S^2 \times \{\ast\}] \) and \( \mu_j = [\{\ast\} \times S^1] \). We may then assume that for every \( \theta_0 \in S^2 \) the loop \((\theta_0, 0) \times S^2 \) is Legendrian, and the fibration \( \pi \) takes the form

\[
\pi(\theta, \rho, \phi) = \begin{cases} 
\phi & \text{for } \rho > 0, \\
\phi & \text{for } \rho < 0.
\end{cases}
\]

Denote the two oriented boundary components of \( \mathcal{N}(T_j) \) by

\[
\partial_+ \mathcal{N}(T_j) = \partial_+ \mathcal{N}(T_j) \cup \partial_- \mathcal{N}(T_j),
\]

where we define the oriented tori \( \partial_+ \mathcal{N}(T_j) = \pm(S^2 \times [\pm 1] \times S^1) \) with corresponding homology bases \( \{ \mu_j^\pm, \lambda_j^\pm \} \subset H_*(\partial_\pm \mathcal{N}(T_j)) \) such that

\[
\lambda_j^+ := \lambda_j \in H_*(\mathcal{N}(T_j)) \quad \text{and} \quad \mu_j^+ := \pm \mu_j \in H_*(\mathcal{N}(T_j)).
\]

Denote the union of all the neighborhoods \( \mathcal{N}(T_j) \) by \( \mathcal{N}(\mathcal{X}_0) \). Then writing two identical copies of \( \Sigma \) as \( \Sigma_\pm \) and choosing a positively oriented coordinate \( s \in S^1 \) for each boundary component \( \partial_j \Sigma_\pm \), we construct new compact oriented manifolds

\[
M^+ = (M \setminus \mathcal{N}(\mathcal{X}_0)) \cup (\Sigma_+ \times S^1) \cup (\Sigma_- \times S^1),
\]

\[
M^- = (M \setminus \mathcal{N}(\mathcal{X}_0)) \cup (\Sigma_- \times S^1) \cup (\Sigma_+ \times S^1),
\]

from \( M \) and \( \mathcal{X}_0 \) by gluing along orientation reversing diffeomorphisms \( \partial_j \Sigma_\pm \times S^1 \rightarrow \partial_\pm \mathcal{N}(T_j) \) that take the form

\[
(s, t) \mapsto (s, \pm 1, \pm t)
\]

in the chosen coordinates. Thus in homology, \( \partial_\pm \mathcal{N}(\mathcal{X}_0) \subset H_1(\partial_\pm \mathcal{N}(T_j) \times S^1) \) is identified with \( \lambda_j^\pm \) and \( [\{\ast\} \times S^1] \in H_1(\partial_\pm \mathcal{N}(T_j) \times S^1) \) with \( \mu_j^\pm \).

Once again the fibration \( \pi : M \setminus (B \cup T) \rightarrow S^1 \) extends smoothly over the glued in region \( (\Sigma_+ \cup \Sigma_-) \times S^1 \) as the projection to \( S^1 \), so \( M^\pm \) inherits from \( \pi \) a natural blown up summand open book \( \pi' \), with interface \( T \cup \mathcal{X}_0 \), binding \( B \) and fibers that are obtained from the fibers of \( \pi \) by attaching \(- (\Sigma_+ \cup \Sigma_-) \) along the boundary components adjacent to \( \mathcal{X}_0 \). We say that \( \pi' \) is obtained by \( \Sigma \)-**decoupling surgery along** \( \mathcal{X}_0 \).

**Remark 1.5.** The choice of the term **decoupling** is easiest to justify in the special case \( \Sigma = \emptyset \): then the surgery cuts open \( M \) along \( T \) and glues in two solid tori that cap off the corresponding boundary components of the page.

Even if \( \Sigma \) is connected, \( M' \) may in general be disconnected, and there is a (possibly empty) component

\[
M'_0 \subset M',
\]

defined as the union of all the closed pages of \( \pi' \). Denote \( M'_{\text{trans}} := M' \setminus M'_0 \), so that

\[
M' = M'_{\text{trans}} \cup M'_0.
\]

On \( M'_{\text{trans}} \), there is a contact structure \( \xi' \) which matches \( \xi \) away from the region of surgery and is supported by \( \pi' \) in \( M'_{\text{trans}} \cap M'_0 \).

The above surgery corresponds topologically to the attachment of a **round handle**: denote the annulus by

\[
\Lambda = [-1, 1] \times S^1
\]

and define

\[
\mathcal{K}_\Lambda = - \Sigma \times \Lambda,
\]

with boundary

\[
\partial_+ \mathcal{K}_\Lambda = - \partial_+ \mathcal{K}_\Lambda \cup \partial_- \mathcal{K}_\Lambda := ( \Sigma \times \Lambda) \cup ( - \Sigma \times \partial_\Lambda ) ,
\]

where we identify the two connected components of \( \partial_\pm \mathcal{K}_\Lambda = \{ \Sigma \times [-1, 1] \} \times S^1 \) with \( - \Sigma \times S^1 \) via the orientation preserving maps

\[
- \Sigma \times S^1 \rightarrow - \Sigma \times \{ \pm 1 \} \times S^1 : (p, \phi) \mapsto (p, \pm 1, \phi),
\]

Using the identifications of the neighborhoods \( \mathcal{N}(T_j) \) with \( S^2 \times [-1, 1] \times S^1 \) chosen above, we can identify

\[
\partial^- \mathcal{K}_\Lambda = \partial_- \mathcal{K}_\Lambda = \bigcup_{j=1}^N \partial_\pm \Sigma_\pm \times [-1, 1] \times S^1
\]

with \( \mathcal{N}(\mathcal{X}_0) \) and use this to attach \( \mathcal{K}_\Lambda \) to \( [0, 1] \times M \) by gluing \( \partial^- \mathcal{K}_\Lambda \) to \( \mathcal{N}(\mathcal{X}_0) \subset \{ 1 \} \times M \), defining an oriented cobordism

\[
W = (0, 1] \times M \cup \mathcal{K}_\Lambda \mathcal{K}_\Lambda
\]

with boundary \( \partial W = M' \cup ( - M') \). Use the coordinates \((\theta, \rho, \phi) \in S^2 \times [-1, 1] \times S^1 \) on each \( \mathcal{N}(T_j) \subset \mathcal{N}(\mathcal{X}_0) \) to define an oriented link \( \mathcal{B}_j \) as the union of all the loops

\[
S^1 \times \{ 0, 0 \} \subset \{ 0, 1 \} \times T_j \subset \{ 0 \} \times \mathcal{X}_0.
\]

Then the core and co-core respectively can be defined as oriented submanifolds by

\[
\begin{aligned}
\mathcal{K}_\Lambda & := (0, 1] \times \mathcal{B}_0 \\
\mathcal{K}_\Lambda & := ( - \Sigma \times \{ 0, 0 \} ) \subset W
\end{aligned}
\]
and
\[ \mathcal{K}_W := \{ p \times h \subset W \} \]
for an arbitrary interior point \( p \subset \Sigma \). We have \( \partial \mathcal{K}_W = -\mathcal{K}_W \subset M \), \( \partial \mathcal{K}_W \subset M' \) and \( \mathcal{K}_W \cap \mathcal{K}_W = 1 \).

**Theorem 6.** Suppose \( \omega \) is a symplectic form on \([0,1] \times M \) which satisfies \( \omega_k > 0 \) and
\[ \frac{N}{\sum_{j=1}^{N} \int_{T_j} \omega = 0} \]
and \( W \) denotes the round handle cobordism of \( \mathcal{K}_W \). Then after a symplectic deformation away from \([0] \times M \), \( \omega \) can be extended symplectically over \( W \) so that it is positive on \( \mathcal{K}_W \), \( \mathcal{K}_W \) and the pages of \( \pi \), and \( \omega \) dominates a supported contact structure \( \xi' \) on \( M'_{\text{ann}} \). In particular, after capping \( M'_{\text{flat}} \) by attaching a Lefschetz fibration over the disk as in \[ \mathcal{K}_W \], this defines a weak symplectic cobordism from \((M, \xi)\) to \((M'_{\text{ann}}, \xi')\).

We will refer to \((W, \omega)\) in this construction from now on as a \( \Sigma \)-decoupling cobordism.

**Remark 1.6.** The homological condition \( \mathcal{K}_W \) is clearly not removable since the 2-cycles \( \sum_{j=1}^{N} [T_j] \) both become unimodular in \( M' \). Note that here the chosen orientations of the tori \( T_j \) play a role, i.e. they cannot in general be chosen arbitrarily unless \( \omega \) is exact. No such issue arose in Theorem \( \ref{thm:main} \) because \( \omega \) is always exact on a neighborhood of a binding circle. This is the reason why the “decoupling” condition is needed for many of the results in \( \ref{thm:main} \) and there are easy examples to show that these theorems are not true without it (cf. Remark \( \ref{rem:decoupling} \)).

For the analog of Theorem \( \ref{thm:main} \) in this context, we shall restrict for simplicity to the case where \( \int_{T_j} \omega \) vanishes for every \( T_j \subset \Sigma_0 \). Note that in this case, the Poincare dual of \( \omega(T_j) \) can be represented by a real 1-cycle in \( M \setminus \Sigma_0 \).

**Theorem 6'.** If \( \int_{T_j} \omega \) is zero for each \( T_j \subset \Sigma_0 \), then the symplectic cobordism \((W, \omega)\) constructed by Theorem \( \ref{thm:main} \) can be arranged so that the following holds: Choose a real 1-cycle \( h \) in \( M \setminus \Sigma_0 \) with \( [h] \) \( \in \pi_0\left[ M(\Sigma) \right] \). Then there is a number \( c > 0 \) with
\[ \text{PD}([\omega]) = [0,1] \times [h] + c[\mathcal{K}_W] \subset H_2(W, \partial W; \mathbb{R}) \]
In particular, if \([0] \times M \subset (W, \omega)\) is (strongly) concave then the following conditions are equivalent:

(i) \( \omega \) is exact.

(ii) \( \mathcal{K}_W = 0 \subset H_2(W, \partial W; \mathbb{R}) \).

(iii) There are no integers \( k, m_1, \ldots, m_N \in \mathbb{Z} \) with \( k > 0 \) and \( \sum_{j=1}^{N} m_j j = 0 \) such that the homology class
\[ h(\lambda_1 + \ldots + \lambda_N) = \sum_{j=1}^{N} m_j j \subset H_2(\Sigma_0) \]
is trivial in \( H_2(M) \).

Further, if \([0] \times M \) is concave and \( M'_{\text{flat}} = 0 \), the following conditions are also equivalent:

(i) \( \mathcal{K}_W \) can be arranged to be a strong symplectic cobordism from \((M, \xi)\) to \((M', \xi')\).

(ii) \( \mathcal{K}_W = 0 \subset H_2(M'; \mathbb{R}) \).

(iii) There are no integers \( k, m_1, \ldots, m_N \in \mathbb{Z} \) with \( k > 0 \) and \( \sum_{j=1}^{N} m_j j = 0 \) such that
\[ k \sum_{j=1}^{N} \lambda_j^k + k \sum_{j=1}^{N} m_j j = 0 \subset H_2(M \setminus \Sigma_0) \]
Finally, if \( M'_{\text{flat}} = 0 \) and \( M'_{\text{ann}} \) are both non-empty, assume the labels are chosen so that \( \Sigma_+ \times S^1 \subset M'_{\text{ann}} \) and \( \Sigma_- \times S^1 \subset M'_{\text{flat}} \), and consider the weak cobordism
\[ \mathcal{K}_W = (W, \omega, \Pi_{M_{\text{ann}}}(X, \omega x)) \]
from \((M, \xi)\) to \((M'_{\text{ann}}, \xi')\) obtained by capping off \( M'_{\text{flat}} \) with a Lefschetz fibration \( X \rightarrow \mathbb{R} \) as in \( \ref{thm:main} \). The following conditions are then equivalent:

(i) \( \mathcal{K}_W \) can be arranged to be a strong symplectic cobordism from \((M, \xi)\) to \((M'_{\text{ann}}, \xi')\).

(ii) \( \mathcal{K}_W = 0 \subset H_2(M'_{\text{ann}}, \mathbb{R}) \).

(iii) There are no integers \( h, m_1, \ldots, m_N \in \mathbb{Z} \) with \( k > 0 \) and \( \sum_{j=1}^{N} m_j j = 0 \) such that the homology class
\[ k \sum_{j=1}^{N} \lambda_j^k + k \sum_{j=1}^{N} m_j j = 0 \subset H_2(M \setminus \Sigma_0) \]
is trivial in \( H_2(M \setminus \Sigma_0) \).

We now discuss some applications of the capping and decoupling cobordisms to Embedded Contact Homology (ECCH). Recall that for a closed contact 3-manifold \((M, \xi)\) and homology class \( h \in H_2(M, \mathbb{R}) \), \( \text{ECH}(M, \xi; h) \) is the homology of a chain complex generated by sets of Reeb orbits with multiplicities whose homology classes add up to \( h \), having a differential counting embedded index 1 holomorphic curves with positive and negative cylindrical ends in the symplectization \( \mathbb{R} \times M \). Similarly, counting embedded index 2 holomorphic curves through a generic point in \( M \) yields the so-called \( \mathcal{U} \)-map,
\[ U: \text{ECH}_*(M, \xi; h) \rightarrow \text{ECH}_*(M, \xi; h) \]
The \( \text{ECH} \) contact invariant
\[ c(M, \xi) \in \text{ECH}_*(M, \xi; 0) \]
is the homology class represented by the “empty orbit set”. It is equivalent via an isomorphism of Taubes \( \text{ECH} \rightarrow \text{ECH} \) to a corresponding invariant in Seiberg-Witten theory, and also to the Onevath-Sinha contact invariant \( \text{ECH} \) by recent work of Colin-Ghiggini-Bond. \( \text{ECH} \) and independently Kothas-Ko-Sinhas-\( \text{ECH} \). Like those invariants, the \( \mathcal{U} \)-map vanishing gives an obstruction to strong symplectic fillings, and a version with twisted coefficients also obstructs weak fillings.

**Remark 1.7.** Technically the definitions of \( \text{ECH}_*(M, \xi; h) \) and \( c(M, \xi) \) depend not just on \( \xi \) but also on a choice of contact form and almost complex structure. However, Taubes’ isomorphism to Seiberg-Witten Floer homology implies that they are actually independent of these choices, thus we are safe in writing \( \text{ECH}_*(M, \xi; h) \) without explicitly mentioning the extra data.
An argument due to Eliashberg shows that $c(M, ξ) = 0$ whenever $(M, ξ)$ is overtwisted, and a much more general computation in [6] established the same result whenever $(M, ξ)$ has planar $k$-genus for any $k ≥ 0$. The latter result can now be recovered as a consequence of Theorem [4] using a result recently announced by Hutchings [6]. That $(M, ξ)$ is overtwisted, and $c(M, ξ) = 0$ implies $c(M, ξ)$ has planar $k$-genus for any $k ≥ 0$. The latter result can now be recovered as a consequence of Theorem [4] using a result recently announced by Hutchings [6].

**Proposition 1.8** [4]. Suppose $(W, ω)$ is a strong symplectic cobordism from $(M, ξ)$ to $(M, ξ)$ such that $ω$ is exact. Then there is a $U$-equivariant map

$$ECH(M, ξ; Ω) → ECH(M, ξ; Ω)$$

that takes $c(M, ξ)$ to $c(M, ξ)$.

**Remark 1.9.** For the example of a 2-handle cobordism constructed from an ordinary open book decomposition, the analogue of Proposition [4] in Heegaard Floer homology has been established by John Baldwin [6].

Let us now discuss a conjectural generalization of Proposition [4] which could remove all conditions on $ω$. Recall that for any closed 2-form $\Omega$ on $M$, one can define $ECH$ with twisted coefficients in the group ring $\mathbb{Z}[H_2(M)/ker \Omega]$, which we shall abbreviate by

$$ECH(M, ξ; h, \Omega) := ECH(M, ξ; h, \mathbb{Z}[H_2(M)/ker \Omega]).$$

Here the differential keeps track of the homology classes in $H_2(M)/ker \Omega$ of the homology curves being counted, see [7]. The $U$-map can again be defined as a degree $−2$ map on $ECH(M, ξ; h, \Omega)$, and the twisted contact invariant $c(M, ξ; \Omega)$ is again the homology class in $ECH(M, ξ; 0, \Omega)$ generated by the empty orbit set. The vanishing results in [7] give convincing evidence that a more general version of the map in Proposition [4] should exist, in particular with the following consequence:

**Conjecture 1.** Suppose $(W, ω)$ is a $Σ$-capping or $Σ$-decapping cobordism from $(M, ξ)$ to $(M, ξ)$, and write $Ω = ω|W$. Then:

1. If $c(M, ξ, Ω)$ vanishes, then so does $c(M, ξ, Ω)$.
2. If $c(M, ξ, Ω)$ is in the image of the map $U^k$ on $ECH(M, ξ; 0, \Omega)$ for some $k ≤ n$, then $c(M, ξ, Ω)$ is in the image of $U^k$ on $ECH(M, ξ; 0, \Omega)$.

The first part of the conjecture would reduce both the untwisted and twisted vanishing results in [7] to the fact, proved essentially by Eliashberg in the appendix of [8], that the fully twisted contact invariant vanishes for every overtwisted contact manifold. The second part is related to another result proved in [9], namely the twisted $ECH$ version of the planarity obstruction of Oancea-Stipsicz-Szabó in Heegaard Floer homology: if $(M, ξ)$ is planar, then $c(M, ξ)$ is in the image of $U^k$ for all $k$ and all $Ω$. If the conjecture holds, then this fact follows from Theorem [10] and the computation of $ECH(S^3, ξ)$.

3In the appendix of [11], Eliashberg sketches an argument to show that every overtwisted contact manifold has trivial contact homology, and this argument also implies the vanishing of the $ECH$ contact invariant.

The obvious way to try to prove Conjecture [1] would be by constructing a $U$-equivariant map

$$ECH(M, ξ; Ω) → ECH(M, ξ; 0, Ω)$$

which takes $c(M, ξ; Ω)$ to $c(M, ξ; 0, Ω)$. Due to the non-exactness of $ω$ and a resulting lack of energy bounds, it seems unlikely that such a map would exist in general, but a more probable scenario is to obtain a map

$$ECH(M, ξ; Ω) → ECH(M, ξ; 0, Ω),$$

where $Λ_0$ is a Novikov completion of $\mathbb{Z}[H_2(W)/ker \omega]$, and we take advantage of the natural inclusions

$$H_2(M)/ker \Omega → H_2(W)/ker \omega$$

to define the $ECH$ of $(M, ξ)$ with coefficients in $Λ_0$. In cases where $M_0$ has connected components with closed leaves, one would expect this map to involve also the Periodic Floer Homology (cf. [12]) of the resulting mapping tori. Defining such a map would require a slightly more careful construction of the weak cobordism $(W, ω)$, such that both boundary components inherit stable Hamiltonian structures which can be used to attach cylindrical ends and define reasonable moduli spaces of finite energy punctured holomorphic curves.

This can always be done due to a construction in [13], which shows that suitable stable Hamiltonian structures exist for any desired homology class on the boundary. It is probably also useful to observe that for an intelligent choice of data, the holomorphic curves in $(W, ω)$ with no positive ends can be enumerated precisely: we will show in Proposition 3.10 that all of them arise from the symplectic core of the handle.

2. **Further Applications, Examples and Discussion**

We shall now give some concrete examples of capping and decapping cobordisms and survey a few more applications, including new proofs of several known results and one or two new ones.

2.1. **The Gromov-Eliashberg theorem using holomorphic spheres.** In [14], holomorphic curve arguments were used to show that planar tori in a filling obstruction, but Theorem [15] makes these proofs much easier by using essentially “soft” methods to reduce them to the well-known result of Gromov-Eliashberg that overtwisted contact manifolds are not weakly fillable. This does not of course make everything elementary, as the Gromov-Eliashberg theorem still requires some technology—the original proof used a “Bishop family” of holomorphic disks with totally real boundary, and these days one can instead use punctured holomorphic curves, Seiberg-Witten theory or Heegaard Floer homology if preferred. While this technological overhead is probably not removable, we can use a decapping cobordism to simplify the level of technology a tiny bit: namely we can reduce it to the following standard fact whose proof requires only closed holomorphic spheres, e.g. the methods used in [15].

**Lemma 2.1.** If $(W, ω)$ is a connected weak filling of a nonempty contact manifold $(M, ξ)$, then it contains no embedded symplectic sphere with vanishing self-intersection.

This lemma follows essentially from McDuff’s results [15], but by today’s standards it is also easy to prove on its own: if one chooses a compatible almost complex structure to make the boundary $\Sigma$-convex and the embedded symplectic sphere $J$-holomorphic, then vanishing self-intersection implies that the latter lives in a smooth 2-dimensional moduli space.
of holomorphic spheres that foliate $W$ (except at finitely many nodal singularities). This forces some leaf of the foliation to hit the boundary tangentially, thus contradicting $J$-convexity.

**Corollary**. Every weakly fillable contact manifold is tight.

**Proof.** A schematic diagram of the proof is shown in Figure 1. Suppose $(W, \omega)$ is a weak filling of $(M, \xi)$ and the latter is twisted. Then $(M, \xi)$ contains a planar 0-torsion domain $M_0$, whose planar piece $M_0^\flat$ is a solid torus with disk-like pages, attached along an interface torus $T = \partial M_0^\flat$ to another subdomain whose pages are not disks. Since $[T] = 0 \in H_2(M)$, $\int_T \omega = 0$ and we can attach a $\mathbb{S}$-decoupling cobordism along $T$, producing a larger symplectic manifold $(W', \omega)$ whose boundary has two connected components

$$\partial W' = M_0^\flat \cup M_0^\#$$

of which the latter carries a contact structure $\xi'$ dominated by $\omega$. The component $M_0^\flat$ has closed sphere-like pages, and is thus the trivial symplectic fibration $S^2 \times S^2 \to S^2$. After capping $M_0^\flat$ by a symplectic fibration $\mathbb{S} \times S^2 \to \mathbb{S}$, we then obtain a weak filling of $(M_0^\flat, \xi')$, containing a symplectic sphere with vanishing self-intersection, contradicting Lemma 4.

**Remark 2.2.** A related argument appears in [Eli90], using the fact that overtwisted contact manifolds always have Gromov torsion; see also [Eli90] below.

---

**22. Eliashberg’s cobordisms from $T^3$ to $S^0 \sqcup \ldots \sqcup S^3$.** Let $T^3 = S^1 \times S^1 \times S^1$ with coordinates $(\eta, \phi, \theta)$ and define for $n \in \mathbb{N}$ the contact structure

$$\xi_n = \ker \left[ \cos(2\pi n) d\phi + \sin(2\pi n) d\theta \right].$$

These contact structures are all tight, but Eliashberg showed in [Eli90] that they are not strongly fillable for $n \geq 2$, which follows from the fact that disjoint unions of multiple copies of $(S^0, \xi_0)$ are not fillable, together with the following:

**Theorem**. For any $n \in \mathbb{N}$, $(T^3, \xi_n)$ is symplectically cobordant to the disjoint union of $n$ copies of the tight $3$-sphere.

**Proof.** The torus $(T^3, \xi_n)$ admits a supporting open book decomposition with 2$n$ irreducible subdomains $M_j$, each having cylindrical pages and trivial monodromy, attached to each other along 2$n$ interface tori $T = \bigcup_i T_i$ such that $T_i = M_i \cap M_{i+1}$ for $j = 2m$. Attaching round handles $\mathbb{D} \times \mathbb{D}$ along every second interface torus $T_i, T_{i+2}, \ldots, T_{i+n-1}$ yields a weak symplectic cobordism into the disjoint union of $n$ copies of the tight $3$-sphere (Figure 2). The latter is also supported by an open book with cylindrical pages and trivial monodromy, so we can attach a 2-handle $\mathbb{D} \times \mathbb{D}$ along one binding component to create a weak cobordism to the tight $3$-sphere. The resulting weak cobordism from $T^3$ to $S^1 \sqcup \ldots \sqcup S^3$ can be deformed to a strong cobordism since the symplectic form is necessarily exact near $S^3 \sqcup \ldots \sqcup S^3$.

**Remark 2.3.** Note that $(T^3, \xi_n)$ is always strongly fillable [Eli90], and indeed, the above cobordism cannot be attached to any weak filling $(W, \omega)$ of $(T^3, \xi_n)$ for which $\int_T \omega \neq 0$. This shows that the homological condition in Theorem H cannot be removed.

**23. Gay’s cobordisms for Giroux torsion.** Recall that a contact manifold $(M, \xi)$ is said to have Giroux torsion $GT(M, \xi) = n \in \mathbb{N}$ if $n$ is the largest integer for which $(M, \xi)$ admits a contact embedding of $(B, [1]) \times T^2, \xi_n)$, where $\xi_n$ is given by (37). We write $GT(M, \xi) = 0$ if there are no such embeddings and $GT(M, \xi) = \infty$ if they exist for arbitrarily large $n$. Every contact manifold with positive Giroux torsion also has planar 1-torsion (see [Gay]), thus as a special case of Theorem H every $(M, \xi)$ with $GT(M, \xi) \geq 1$ is symplectically cobordant to something overtwisted; this was proved by David Gay in [Gay] for $GT(M, \xi) \geq 2$.
to derive a contradiction if \((M, \xi)\) has a filling, but one can just as well use Lemma 2.1 above. A close relative of Gay’s cobordism construction is easily obtained from the above picture: attaching round handles \(D \times \mathbb{R}\) along both \(T_+\) and \(T_-\), the top of the cobordism contains a connected component with closed sphere-like pages (the top picture in Figure 3), which can be capped by \(S^3 \times S^3\) to produce a cobordism that contains symplectic spheres of self-intersection number 0.

24. Some new examples with \(M_2 \not\cong M_3\) but \(M_2 \not\cong M_4\). Gromov’s theorem on the non-existence of exact Lagrangians in \(\mathbb{R}^{2n}\) provides perhaps the example of a pair of contact manifolds that are strongly but not exactly cobordant: indeed, viewing \((T^3, \xi_{\text{std}})\) as the boundary of a Weinstein neighborhood of any Lagrangian torus in the standard strong filling of the tight 3-sphere \((S^3, \xi_{\text{std}})\), we obtain

\[
\left( (T^3, \xi_{\text{std}}) \right) \not\cong \left( (S^3, \xi_{\text{std}}) \right) \quad \text{but} \quad \left( (T^3, \xi_{\text{std}}) \right) \not\cong \left( (S^3, \xi_{\text{std}}) \right).
\]

The nonexistence of the exact cobordism here can also be proved by the argument of Hofer-H(getApplicationContext()).

A subtle obstruction to exact cobordisms is defined in joint work of the author with Aniko Latschev via Symplectic Field Theory, leading to the following example. For any integer \(k \geq 1\), suppose \(\Sigma\) is a closed, connected and oriented surface of genus \(g \geq k\), and \(\Gamma \subset \Sigma\) is a multimmune consisting of \(k\) disjoint embedded loops which divide \(\Sigma\) into exactly two connected components

\[
\Sigma = \Sigma_+ \cup \Sigma_-,
\]

such that \(\Sigma_+\) has genus 0 and \(\Sigma_-\) has genus \(g - k + 1 > 0\). By a construction due to Lutz-Strömer, the product

\[
M_{\mathcal{A}} := S^2 \times \Sigma
\]

then admits a unique (up to isotopy) \(S^2\)-invariant contact structure \(\xi_{\mathcal{A}}\) such that the convex surfaces \(\Sigma\) \times \Sigma have dividing set \(\Gamma\). The contact manifold \((M_{\mathcal{A}}, \xi_{\mathcal{A}})\) then has planar \((k - 1)\)-torus, as the two subsets \(S^2 \times \Sigma_+\) can be regarded as the irreducible subdomains of a supporting open book with pages \(\Sigma\) \times \Sigma_+\), so we view \(S^2 \times \Sigma_+\) as the planar page and \(S^2 \times \Sigma_-\) as the regular part (see Definition 2.2). In particular, \((M_{\mathcal{A}}, \xi_{\mathcal{A}})\) is overdetermined and if \(k = 1\) and \(g \geq 2\) it has a Reeb vector field with no contractible periodic orbits. It turns out in fact that each increment of \(k\) contains an obstruction to exact fillings that is invisible in the non-exact case.

**Theorem 7.** If \(k > 0\) then for any \(g \geq k\) and \(g' \geq \ell\),

\[
(M_{\mathcal{A}}(k), \xi_{\mathcal{A}}) \not\cong (M_{\mathcal{A}}(k') \xi_{\mathcal{A}}') \quad \text{but} \quad (M_{\mathcal{A}}(k), \xi_{\mathcal{A}}) \not\cong (M_{\mathcal{A}}(k' \xi_{\mathcal{A}}'))
\]

Proof The nonexistence of the exact cobordism is a result of [ают]. The existence of the non-exact cobordism follows immediately from Corollary 1 above but in certain cases one can construct it much more explicitly as in Figure 3. In particular, \((M_{\mathcal{A}}(k), \xi_{\mathcal{A}})\) is supported by a summed open book consisting of the two irreducible subdomains \(S^2 \times \Sigma_+\) with pages \(\Sigma\) \times \Sigma_+\) attached along \(k\) parallel tori. Attaching \(D \times \mathbb{R}\) along one of the interface tori gives a weak \(\mathbb{R}\)-decoupling cobordism to \((M_{\mathcal{A}}(k), \xi_{\mathcal{A}})\).
25. Open books with reducible monodromy. Any compact, connected and oriented surface $\Sigma$ with boundary, together with a diffeomorphism $\varphi : \Sigma \to \Sigma$ fixing the boundary, determines a contact 3-manifold $(M_\varphi, \xi_\varphi)$, namely the one supported by the open book decomposition with page $\Sigma$ and monodromy $\varphi$. Recall that the mapping class group of the monodromy map $\varphi$ is said to be reducible if it has a representative that preserves some multicurve $C \subset \Sigma$ such that no component of $\Sigma \setminus C$ is a disk or an annulus. Consider the simple case in which $\varphi$ preserves each individual connected component $C \subset \Sigma$ and also preserves its orientation (note that this is always true for some iterate of $\varphi$). In this case we may assume after a suitable isotopy that $C$ is the identity on a neighborhood of $\partial \Sigma \cup \Sigma$, so that for some open annular neighborhood $C \subset \Sigma(\gamma) \subset \Sigma$ of each curve $\gamma \subset C$, $M_\varphi$ contains a thickened torus

$$S^2 \times \Sigma(\gamma) \subset M_\varphi$$

on which the open book decomposition is the projection to the first factor. Let $N(\Gamma) \subset \Sigma$ denote the union of all the neighborhoods $\Sigma(\gamma)$ and define the possibly disconnected compact surface

$$\Sigma' = \Sigma \setminus N(\Gamma)$$

with boundary; then $\varphi$ restricts to this surface as an orientation preserving diffeomorphism $\varphi' : \Sigma' \to \Sigma'$ that preserves each connected component and equals the identity near $\partial \Sigma'$. Denote the connected components of $\Sigma'$ by

$$\Sigma'_j \subset \Sigma'$$

and the corresponding restrictions of $\varphi'$ by

$$\varphi'_j : \Sigma'_j \to \Sigma'_j$$

for $j = 1, \ldots, N$. Since each $\Sigma'_j$ necessarily has nonempty boundary, each gives rise to a connected contact manifold $(M_{\varphi'_j}, \xi_{\varphi'_j})$.

Theorem 8. Given a reducible monodromy map $\varphi : \Sigma \to \Sigma$ as described above, there exists a weak symplectic cobordism $(W, \omega)$ from

$$(M_{\varphi'_1}, \xi_{\varphi'_1}) \cup \ldots \cup (M_{\varphi'_N}, \xi_{\varphi'_N})$$

to

$$(M_{\varphi}, \xi_{\varphi})$$.

which is (strongly) concave at the negative boundary and such that the restriction of $\omega$ to the positive boundary is Poincaré dual to a positive multiple of

$$\sum_{j \in F} [S^3 \times \{ p_j \}] \in H_3(M_{\varphi'; \mathbb{R}}),$$

where the summation is over the connected components of $\Gamma$ and $p_j \in \mathcal{N}(\gamma)$ denotes an arbitrarily chosen point.

Moreover, given any closed 2-form $\Omega$ on $M_{\varphi'; \mathbb{R}} \cup \ldots \cup M_{\varphi'^N}$ that dominates the respective contact structures, one can also construct a weak cobordism between the contact manifolds above such that $\omega$ matches $\Omega$ at the negative boundary.

Proof. The cobordism is a stack of $\omega$-capping cobordisms, constructed by attaching handles to the form $[-1,1] \times S^3 \times \mathbb{R}$ via Theorem F along all pairs of binding circles in $M_{\varphi'; \mathbb{R}} \cup \ldots \cup M_{\varphi'^N}$ that correspond to the same curve in $\Gamma$. The core of each of these handles is a disk with boundary of the form $S^3 \times \{ \ast \} \subset S^3 \times \mathcal{N}(\gamma) \subset M_{\varphi'}$, thus the cobordism class of $\omega$ at the positive boundary follows immediately from Theorem F.

Corollary 3. If the contact manifolds $(M_{\varphi', \xi_{\varphi'}})$ for $j = 1, \ldots, N$ are all weakly fillable, then so is $(M_{\varphi}, \xi_{\varphi})$.

Remark 24. John Bokstev has observed that topologically, the cobordism of Theorem 9 can also be obtained by performing boundary connected sums on the pages and then using $\omega$-capping cobordisms to remove extra boundary components; in [23] this is used to deduce a relation between the Ozsvath-Szabo contact invariants of $(M_{\varphi'}, \xi_{\varphi'})$ and the pieces $(M_{\varphi'_1}, \xi_{\varphi'_1}), \ldots, (M_{\varphi'_N}, \xi_{\varphi'_N})$. Additionally, Jeremy Van Horn-Morris and John Etnyre have pointed out to me that if one also assumes every component of $\Sigma \setminus \Gamma$ to intersect $\partial \Sigma$, then one can replace the weak cobordism of Theorem 8 with a Stein cobordism. This does not appear to be possible if any component of $\Sigma \setminus \Gamma$ has its full boundary in $\Gamma$.

26. Etnyre’s planarity obstruction. Let us say that a connected contact 3-manifold $(M, \xi)$ is maximally cobordant to $S^3$ if there exists a connected compact 4-manifold $W$ with $\partial W = S^3 \cup (-M)$ such that for every closed 2-form $\omega$ on $M$ with $[\omega] > 0$, there is a symplectic form $\omega'$ on $W$ with $\omega|_{\partial W} = \Omega$ defining a weak symplectic cobordism from $(M, \xi)$ to $(S^3, \xi_S)$. Theorem 8 says that every planar contact manifold is maximally cobordant to $S^3$. It turns out that this suffices to give an alternative proof of the planarity obstruction in [Etnyre 4.1].

Theorem 9. Suppose $(M, \xi)$ is maximally cobordant to $S^3$. Then every connected weak or semi-filling of $(M, \xi)$ has connected boundary and a negative-definite intersection form.

Proof. Let $W$ be the compact 4-manifold with $\partial W = S^3 \cup (-M)$ guaranteed by the assumption, and suppose $(W_0, \omega)$ is a weak filling of $(M, \xi) \cup (M', \xi')$ where $(M', \xi')$ is some other contact manifold, possibly empty. If $W = W_0 \cup W_1$, $W_1$ is defined by giving these two along $M$, then by assumption $\omega$ is nullhomologous. $W_1$ becomes a weak filling of $(S^3, \xi_S) \cup (M', \xi')$, implying that $M'$ must be empty since $(S^3, \xi_S)$ is not weakly co-fillable.

Now $\omega$ is exact near $\partial W = S^3$, so without loss of generality we may assume $(W, \omega)$ is a strong filling of $(S^3, \xi_S)$.

We claim that the map induced on homology $H_2(W_0; \mathbb{Q}) \to H_2(W; \mathbb{Q})$ by the inclusion $i : W_0 \to W$ is injective. Indeed, if $A \in H_2(W_0; \mathbb{Q})$ satisfies $\int_A \omega = 0$, then obviously $\int_A \omega$
is also nonzero and thus $\alpha A \not\in H^2(W;\mathbb{Q})$. If $\int_w \omega = 0$ but $A \in H_2(W;\mathbb{Q})$ is nontrivial, we can pick any closed 2-form $\sigma$ on $W$ with $\int_w \sigma = 0$ and replace $\omega$ by $\omega + \sigma$ for any $\epsilon > 0$ sufficiently small so that $(W,\omega + \sigma)$ remains a weak filling of $(M,\xi)$. Then $\omega + \sigma$ also extends over $W$, so that the above argument goes through again to prove that $\alpha A$ is nontrivial.

Finally, we use the fact that the strong fillings of $(S^3,\xi_0)$ have been classified: by a result of Gromov and Eliashberg, $W$ is necessarily diffeomorphic to a symplectic blow-up of the 4-ball, i.e., $W \cong B^4 \# \mathbb{CP}^2 \# \ldots \# \mathbb{CP}^2$.

Since the latter has a negative-definite intersection form and $\alpha : H_2(W;\mathbb{Q}) \to H_2(W;\mathbb{Q})$ is injective, the result follows. \hfill \Box

Our proof of Theorem 2 combined with Conjecture 1 would also resolve the algebraic planarity obstruction established in \cite{Wen}, which is the twisted ECH version of a Heegaard Floer theoretic result due to Ozsváth, Stipsicz and Szabó \cite{OSS}. Note that the condition of being maximally coorientable to $S^3$ does not require $(M,\xi)$ to be fillable. It is also not clear whether there can exist non-planar contact manifolds that also satisfy this condition; the author is unaware of any known invariants that would be able to detect this distinction.

**Question 3.** Is there a non-planar contact 3-manifold which is maximally coorientable to $S^3$?

Note that if the assumption of Theorem 2 is relaxed to $(M,\xi) \cong (S^3,\xi_0)$, then the result becomes false: a counterexample is furnished by the standard 3-torus $(T^3,\xi)$, which admits a coorientation to $(S^3,\xi_0)$ by Theorem 2 but also is strongly filled by $T^2 \times D^2$, whose intersection form is indefinite. Assuming a strong filling $(W,\omega)$ of $(M,\xi)$, the proof above fails precisely at the point where the inclusion $W \to W$ is required to induce an injective map $H_2(W;\mathbb{Q}) \to H_2(W;\mathbb{Q})$. However, it still follows by the same argument that $H_2(W;\mathbb{Q})$ cannot contain any class with strictly positive square, hence we obtain the following weaker result with more general assumptions—in particular to all the contact manifolds covered by Theorem 2.

**Theorem 10.** Suppose $(M,\xi)$ is a closed connected contact 3-manifold with $(M,\xi) \cong (S^3,\xi_0)$. Then every strong filling $(W,\omega)$ of $(M,\xi)$ has connected boundary and $\int_w \omega = 0$.

### 2.7. Some remarks on planar torsion

The filling obstruction known as planar torsion was introduced in \cite{Wen} with mainly holomorphic curves as motivation, as it provides the most general setting known so far in which the existence and uniqueness of certain embedded holomorphic curves leads to a vanishing result for the ECH contact invariant. In light of our cobordism construction, however, one can now provide an alternative motivation for the definition in purely symplectic topological terms. The first step is to understand what kinds of blow up sumed open books automatically support overtwisted contact structures: using Eliashberg’s classification theorem \cite{Eli} and Giroux’s criterion (cf. \cite{Gir}), this naturally leads to the notion of planar 0-torsion. Then a more general blow up sumed open book defines a planar 0-torsion domain for some $k \geq 1$, and only if it can be transformed into a planar 0-torsion domain by a sequence of $\mathbb{Z}$-capping and $\mathbb{Z}$-decapping surgeries; this is the essence of Proposition \ref{prop:planar} proved below. From this perspective, the definition of planar torsion and the crucial role played by blow up sumed open books seem completely natural.

More generally, the partially planar domains are precisely the blow up sumed open books for which a sequence of $\mathbb{Z}$-capping and $\mathbb{Z}$-decapping cobordisms can be used to construct a symplectic cap that contains a symplectic sphere with square 0. As far as the author is aware, almost all existing uniqueness or classification results for symplectic fillings (e.g. \cite{Wen} [last 2000]) apply to contact manifolds that admit caps of this type. However, it does not always suffice to construct an appropriate cap and then apply McDuff’s results \cite{McD}, e.g. the classification of strong fillings of planar contact manifolds in terms of Lefschetz fibrations \cite{Wen} truly relies on punctual holomorphic curves, as there is no obvious way to produce a Lefschetz fibration with bounded fibers out of a family of holomorphic spheres in a cap.

Finally, we remark that while Theorems 1 and 2 substantially simplify the proof that planar torsion is a filling obstruction, they do not reproduce all of the results in \cite{Wen}; in particular the technology of Embedded Contact Homology is not yet far enough along to deduce the vanishing of the contact invariant from a non-exact cobordism. Moreover, a proof using capping and decapping cobordisms simplifies the technology needed but does not remove it, as a simplified version of the very same technology is required to prove the Gromov-Eliashberg theorem (cf. \cite{G}). From the author’s own perspective, the idea for constructing symplectic cobordisms out of these types of handles would never have emerged without a holomorphic curve picture in the background (cf. Figure 1), and as we will discuss in \cite{Wen}, after one has constructed the symplectic structure, it is practically no extra effort to add a foliation by embedded $J$-holomorphic curves which reproduces the $J$-holomorphic blown up open books of \cite{Wen} on both boundary components. The moral is that whether one prefers to prove non-embeddability results by direct holomorphic curve arguments or by constructing cobordisms to reduce them to previously known results, it is essentially the same thing: neither proof would be possible without the other.

### 3. The details

The plan for proving the main results is as follows. We begin in \ref{sec:main} by reviewing the fundamental definitions involving blown up sumed open books and planar torsion, culminating with the (more or less obvious) observation that one can always use capping or decapping surgery to decrease the order of a planar torsion domain. In \ref{sec:main} we introduce a useful concrete model for a blown up sumed open book and its support contact structure. This is applied in \ref{sec:main} to write down a model of a weak $\mathbb{Z}$-decapping cobordism, and more generally when the given symplectic form $\omega$ in Theorem \ref{thm:planar} (or exact), see Remark \ref{rem:planar}. For the general case, we need to show additionally that any given symplectic form on $[0,1] \times M$ satisfying the necessary cobordismal condition can be deformed so as to attach smoothly to the model cobordisms we’ve constructed; this is shown in \ref{sec:main} thus completing the proofs of Theorems 2 and 3. We prove Theorems 5 and 7 in \ref{sec:main} answering the essentially cobordismal question of when the weak cobordism can be made strong, and when its symplectic form is exact. With these ingredients all in place, the proofs of the main results from \ref{sec:main} are completed in \ref{sec:main}. Finally, \ref{sec:main} gives a brief discussion of the existence and uniqueness of holomorphic curves in the cobordisms we’ve constructed.

#### 3.1. Review of sumed open books and planar torsion

The following notions were introduced in \cite{Wen}, and we refer to that paper for more precise definitions and further discussion.
Assume $M$ is a compact oriented 3-manifold, possibly with boundary, the latter consisting of a union of 2-tori. A blown up summed open book $\pi$ on $M$ can be described via the following data:

1. An oriented link $B \subset M \setminus \partial M$, called the **binding**.
2. A disjoint union of 2-tori $T \subset M \setminus \partial M$, called the **interface**.
3. For each interface torus $T \subset T$ a distinguished basis $(\lambda, \mu)$ of $H_1(T)$, where $\mu$ is defined only up to sign.
4. For each boundary torus $T \subset \partial M$ a distinguished basis $(\lambda, \mu)$ of $H_1(T)$.
5. A fibration $\pi : M \setminus (B \cup T) \to S^1$

whose restriction to $\partial M$ is a submersion.

The distinguished homology classes $\lambda, \mu \in H_1(T)$ associated to each torus $T \subset B \cup T \cup \partial M$ are called **longitude**s and **meridians** respectively, and the oriented connected components of the fibers $\pi^{-1}(\text{const})$ are called **pages**.

We assume moreover that the fibration $\pi$ can be expressed in the following normal forms near the components of $B \cup T \cup \partial M$. As in an ordinary book decomposition, each binding circle $\gamma \subset B$ has a neighborhood admitting coordinates $(\theta, \rho, \phi) \in S^1 \times [0,1] \times S^1$ such that $\gamma = \{\theta = 0\}$ and

$$\pi(\theta, \rho, \phi) = \rho.$$  

Near an interface torus $T \subset \mathcal{I}$, we can find a neighborhood with coordinates $(\theta, \rho, \phi) \in S^1 \times [0,1] \times S^1$ such that $T = \{\theta = 0\} \times [0,1] \times S^1$ with $(\lambda, \mu)$ matching the natural basis of $H_1(S^1 \times [0,1] \times S^1)$, and

$$\pi(\theta, \rho, \phi) = \begin{cases} 0, & \text{for } \rho > 0, \\ \phi, & \text{for } \rho < 0. \end{cases}$$

A neighborhood of a boundary torus $T \subset \mathcal{O}$ similarly admits coordinates $(\theta, \rho, \phi) \in S^1 \times [0,1] \times S^1$ with $T = \{\theta = 0\} \times [0,1] \times S^1$ and

$$\pi(\theta, \rho, \phi) = -\phi.$$  

Observe that unlike the normal form [23], the map [23] is well defined at $\rho = 0$, since there are no polar coordinates and hence no coordinate singularity. The above conditions imply that the closure of each page is a smoothly immersed surface, whose boundary components are each embedded submanifolds of $B$, $T$ or $\partial M$, and in the last two cases homologous to the distinguished longitude $\lambda$. The "generic" page has an embedded closure, but in isolated cases there may be pairs of boundary components that are identical to 1-dimensional submanifolds in $\mathcal{I}$.

In general, any all of $B$, $T$ and $\partial M$ may be empty, and $M$ may also be disconnected. If $B \cup T \cup \partial M = \emptyset$ we have simply a fibration $\pi : M \to S^1$ whose fibers are closed oriented surfaces. If $B \cup \partial M = \emptyset$ but $B \neq \emptyset$ and $M$ is connected, we have an ordinary book.

We say that $\pi$ is **irreducible** if the fibers $\pi^{-1}(\text{const})$ are connected, i.e. there is only one $S^1$-parametrized family of pages. More generally, any blown up summed open book can be presented uniquely as a union of irreducible subdomains $M = M_1 \cup \ldots \cup M_N$,

which each inherit irreducible blown up summed open books and are attached together along boundary tori (which become interface tori in $M$).

The notion of a contact structure supported by an open book generalizes in a natural way: we say that a contact structure $\xi$ on $M$ is **supported** by $\pi$ if it is the kernel of a Giroux form, a contact form whose Reeb vector field is everywhere positively transverse to the pages and positively tangent to their boundaries, and which induces a characteristic foliation on $T \cup \partial M$ with closed leaves parallel to the distinguished meridians. A Giroux form exists and is unique up to homotopy through Giroux forms on any connected manifold with a blown up summed open book, except in the case where the pages are closed, i.e. $B \cup T \cup \partial M = \emptyset$.

The binding is then positively transverse, and the interface and boundary are disjoint unions of pre-Lagrangian tori.

**Definition 3.1.** An irreducible blown up summed open book is called **planar** if its pages have genus zero. An arbitrary blown up summed open book is then called **partially planar** if its interior contains a planar irreducible subdomain, which we call a **planar piece**. A **planarly planar domain** is a contact 3-manifold $(M, \xi)$, possibly with boundary, together with a supporting blown up summed open book that is planarly planar. For a given closed 2-form $\Omega$ on $M$, and a partially planar domain $(M, \xi)$ with planar piece $M^P \subset M$, we say that $(M, \xi)$ is $\Omega$-**separating**, if $\int_T \Omega = 0$ for all interface tori $T$ of $M$ that lie in $M^P$, and **fully separating** if this is true for all $\Omega$.

**Definition 3.2.** A blown up summed open book is called **symmetric** if it has empty boundary, all its pages are diffeomorphic to each other, and it contains exactly two irreducible subdomains $M = M_1 \cup M_2$.

The simplest example of a symmetric summed open book is the one whose pages are disks: this supports the tight contact structure on $S^3 \times S^1$ (cf. Figure 2 right).

**Definition 3.3.** For any integer $k \geq 0$, an $\Omega$-separating partially planar domain $(M, \xi)$ with planar piece $M^P \subset M$ is called an $\Omega$-**separating planar $k$-torsion** domain if it satisfies the following conditions:

- $(M, \xi)$ is not symmetric.
- $\partial M^P \neq \emptyset$.
- The pages in $M^P$ have $k + 1$ boundary components.

The (necessarily nonempty) subdomain $M \setminus M^P$ is then called the **padding**.

We say that a contact manifold $(M, \xi)$ with closed 2-form $\Omega$ has $\Omega$-separating planar $k$-torsion if it contains an $\Omega$-separating planar $k$-torsion domain. If this is true for all closed 2-forms $\Omega$ on $M$, then we say $(M, \xi)$ has **fully separating** planar $k$-torsion.

It was shown in [25] that a contact manifold is $\Omega$-separating if and only if it has planar $\Omega$-torsion, which is always fully separating since the interface then intersects the planar piece only at its boundary, a single nullhomologous torus. The proofs of Theorems 1 and 2 thus rest on the following easy consequence of the preceding definitions.

**Proposition 3.4.** If $M$ is a planar $k$-torsion domain for some $k \geq 1$, then it contains a binding circle $\gamma$ or interface torus $T$ in its planar piece such that the following is true. Let $M'$ denote the manifold with corresponding blown up summed open book obtained from $M$ by $\gamma$-rapping surgery along $\gamma$, or $\Omega$-decoupling surgery along $T$ respectively. Then some connected component of $M'$ is a planar $\xi$-torsion domain for some $\xi \in \{k - 2, k - 1\}$.
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Proof. By assumption, $M$ contains a planar piece $M^p$ with nonempty boundary, and if $T_0 \subset \partial M^p$ denotes a boundary component, then the pages in $M^p$ have exactly one boundary component adjacent to $T_0$. The pages in $M^H$ have $k + 1$ boundary components, and without loss of generality we may assume no other irreducible subdomain in the interior of $M$ has planar pages with fewer boundary components than this. Since $k \geq 1$, these pages have at least one boundary component adjacent to some bounding circle or interface torus $T$ distinct from $T_0$. Performing $\pm$-capping surgery to remove $\gamma$ or $\mp$-decoupling surgery to remove $T$ produces a new manifold $M'$ containing a planar irreducible subdomain $M'^p$ whose pages have $\ell$ boundary components where $\ell$ is either $k + 1$ or $k - 1$; the latter can only result from a decoupling surgery along $T \subset M^H \setminus \partial M^H$. Thus $M'$ is a planar $(\ell + 1)$-torus domain unless it is symmetric. The latter would mean $\partial M' = \emptyset$, hence also $\partial M^H = \emptyset$, and $M \setminus M^H$ is also irreducible and has planar pages with $\ell$ boundary components. This cannot arise from decoupling surgery along a binding circle or decoupling surgery along a torus in the interior of $M^H$, as we assumed all planar pages in the interior of $M$ outside of $M^H$ to have at least $k + 1 \geq \ell + 1$ boundary components. The only remaining possibility would be decoupling surgery along $T \subset \partial M^H$, but then symmetry of $M'$ would imply that $M$ must also have been symmetric, hence a contradiction. \hfill □

3.2. A model for a blown up sumed open book. Assume $(M_0, \xi)$ is a compact contact 3-manifold, possibly with boundary, supported by a blown up sumed open book $\pi$ with binding $B$, interface $I$ and fibration

$$\pi : M_0 \setminus (B \cup I) \to S^1.$$  

We assume that each connected component of $M_0$ contains at least one component of $B \cup I \cup \partial M_0$, so that $\pi$ will support a contact structure everywhere. It will be useful to identify this with the following generalization of the notion of an abstract open book (cf. [15]).

The closure of a fiber $\pi^{-1}(\text{const}) \subset M_0$ is the image of some compact oriented surface $S$ with boundary under an immersion

$$i : S \to M_0,$$

which is an embedding on the interior. The monodromy of the fibration then determines (up to isotopy) a diffeomorphism $\psi : S \to S$ which preserves connected components and is the identity in a neighborhood of the boundary, and we define the mapping torus

$$S_\psi = (S \times [0,1]) / \sim$$

with $(z,t+1) \sim (\psi(z),t)$ for all $t \in \mathbb{R}$, $z \in S$. Denote by

$$\phi : S_\psi \to \mathbb{R}/\mathbb{Z} = S^1$$

the natural fibration.

Let us label the connected components of $\partial S$ by

$$\partial S = \partial_0 S \cup \ldots \cup \partial_n S,$$

and for each $i = 1, \ldots, n$ choose an open collar neighborhood $U^i \subset \partial_i S$ on which $\psi$ is the identity. Denote the union of all these neighborhoods by $U \subset S$. Now for each $i = 1, \ldots, n$, choose positively oriented coordinates

$$(\theta, \rho) : U^i \to S^1 \times [\rho,1)$$

for some $\rho \in (0,1)$. These neighborhoods give rise to corresponding collar neighborhoods of $\partial S_\psi$,

$$U^i_\rho - U^i \times [\rho,1) \subset S_\psi,$$

which can be identified with $S^1 \times [\rho,1) \times S^1$ via the coordinates $(\theta, \rho, \phi)$. The index set $I = \{1, \ldots, n\}$ comes with an obvious partition

$$I = I_B \cup I_I \cup I_0,$$

where

$$I_B = \{i \in I \mid (\partial_i S) \subset B\},$$

$$I_I = \{i \in I \mid (\partial_i S) \subset I\},$$

$$I_0 = \{i \in I \mid (\partial_i S) \subset \partial M_0\}.$$

There is also a free $\mathbb{Z}_2$-action on $I_B$ defined via an involution

$$\sigma : I_B \to I_B$$

such that $j = \sigma(i)$ if and only if $i(\partial_i S)$ and $j(\partial_j S)$ lie in the same connected component of $I$. Now define for each $i \in I$ the domain

$$N_i = \begin{cases} S^1 \times \mathbb{R} & \text{if } i \in I_B, \\ S^1 \times [-1,1] \times S^1 & \text{if } i \in I_I, \\ S^1 \times [0,1] \times S^1 & \text{if } i \in I_0, \end{cases}$$

and denote by $(\theta, \rho, \phi)$ the natural coordinates on $N_i$, where for $i \in I_B$ we view $(\rho, \phi)$ as polar coordinates on the disk with the angle normalized to take values in $S^1 = \mathbb{R}/\mathbb{Z}$. Denote the subsets $(\rho = 0)$ by

$$E_{iB} = \bigcup_{i \in I_B} S^1 \times \{0\} \subset \bigcup_{i \in I_B} N_i, \quad T_{iB} = \bigcup_{i \in I_B} S^1 \times \{0\} \times S^1 \subset \bigcup_{i \in I_B} N_i.$$

The chosen coordinates on the neighborhoods $U^i_\rho$ then determine a gluing map

$$\Phi : \bigcup_{i \in I_B} U^i_\rho \to \bigcup_{i \in I_B} N_i$$

which takes $U^i_\rho$ to $N_i$, and we use this to define a new compact and oriented manifold, possibly with boundary,

$$M_{iB} = S_\psi \cup \left( \bigcup_{i \in I_B} N_i \right) / \sim,$$

where the equivalence relation identifies $(\theta, \rho, \phi) \in N_i$ for $i \in I_B$ with $(\theta, -\rho, -\phi) \in N_j$. This naturally contains $E_{iB}$ and $T_{iB}$ as submanifolds, and the fibration $\phi : S_\psi \to S^1$ can be extended over $M_{iB} \setminus (E_{iB} \cup T_{iB})$ so that it matches the canonical $\phi$-coordinate on $N_i$ wherever $\rho > 0$. Now $M_0$ can be identified with $M_{iB}$ via a diffeomorphism that maps $B$ to $E_{iB}$ and $I$ to $T_{iB}$, and transforms the fibration $\pi : M_0 \setminus (B \cup I) \to S^1$ to $\phi$.  

A supported contact structure on $M_{iB}$ can be defined as follows. First, define a smooth 1-form of the form

$$\lambda_0 = f_i(\rho) \, \partial \theta + g_i(\rho) \, d\phi$$

on $N_i$, $i \in I$, where $f_i, g_i : [0,1] \to \mathbb{R}$ are smooth functions chosen to have the following properties:
(1) As \( r \) moves from 0 to 1, \( \rho \mapsto (f(\rho), g(\rho)) \in \mathbb{R}^2 \setminus \{0\} \) defines a path through the first quadrant from (1,0) to (0,1).

(2) \( \lambda_0 \) is contact on \([0, \rho < r] \subset N_i\).

(3) \( f(\rho) = 0 \) for \( \rho \in [r, 1]\).

(4) \( g(\rho) = -1 \) for \( \rho \in [r', 1] \), for some positive number \( r' < r \).

(5) \( g(\rho) > 0 \) for \( \rho \in (0, r') \).

Remark 3.5. The contact condition is satisfied if and only if \( f(\rho) + \rho g(\rho) \neq 0 \), except at \( f(1) = 0 \), where the coordinate singularity changes the condition to \( g(1) \neq 0 \). One consequence is that \( f(\rho) < 0 \) for \( \rho \in [r', r] \), hence \( f(\rho) > 0 \). The assumption that \( \lambda_0 \) is a smooth 1-form imposes some additional conditions; namely, for \( \iota \in I_i \), \( (\rho, \phi) \mapsto f(\rho) + (\rho, \phi) \mapsto g(\rho)/\rho^2 \) must define smooth functions at the origin in \( \mathbb{R}^2 \) (in polar coordinates), and for \( \iota \in I_i \), \( f_\iota \) and \( g_\iota \) can be extended smoothly over \([-1, 1]\) such that

\[
(f(\rho) + \rho g(\rho)) = (0, -1) \quad \text{for} \quad \rho < [-1, -r].
\]

In particular this implies \( f(\rho) + \rho g(\rho) = (0, -1) \) for \( \rho \in [-1, -r] \). We will assume these conditions are always satisfied without further comment.

The co-oriented distribution

\[
\tilde{\lambda}_i := \ker \lambda_i
\]

is a cobinormal on \( M_{\text{lib}} \), which is integrable on the mapping torus \( S_\phi \), and outside of this is a positive contact structure. To perturb it to a global contact structure, choose a 1-form \( \alpha \) on \( S \) which satisfies \( d\alpha > 0 \) and takes the form

\[
\alpha = (2 - \rho) d\theta
\]

on \( H^i \). By a simple interpolation trick (cf. [15], [16]), \( \alpha \) can be used to construct a 1-form \( \alpha_\phi \) on \( S_\phi \) that satisfies

\[
\alpha_\phi \big|_{S_\phi} > 0 \quad \text{and} \quad \alpha_\phi = (2 - \rho) d\theta \quad \text{on} \quad H^i.
\]

Choosing \( \epsilon > 0 \) sufficiently small, we can bring \( \ker (d\phi \circ \alpha_\phi) \) sufficiently \( C^\omega \)-close to \( \tilde{\lambda}_i \) on \( S_\phi \), so that \( \ker (d\phi \circ \alpha_\phi) \) can be chosen to equal \( \lambda_i \) near \( \partial M_{\text{lib}} \), which can be written

\[
\alpha_\phi = \begin{cases}
\phi + c_\phi & \text{on} \ S_\phi, \\
\phi, & \text{on} \ [\rho \in [r', r]] \subset N_i,
\end{cases}
\]

where the fact \( f(\rho) > 0 \) for \( \rho < 0 \) sufficiently small to choose smooth functions \( f_\phi \in [0,1] \to \mathbb{R} \) satisfying

- \( f_\phi(\rho) = f(\rho) \text{ for } \rho \in [0, r'] \),
- \( f_\phi(\rho) = \rho g(\rho) \text{ for } \rho \in [r, r'] \),
- \( f_\phi(\rho) = \rho g(\rho) \text{ for } \rho \in [0, r] \).

Note that for \( \iota \in I_i \), \( f_\iota(\rho) \) also extends naturally over \([-1, 1]\) with \( f_\iota(\rho) = \rho g(\rho) \text{ for } \rho \in [0, r'] \), and \( f_\iota(\rho) = \rho g(\rho) \text{ for } \rho \in [0, r] \).

All contact forms that one can construct in this way are homotopic to each other through families of contact forms, so the resulting contact structure

\[
\xi_i := \ker \lambda_i
\]

is uniquely determined up to isotopy. Moreover, it is easy to check that the Reeb vector field determined by \( \lambda_i \) is everywhere positively transverse to the pages: in particular, \( \lambda_i \) is a Gromov form for the blown up summed open book we’ve constructed on \( M_{\text{lib}} \), thus \( (M_{\text{lib}}, \xi_i) \) is contactomorphic to (\( M, \xi \)).

3.3. A model decoupling cobordism. Assume now that the manifold \( M_i \) from the previous section is embedded into a closed contact 3-manifold \((M, \xi)\) such that \( \xi \) is an extension of the contact structure that was given on \( M_i \). Without loss of generality, we can identify \((M, \xi)\) with the abstract model \((M_{\text{lib}}, \xi_i)\), and assume in particular that \( \lambda_0 \) and \( \lambda_i \) are 1-forms on \( M \) which restrict on \( M_i \) to the models constructed above, and on a neighborhood of \( M \setminus M_i \) define contact forms whose kernels are \( \xi \).

Our goal in this section is to construct a weak symplectic cobordism that realizes a \( \Sigma \)-decoupling surgery along some set of oriented interface tori

\[
\partial_i = T_1 \cup \ldots \cup T_N \subset \mathbb{Z}.
\]

The chosen orientation of each \( T_j \) splits a tubular neighborhood \( N(T_j) \subset M_j \) naturally into positive and negative parts

\[
N(T_j) = N_j \cup N_j^\alpha \subset N_j \cup N_j^\alpha \subset M_j
\]

whose intersection is \( T_j \). To simplify notation in the following, let us assume these neighborhoods are chosen and the page boundary components \( \partial S = \partial_1 S \cup \ldots \cup \partial_N S \) are ordered so that for each \( j = 1, \ldots, N \),

\[
N(T_j) = N_j^\alpha \times [-1, 1] \times S^1 \quad \text{and} \quad N_j^\alpha \cup N_j = S^1 \times [0, 1] \times S^1.
\]

We will fix on \( N(T_j) \) the standard coordinates \((\theta, \rho, \phi)\) of \( N_j \), and assume all the functions chosen to define \( \lambda_0 \) and \( \lambda_i \) are the same for all of these neighborhoods, so we can write

\[
f = f_j \quad g = g_j \quad f = f_j
\]

for \( j = 1, \ldots, N \).

For Theorems 3.1 and 3.2 the cobordism we construct will be needed to be attached to a trivial cobordism of the form \([0,1] \times M \cup \omega \), which will be impossible if our model symplectic form does not match \( \omega \) at least cohomologically at \([1] \times M \). In order to realize the right cobordism class in the model, we choose a closed 2-form \( \omega_\phi \) on \( M \) representing an arbitrary cobordism class for which the condition (3.7) is satisfied. Since we only care about \( \omega_\phi \) up to cobordism, we are free to add an exact 2-form and thus assume \( \omega_\phi \) satisfies

\[
\omega_\phi = c_j d\phi \quad \text{on} \quad N(T_j)
\]

for each \( j = 1, \ldots, N \), where \( c_j \in \mathbb{R} \) are constants satisfying

\[
\sum_{j=1}^N c_j = 0.
\]

Since \( \lambda_0 \wedge d\lambda_0 > 0 \) everywhere on \( M \), we can define an exact symplectic form on the trivial cobordism \([0,1] \times M \cup \omega \), which will be impossible if our model symplectic form does not match \( \omega \) at \([0,1] \times M \). With \( \omega(0) = 0 \) and \( \omega(1) \) uniformly small, and set

\[
\omega_\phi = \omega(\phi) \quad \text{and} \quad \lambda_j = \lambda(\phi).
\]

If \( |\phi| \) is sufficiently small then \( \omega_\phi \) is symplectic and restricts positively to both \( \xi \) and the pages of \( \Sigma \). Now if \( c \) is a sufficiently large constant, then the 2-form

\[
\omega_\Sigma := c \omega_\phi + \omega_\phi
\]
also has these properties. In the following we shall always assume $C$ is arbitrarily large whenever convenient. Note that for the case of Theorem 3.2 we may assume without loss of generality that $\mathbf{1}_\text{H} = 0$, see Remark 3.2.

To construct a cobordism corresponding to the round handle attachment, we shall first "dig a hole" in the trivial cobordism $[0,1] \times M$ near each of the tori $\{1\} \times T_j$. In order to find nice coordinates near the boundary of the hole, it will be useful to consider the vector field $X_\mathbf{1}$ on $[0,1] \times \mathcal{N}(T_j)$ defined by the condition

$$\omega(X_\mathbf{1}, \cdot) = -d\theta.$$  

**Lemma 3.6.** The vector field $X_\mathbf{1}$ is locally Hamiltonian with respect to $\omega_\mathbf{1}$ and takes the form

$$X_\mathbf{1} = A(t, \rho) \partial_t + B(t, \rho) \partial_\rho$$

for some smooth functions $A, B : [0,1] \times [0,1] \to \mathbb{R}$ with the following properties:

1. For $\pm \rho \in [0,1]$, $A(t, \rho) = 0$ and $B(t, \rho) = \pm \rho$.
2. For $\pm \rho \in [0,1]$, $A(t, \rho) = 0$ and $\pm B(t, \rho) > 0$.
3. For $\rho \in (0, 1)$, $A(t, \rho) < 0$.

**Proof.** By a direct computation, $X_\mathbf{1}$ takes the form $\mathbf{29}$ with $A$ and $B$ satisfying the linear system

$$\begin{pmatrix} -\partial_t f(\rho) - \partial_\rho f'(\rho) + f''(\rho) \\ \partial_\rho f(\rho) + f'(\rho) \end{pmatrix} \begin{pmatrix} A(t, \rho) \\ B(t, \rho) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$ 

The determinant $\Delta(t, \rho)$ of this matrix is always negative since the contact condition requires $f'(\rho)f''(\rho) - f'(\rho)f''(\rho) > 0$ for $|A| < \rho$, and for $\pm \rho \in [0,1]$ we have $g(\rho) = \pm 1$, $\pm f'(\rho) \leq 0$ and $\partial_\rho f(\rho) < 0$. The general solution for $A$ and $B$ can thus be written as

$$A(t, \rho) = \frac{1}{\Delta(t, \rho)} \begin{pmatrix} |(f + 1)g'(\rho)| \\ -\partial_\rho f(\rho) \end{pmatrix}.$$ 

The stated conditions on $A(t, \rho)$ and $B(t, \rho)$ then follow immediately from the conditions we've placed on $f, g, f_\rho$ and $\varphi$.

In light of (29), $X_\mathbf{1}$ is in the kernel of $d\omega_\mathbf{1} \wedge d\theta$, and we conclude easily that it is locally Hamiltonian since

$$L_{X_\mathbf{1}} \omega_\mathbf{1} = d\omega_\mathbf{1}(C_{\omega_\mathbf{1}} + c_1 d\omega_\mathbf{1} \wedge d\theta) = d(\omega_\mathbf{1} \wedge C_\theta) = 0.$$ 

Due to the lemma, we can choose a smoothly embedded curve

$$[-1,1] \to [1/2,1] \times [-1,1] ; \tau \mapsto (t(\tau), \rho(\tau))$$

that is everywhere transverse to the vector field $\mathbf{29}$ and also satisfies $(t(0), \rho(0)) = (1/2,0)$ and

$$(t(\tau), \rho(\tau)) = (\mp \tau, \pm 1)$$

near $\tau = \pm 1$ (see Figure 2). Writing the annulus as $A = [-1,1] \times S^1 \setminus \mathbf{1}$, use the curve just chosen to define an embedding

$$\Psi : S^1 \times A \to [0,1] \times \mathcal{N}(T_j) ; (\theta, \tau, \rho) \mapsto (t(\tau), \rho(\tau), \rho),$$

which traces out a smooth hypersurface $\mathcal{H}_J \subset [0,1] \times \mathcal{N}(T_j)$ that meets $\mathbf{1} \times M$ transversally at the pair of tori $\{1\} \times \partial \mathcal{N}(T_j)$. Denote by

$$\mathcal{H}_J \subset [0,1] \times M$$

the interior of the component of $([0,1] \times M) \setminus \mathcal{H}_J$ that contains $\{1\} \times T_j$ (see Figure 2). Observe that by construction, $\mathcal{H}_J$ lies entirely within $[1/2,1] \times \mathcal{N}(T_j)$, and the locally Hamiltonian vector field $X_\mathbf{1}$ points transversely outward at $\partial \mathcal{H}_J \cap \partial \mathcal{N}(T_j)$. Thus for sufficiently small $\delta > 0$, we can use the flow $\phi_{X_\mathbf{1}}^\delta$ of $X_\mathbf{1}$ to parametrize a neighborhood of $\mathcal{H}_J$ in $\mathcal{N}(T_j)$ by an embedding

$$\Psi : (1 - \delta, 1) \times S^1 \times A \to [0,1] \times M$$

such that $\eta$ is an $S^1$-invariant 1-form on $S^1 \times A$ that satisfies

$$\eta = \pm \omega(\pi t) + d\omega$$

near $\tau = \pm 1 \to \partial(S^1 \times A)$, and $d\eta \wedge d\theta > 0$ everywhere.

**Lemma 3.7.** We have

$$\Psi^* \omega_\mathbf{1} = -d(\sigma d\theta) + d\eta,$$

where $\eta$ is an $S^1$-invariant 1-form on $S^1 \times A$ that satisfies

$$\eta = \pm \omega(\pi t) + d\omega$$

near $\tau = \pm 1 \to \partial(S^1 \times A)$, and $d\eta \wedge d\theta > 0$ everywhere.

**Proof.** In $[0,1] \times \mathcal{N}(T_j)$ we can write $\omega_\mathbf{1} = d\lambda$, where

$$\lambda := \psi(\iota_{X_\mathbf{1}} \lambda_0) + \epsilon d\theta.$$ 

Then defining $\eta := \Psi^* \lambda$ on $S^1 \times A$, we have $d\eta = \Psi^* d\lambda_0$ and can write $\eta$ explicitly near $\tau = \pm 1$ by plugging in $t = \pi \tau$, $\rho = \pm 1$, $f(\rho) = 0$, $\psi(\rho) = \pm 1$ and $f_\rho(\rho) = \epsilon(2 \rho - 1)$, hence

$$\eta = \psi(\iota_{X_\mathbf{1}} \lambda_0) + \epsilon d\theta = \psi(\iota_{X_\mathbf{1}} \lambda_0 + d\omega) + \epsilon d\theta \pm \partial_\theta d\omega.$$ 

As desired. Since $\lambda$ is invariant under the $S^1$-action by translation of $\theta$, $\eta$ is also $S^1$-invariant. The claim $d\theta \wedge d\eta > 0$ is a consequence of the fact that $H_{\mathcal{H}_J}$ is transverse to the vector field $X_{\mathbf{1}}$, which is $\omega_\mathbf{1}$-dual to $-d\theta$; indeed, ignoring combinatorial factors we find

$$d\eta = d\eta(\partial_\theta, \partial_\rho, \partial_\lambda) = -\omega(\iota_{X_\mathbf{1}} \omega_\mathbf{1}) + \omega(\iota_{X_\mathbf{1}} \partial_\theta, \partial_\rho, \partial_\lambda) \neq 0.$$ 

It follows that $d\theta \wedge d\eta$ is positive since this is obviously true near $\tau = \pm 1$. The formula (3.10) now follows from the fact that $-d\theta = \iota_{X_\mathbf{1}} d\lambda_0$ and $X_\mathbf{1}$ has a symplectic flow. \hfill $\square$

**Figure 5.** The path $(t(\tau), \rho(\tau))$ transverse to the vector field of (3.9).
for any $(\tau, \phi) \in \Lambda$, and makes $T(\Sigma \times \{\ast\})$ and $T(\{\ast\} \times \Lambda)$ into symplectically orthogonal symplectic subspaces everywhere along $\Sigma \times \partial \Sigma$.

Proof. We will use a standard deformation trick to simplify $\nu$ on each of the regions $V_j \times \Lambda$ so that it can be extended as a split symplectic form. Choose a 1-form $\eta_0$ on $\Lambda$ with $d\eta_0 > 0$ and lift it in the obvious way to $S^1 \times \Lambda$. Since

$$\int_{[\sigma_0, \sigma_1]} \eta = 2 \left[ \sigma_1 (1 + \delta) \right] > 0$$

and $\eta$ has no $d\delta$-term near $S^1 \times \partial \Sigma$, we can also arrange for $\eta_0$ to match $\eta$ on a neighborhood of $S^1 \times \partial \Sigma$. Next choose a smooth cutoff function $\beta : (1 - \delta, 1] \to [0, 1]$ that satisfies $\beta(\sigma) = 0$ near $\sigma = 1 - \delta$ and $\beta(\sigma) = 1$ near $\sigma = 1$, and use this to define a smooth function $\beta : \Sigma \to [0, 1]$ which satisfies

$$\beta(\sigma, \tau, \phi) = \beta(\sigma)$$

on $V_j$. Choose also a smooth function $\psi : (1 - \delta, 1] \to [0, 1]$ satisfying $\psi(0) > 0$ and $\psi(\sigma) = 1$ near $\sigma = 1$, and a 1-form $\mu$ on $\Sigma$ such that

$$\mu = -\psi(\sigma) d\sigma$$

in $V_j$, $d\mu > 0$ everywhere.

A suitable symplectic form on $\Sigma \times \Lambda$ can then be defined by

$$\omega_{\mu} = \omega_{\eta} + d([\psi_{\tau} \sigma \psi_{\delta} \eta_{\delta}] + (1 - \beta)\eta_0).$$

By construction, $\omega_{\mu}$ matches $\omega_{\eta}$ near $\partial \Sigma \times \Lambda$, while near $\Sigma \times \partial \Lambda$ and outside of the regions $V_j \times \Lambda$ it takes the split form

$$-d\sigma + \eta,$$

which is symplectic and makes each of $T(\Sigma \times \{\ast\})$ and $T(\{\ast\} \times \Lambda)$ into symplectically orthogonal to each other. To test whether $\omega_{\mu}$ is symplectic on $V_j \times \Lambda$, we compute

$$\left[ \frac{1}{2} \cdot \omega_{\mu} \right],$$

the first term is always nonzero since $\partial \Sigma \times \Lambda$ and $\partial \Lambda \times \partial \Lambda$ are both positive. The whole expression is therefore nonzero whenever either $\beta(\sigma) = 0$ or $\psi(\sigma)$ is sufficiently large, and we are free to choose $\psi$ so that it increases fast in the region where $\beta$ is not constant. This choice also ensures $\omega_{\mu}(d\sigma, d\phi) > 0$ everywhere on $V_j \times \Lambda$.

To find a symplectic extension of $\omega_{\eta}$ over $\Sigma \times \Lambda$, choose now a closed 1-form $\kappa$ on $\Sigma$ which takes the form

$$\kappa = c d\phi,$$

near each boundary component $\partial \Sigma^2$. This is possible due to the homological condition $\kappa$. Then if $\omega_{\kappa}$ denotes the extension of $\omega_{\mu}$ given by Lemma 3.9, we extend $\omega_{\eta}$ as

$$\omega_{\eta} := C \omega_{\mu} + d\phi \wedge \kappa.$$
Whenever \( C \) is sufficiently large, Lemma 3.3 implies that this form is also symplectic and restricts positively to the surfaces \( -\Sigma \times \{(r, \phi)\} \) and \( \{p\} \times \mathbb{R}_+ \) if \( p \notin \Sigma \) lies outside a neighborhood of the boundary. This implies that it is positive on the pages of \( \pi' \), as well as on the core \( K_{\Sigma} = (0, 1] \times \Sigma_0 \cup (-\Sigma \times \{(0, 0)\}) \subset W \) and co-core \( K_{\Sigma} = \{p\} \times \mathbb{R}_+ \subset W \) (cf. 3.3).

To summarize: we have constructed a smooth cobordism \( W \) with symplectic form \( \omega'_0 \) that matches \( \omega_0 \) near \( M = \{0\} \times M \) and is positive on the core and co-core and on the pages of the induced blow up summed open book at the other boundary component \( M' \). An appropriate conflation \( 1 \)-form \( \lambda'_0 \) can now be defined on \( M' \) by

\[
\lambda'_0 = \begin{cases} 
\lambda_0 & \text{on } M \setminus N(\Sigma_0), \\
\phi_0 & \text{on } \Sigma_0 \times S^1,
\end{cases}
\]

where we use \( \phi_0 \) to denote the natural \( S^1 \)-coordinate on \( \Sigma_0 \times S^1 \). The distribution \( \mathcal{D}_0 := \ker \lambda'_0 \) is then tangent to the pages on \( \phi_0 = 0 \), hence \( \omega'_0|_{\mathcal{D}_0} > 0 \). It follows that on any connected component of \( M' \) that does not contain closed pages, \( \omega'_0 \) has a perturbation to a contact structure \( \xi' \) that is supported by \( S' \) and dominated by \( \omega_0'. \)

**Remark 3.10.** It will be useful later to observe that \( \int_{\partial M} \omega'_0 = 0 \) is not only positive but can be assumed to be arbitrarily large. In fact it must in general be large due to the deformation trick used in the proof of Lemma 3.3.

**Remark 3.11.** If the constants \( c_j \) all vanish, i.e. \( \theta_0 = 0 \) on \( N(\Sigma_0) \), then one can choose the \( 1 \)-form \( \kappa \) in the above construction to be identically zero. This has the useful consequence that for any \( \tau \in [-1, 1] \) and any closed embedded loop \( \ell \subset \Sigma \) outside a neighborhood of \( \partial \Sigma \), the forms \( \tau \times \{\ell\} \times S^1 \subset \Sigma \times \mathbb{R}_+ \) is Lagrangian. More generally, if \( \tau \subset \Sigma \) is any properly embedded compact \( 1 \)-dimensional submanifold transverse to \( \partial \Sigma \), then

\[
\int_{\partial M} \omega'_0 = 0.
\]

Indeed, with \( \kappa = 0 \) it is equivalent to show that the integral of \( \omega'_0 \) vanishes, and using 3.3 we find

\[
\int_{\partial M} \omega'_0 = \int_{\partial M} \omega_0 = \int_{[\delta, 1]} \eta = 0
\]

since \( \mu \) vanishes on the \( S^1 \)-factor in \( \partial M \times \{\tau\} \times S^1 \). Since \( \eta \) is \( S^1 \)-invariant on \( S^1 \times \mathbb{R}_+ \), this integral doesn’t depend on the position of any point in \( \partial M \times \mathbb{R}_+ \) but only on the algebraic count of these points, which is zero, thus

\[
\int_{[\delta, 1]} \eta = 0.
\]

**Remark 3.12.** The reader who is only interested in strong cobordisms, or more generally the case where the negative boundary of the cobordism is (strongly) concave, may assume throughout this section that \( \theta_0 = 0 \). In this case, the symplectic form \( \omega \) defined on \( W \) is exact near \( M \subset \partial W \) and has a primitive there which restricts to a constant multiple of the contact form \( \lambda_0 \) so this boundary component is concave. The contents of the rest of this section therefore suffice to complete the proofs of Theorems 3.2 and 3.4 respectively if the given \( \omega \) on \( \{0, 1\} \times M \) is exact: indeed, by 3.2 and Proposition 3.1, \( \omega \) can then be deformed to make it (strongly) convex at the positive boundary, after a further deformation to match the contact forms, the Liouville flow can be used to attach it smoothly to our model as long as the constant \( C > 0 \) is chosen sufficiently large. The case where \( \omega \) is not exact requires the additional deformation argument of 3.3 below.

**3.4. Modifications for the capping cobordism.** The above construction works essentially the same way for the handle \( \Sigma \times \mathbb{D} \), so we will be content to briefly summarize the differences. Here we pick binding components

\[
B_0 = \gamma_1 \cup \ldots \cup \gamma_n \subset B
\]

denote the corresponding solid torus neighborhoods by \( N(\gamma_j) \subset S^1 \times \mathbb{D} \) with coordinates \( (\theta, \rho, \phi) \), viewing \( (\rho, \phi) \) as polar coordinates, and denote the union of these neighborhoods by \( N(\partial B_0) \). The model symplectic form \( \omega_0 \) on the trivial cobordism \( \{0\} \times M \) is again defined via 3.3 and 3.4, with the difference that since every closed 2-form on \( N(\gamma_j) \) is exact, we can assume (after adding an exact 2-form) that \( \theta_0 \) vanishes on all of these neighborhoods. The role of \( H_{\gamma_j} \) is now played by a hypersurface

\[
H_{\gamma_j} \subset \{0, 1\} \times N(\gamma_j)
\]

parameterized by an embedding

\[
\Psi : S^1 \times \mathbb{D} \rightarrow \{0, 1\} \times M,
\]

defining exactly as before, and then use its flow to parametrize a neighborhood of \( H_{\gamma_j} \) in the region \( H_{\gamma_j} \) that it bounds via a map

\[
\Psi : (1 - \delta, 1] \times S^1 \times \mathbb{D} \rightarrow \{0, 1\} \times N(\gamma_j) : (\theta, \rho, \phi) \mapsto \Psi(\theta, \rho, \phi)
\]

for which \( \widetilde{\Psi}'_{\omega_0} \), again takes the form \( -d(\sigma \partial \theta + \eta) + d\eta \) for some \( 1 \)-form \( \eta \) on \( S^1 \times \mathbb{D} \) that satisfies

\[
\eta = \sigma(\tau + 1) \omega_0
\]

near \( S^1 \times \partial \mathbb{D} \) and \( \partial \mathbb{D} \times \mathbb{D} \) everywhere. Denote the image of \( \widetilde{\Psi}'_{\omega_0} \) corresponding to each \( \gamma_j \) by \( W_j \), with negatively oriented coordinates \( \sigma(\theta, \tau, \phi) \in (1 - \delta, 1] \times S^1 \times \mathbb{D} \). Writing the union of the regions \( H_{\gamma_j} \) as \( H_{\partial B_0} \), the smooth cobordism is then defined by

\[
W = \{0, 1\} \times M \cup (-\partial W_0) \cup (-\Sigma \times \mathbb{D})
\]

where \( \Sigma \times \mathbb{D} \) is glued in by identifying \( \partial W_j \times \mathbb{D} \) so that the coordinates match. This has boundary \( \partial W = M' \cup (-M) \), where \( M' = \{0\} \times M \) and

\[
M' = \{0\} \times M \cup \{0\} \times S^1 \cup \{0\} \times \mathbb{D}
\]

hence the glued in region \( \Sigma \times S^1 \) carries the coordinates \( \sigma(\theta, \phi) \) near its boundary. Choosing a \( 1 \)-form \( \eta \) on \( \mathbb{D} \) that matches \( \eta \) near \( \partial \mathbb{D} \) and satisfies \( \partial \eta > 0 \), the interpolation trick 3.3 can again be used to deform \( \omega_0 \) in a collar neighborhood of \( \partial \Sigma \times \mathbb{D} \) so that it admits a symplectic extension over the rest of \( \Sigma \times \mathbb{D} \) in the form \( \omega_0 = -d\phi + d\eta \). The resulting form \( \omega'_0 = C \omega'_0 + \phi_0 \) is symplectic everywhere on \( W \) and is also positive on the pages of \( \pi' \) at \( M' \) if \( C \) is sufficiently large, as well as on the core

(3.14) \( K_{\Sigma} = (0, 1] \times \partial B_0 \cup (-\Sigma \times \{0\}) \subset W \)

and the co-core

(3.15) \( K_{\Sigma} = \{p\} \times \partial B_0 \subset W \).
for an appropriate choice of $p \in \Sigma$. The conﬁguration 1-form extends smoothly over $\Sigma \times S^1$ as $\lambda' = \phi_0$, so that $\lambda'$ is also positive on $\phi_0^{-1}(\lambda')$ and thus dominates any contact form obtained as a small perturbation.

3.5. Symplectic deformation in a collar neighborhood. To apply the constructions of the previous sections in proving Theorems 2 and 8 when the given symplectic form $\omega$ on $[0,1] \times M$ dominating $\xi$ is non-exact, we must show that $\omega$ can be deformed away from $[0] \times M$ to reproduce the model

$$\omega_\varepsilon - C d(\varphi_t(\lambda_0) + \lambda + \Omega_0),$$

where $\lambda_0$ and $\lambda_0$ are 1-forms as described at the beginning of Sections 3.3, $\varphi : [0,1] \to \mathbb{R}$ is a smooth function with $\varphi' > 0$, $\varphi(0) = 0$ and $|\varphi'|$ small, $\Omega_0$ is some closed 2-form on $M$ in the appropriate cohomology class, and $C > 0$ is a constant that we can assume to be as large as necessary. The following application of a standard Moser deformation argument (cf. [MSTI] Lemma 2.3) will be useful.

**Lemma 3.13.** Suppose $(W, \omega)$ is a symplectic 4-manifold, $M$ is a closed oriented 3-manifold with an embedding $\Phi : M \to W$ and $\lambda$ is a 1-form on $M$ that satisﬁes $\lambda \wedge \Phi^* \omega > 0$. Then for sufficiently small $\varepsilon > 0$, $\Phi$ extends to an embedding

$$\Phi : (\mathbb{R}, \varepsilon) \times M \to W$$

such that $\Phi(0, \cdot) = \Phi$ and $\Phi^* \omega = d(\lambda) + \Phi^* \omega$.

Observe that if $M \subset W$ is an oriented hypersurface in a symplectic 4-manifold $(W, \omega)$ with a positive contact structure $\xi$, then $\omega$ dominates $\xi$ if and only if it satisﬁes $\lambda \wedge |\xi| > 0$.

for every contact form $\lambda$ on $(M, \xi)$. Using the obvious variants of Lemma 3.13 when the hypersurface is a positive or negative boundary component of $W$, we obtain the following useful consequence:

**Lemma 3.14.** Suppose $(M, \xi)$ is a closed contact 3-manifold and $(\{t \leq 1\} \times M, \omega)$ on $[0,1] \times M$, $\omega_t$ is a 2-ﬁeld on $[0,1] \times M$, then for any small $\varepsilon > 0$, $(\{t \leq 1\} \times M, \omega)$ satisﬁes $\lambda \wedge |\xi| > 0$ on any contact form $\lambda$ on $(M, \xi)$. Using the obvious variant of Lemma 3.13 when the hypersurface is a positive or negative boundary component of $W$, we obtain the following useful consequence:

** Proposition 3.15.** Suppose $(M, \xi)$ is a closed contact 3-manifold with contact form $\lambda$, $\lambda_0$ is a 1-form on $M$ satisﬁes $\lambda_0 \wedge |\xi| > 0$, $\omega = d(\lambda_0) + \lambda$ on $\Omega_0$ is a closed 2-form on $M$ with $\Omega_0 = \lambda_0 \wedge |\xi|$ is symplectic on $[0,1] \times M$ with $\omega_t > 0$, and $\Omega_0$ is a closed 2-form on $M$ with $\Omega_0 = \lambda_0 \wedge |\xi|$ is symplectic on $[0,1] \times M$ with $\omega_t > 0$, and $\Omega_0$ is a closed 2-form on $M$ with $\Omega_0 = \lambda_0 \wedge |\xi|$ is symplectic on $[0,1] \times M$ with $\omega_t > 0$ and $\Delta_0$ is a closed 2-form on $M$ with $\Delta_0 = \phi_0^{-1}(\lambda')$ and $\lambda = \phi_0^{-1}(\lambda')$.

Proof. By Lemma 3.13, we can assume without loss of generality that $\omega$ has the form

$$\omega = d(\lambda_0) + \Omega$$

near $[0] \times M$, where $\Omega$ is the closed 2-form on $M$ deﬁned as the restriction of $\omega$ to $[0] \times M$.

The proof now proceeds in two steps, of which the ﬁrst is to put the symplectic structure $\omega$ of (3.15) into a slightly simpler form via a coordinate change near $[0] \times M$. Deﬁne the 1-form

$$\lambda_0 = \varphi(t)\lambda_0 + \lambda$$

on $[0,1] \times M$ and write $\omega_\varepsilon = d\lambda_0 = \omega_\varepsilon - C_{\Omega_0} + \Omega_0$. Let $\Gamma$ denote the vector ﬁeld that is $\omega^\varepsilon$-dual to $\Omega_0$, i.e. $\omega(\Gamma) = \Omega_0$. For $C$ suﬃciently large, $\Gamma$ is then a small perturbation of the vector ﬁeld that is $\omega_\varepsilon$-dual to $\lambda_0$, which is a Liouville (with respect to $\omega_\varepsilon$) vector ﬁeld positively transverse to $[0] \times M$ since $\lambda_0$ dominates $\xi$ and $\omega_\varepsilon$ is also positive transverse to $[0] \times M$ and use its ﬂow $\varphi_\varepsilon$ to deﬁne an embedding

$$\varphi : [0,\varepsilon] \times M \to [0,1] \times M : (t, m) \to \varphi_\varepsilon(m)$$

for $\varepsilon > 0$ suﬃciently small. If $\lambda_0$ then the Reeb vector ﬁeld determined by $\lambda_0$ along $[0] \times M$ we then have

$$\lambda_0(\varphi_\varepsilon^* \omega_\varepsilon) = C_{\Omega_0} \omega_\varepsilon$$

and

$$\lambda_0(\varphi_\varepsilon^* \omega_\varepsilon) = C_{\Omega_0} \omega_\varepsilon$$

Hence $\lambda_0(\varphi_\varepsilon^* \omega_\varepsilon)$ matches the symplectic form $d(\theta) + C d\lambda_0 + \Omega_0$ pointwise at $[0] \times M$, and another Moser deformation argument thus allows us to isotope the embedding $\varphi_\varepsilon$ so that $\lambda_0(\varphi_\varepsilon^* \omega_\varepsilon)$ takes this form on some neighborhood of $[0] \times M$. Equivalently, this means $\lambda_0(\varphi_\varepsilon^* \omega_\varepsilon)$ admits a deformation to a new symplectic form $\omega_\varepsilon'$, which takes the form

$$\omega_\varepsilon' = d(\theta) + C d\lambda_0 + \Omega_0$$

on an arbitrary small neighborhood of $[0] \times M$ and matches the original $\omega_\varepsilon$ outside a slightly larger neighborhood. Indeed, choose a constant $C' > 0$ large enough so that

$$\|C'\lambda_0 + \Omega_0\| > 0,$$

and since $\Omega$ and $\Omega_0$ are cohomologous by assumption, choose a 1-form $\eta$ on $M$ such that $C'\lambda_0 + \Omega_0 = C d\lambda_0 + \eta$. For some $\delta > 0$ small, choose a cutoff function $\theta(t)$ that equals 0 near $t = 0$ and $t = \delta$, and deﬁne

$$\omega' = d(\theta(t) \lambda_0) + \Omega + d(\theta(t) \eta)$$

with $\theta : [0,\delta] \to [0,\infty)$ a smooth function satisfying

$$\theta(t) = t \text{ near } t = 0,$$

$$\theta'(t) > 0,$$

$$\theta(t) + C' = C(\delta + 1).$$

If $\theta$ is chosen to increase suﬃciently fast, then $\omega'$ is symplectic, and this can always be arranged if $C > 0$ is made suﬃciently large. This depends in particular on the fact that the 2-forms $\Omega$ and $C d\lambda_0 + \Omega_0$ are both positive on $\xi$. The restrictions of $\omega'$ and $\omega_\varepsilon'$ to the hypersurface $\{t = \delta\} \times M$ now match, thus the two can be glued together smoothly by Lemma 3.13.

Combining Proposition 3.15 with the cobordism constructions of 3.14, completeness of the proofs of Theorems 2 and 8.


3.6. Cohomology. We now prove Theorems 3.7 and 3.8 by characterizing the situations in which $\omega$ can be made exact on $W$ or on $M_{\text{non}}$.

Assume first that $(W, \omega)$ is a $\Sigma$-capping cobordism $(\emptyset, 1) \times M \cup N \cup C$, with $N \cup C = \Sigma \times \emptyset$ attached along a neighborhood $N'(B_0)$ of $B_0 = \gamma_1 \cup \cdots \cup \gamma_N$. Write $W = M' \cup (-M)$ and

$$\Omega := \omega|_{M'}, \quad \Omega' := \omega|_{M''}.$$

Due to 3.19 we may assume without loss of generality that $\Omega$ has the form

$$\Omega = C M + \Omega_0,$$

where $C > 0$ is arbitrarily large, $\lambda$ is the usual contact form on $M$ and $\Omega_0$ vanishes on $N'(B_0)$.

By Remark 3.10 we can also assume in the following the $\Omega_0$ has arbitrarily large.

The decomposition of $W$ into $(\emptyset, 1) \times M$ and $N \cup C$, which intersect at $N'(B_0) \subset (\emptyset, 1) \times M$, gives rise to the Mayer-Vietoris sequence,

$$\cdots \to H_2(N'(B_0)) \to H_2(M) \oplus H_2(N) \to H_2(W) \to H_1(N'(B_0)) \to H_1(M) \oplus H_1(N) \to \cdots$$

in which $H_2(N'(B_0)) = H_2(M) \oplus H_2(N) = H_1(N'(B_0)) = H_1(M) \oplus H_1(N)$ and $H_1(N'(B_0)) = H_1(B_0) = \mathbb{Z}^N$.

Thus there is an isomorphism

$$H_2(W) \cong \text{im}(H_2(M) \to H_2(W)) \oplus \ker (H_1(N'(B_0)) \to H_1(M) \oplus H_1(N)),$$

in which the first summand is an isomorphic copy of $H_2(M)$. Denote by $f : N'(B_0) \to M$ and $f' : N'(B_0) \to N$ the natural inclusions. Then $f'^*(\gamma) = \emptyset, \Sigma \subset H_1(M) \oplus H_1(N)$, so since $\Sigma$ is connected, $\ker f'^* = \mathbb{Z}$ is isomorphic to $\mathbb{Z}$ and is generated by $[\gamma_1] + \cdots + [\gamma_N]$. It follows that the second summand in 3.18 consists of all integer multiples of $[\gamma_1] + \cdots + [\gamma_N]$ which are also in $\ker f'$, i.e., it is isomorphic to $\mathbb{Z}$ if $[\gamma_1] + \cdots + [\gamma_N]$ is torsion in $H_1(M)$, and is otherwise trivial. In the former case, let $k_0 \in \mathbb{N}$ be the smallest integer for which $h_0([\gamma_1] + \cdots + [\gamma_N]) = 0 \in H_1(M)$, and construct a cycle $A_{k_0} \in H_2(W)$ in the form

$$A_{k_0} = C M + h_0 K_{\Sigma^2},$$

where $C_M$ is any 2-chain in $(\emptyset, 1) \times M$ with $\partial f_M = h_0([\gamma_1] + \cdots + [\gamma_N])$ and $K_{\Sigma^2} \subset W$ is the core of $M'$.

The isomorphism 3.19 implies that everything in $H_2(W)$ is an element of $H_2(M)$ plus an integer multiple of $h_0 K_{\Sigma^2}$.

Let $\nu$ denote a real 1-cycle in $M \setminus N'(B_0)$ such that $[\nu] = PD(\Omega) \in H_1(M; \mathbb{R})$; note that this is always possible since $\Omega$ is necessarily exact on $N'(B_0)$. The product $[\nu, 1] \times \emptyset$ then represents a relative homology class in $H_2(W, W \setminus \Sigma \times \emptyset)$.

Proposition 3.16. There is a number $c > 0$ such that $PD(\Omega) = [\nu, 1] \times [\nu] + c K_{\Sigma^2}$.

Proof. It suffices to show that for every $A \in H_2(W)$, the evaluation of $\omega$ on $A$ matches the intersection product

$$\int_A \omega = A \cdot ([\nu, 1] \times [\nu] + c K_{\Sigma^2}).$$

For any $A \in \text{im}(H_2(M) \to H_2(W))$ this is immediately clear since

$$\int_A \omega = \int_A \Omega - A \cdot [\nu].$$

where the latter is the intersection product in $M$, and $A$ does not intersect anything in the handle. By 3.15, either the image of $H_2(M) \to H_2(W)$ is the entirety of $H_2(W)$ or there is one more generator $A_{k_0} = C M + h_0 K_{\Sigma^2}$. For the latter we have

$$\int_{A_{k_0}} \omega = \int_{C M} \Omega + h_0 \int_{K_{\Sigma^2}} \omega,$$

and

$$A_{k_0} \cdot ([\nu, 1] \times [\nu] + c K_{\Sigma^2}) = C M \cdot [\nu] + h_0 c,$$

so 3.20 is satisfied if and only if

$$c = \int_{K_{\Sigma^2}} \omega = \sum_{i=1}^{N} \lambda_i h_i \int_{K_{\Sigma^2}} \omega > 0,$$

This is positive without loss of generality since $\int_{K_{\Sigma^2}} \omega$ was assumed to be arbitrarily large.

The above argument also shows that if $[\emptyset] \times M \subset (W, \omega)$ is concave, then $\omega$ cannot be exact if $[\gamma_1] + \cdots + [\gamma_N]$ is torsion, even without assuming $\int_{K_{\Sigma^2}} \omega$ to be arbitrarily large. Indeed, in this case we have $\Omega = -\partial f$ for a contact form $\mu$ on $(M, \xi)$, and $[\nu] = 0$, hence

$$\int_{A_{k_0}} \omega = \int_{C M} \mu + h_0 \int_{K_{\Sigma^2}} \omega = \sum_{i=1}^{N} \lambda_i h_i \int_{K_{\Sigma^2}} \omega,$$

and

$$c = \int_{K_{\Sigma^2}} \omega = \sum_{i=1}^{N} \lambda_i h_i > 0.$$

On the other hand, if $[\gamma_1] + \cdots + [\gamma_N] \in H_1(M)$ is not torsion, then $H_2(M)$ generates everything in $H_2(W)$, so $\int_{K_{\Sigma^2}} \omega$ always vanishes since $\Omega$ is exact. This proves the first half of Theorem 3.7.

We also conclude from the above that if $[\emptyset] \times M \subset (W, \omega)$ is concave, then there is a constant $c > 0$ such that

$$PD(\Omega') = c K_{\Sigma^2} \subset H_2(M'; \mathbb{R}),$$

the second half of the theorem is proved by showing that $[\nu, 1] \times M' \subset H_2(M'; \mathbb{R})$ if and only if the stated homological condition on $[\gamma_1, \ldots, \gamma_N]$ is satisfied. Writing $M' = (M \setminus N'(B_0)) \cup (-\Sigma \times S^1)$, we obtain the Mayer-Vietoris sequence

$$\cdots \to H_2(M') \to H_1(\partial N'(B_0)) \to H_1(M \setminus B_0) \to H_1(\Sigma \times S^1) \to \cdots.$$

where $H_1(\partial N'(B_0)) \cong \mathbb{Z}^N$, with each component $\partial f \gamma_j$ carrying the two distinguished generators $\mu_j$, $\nu_j$ defined in 3.15 Denote the inclusions $f' : N'(B_0) \to M \setminus B_0$ and $f' : N'(B_0) \to \Sigma \times S^1$. Then $f' \gamma_j = \partial f \gamma_j \gamma_j \in H_1(\Sigma \times S^1)$ and $f' \gamma_j = [\gamma_j] \times S^1 \in H_1(\Sigma \times S^1)$, so $K_{\Sigma^2}$ consists of all classes of the form

$$k \sum_{j=1}^{N} \lambda_j + m \sum_{j=1}^{N} \mu_j$$

with $k, m_1, \ldots, m_N \in \mathbb{Z}$ and $\sum m_j = 0$. Now, $[\Sigma^2] \cdot [\Sigma \times S^1] \in M'$ and it vanishes in $H_1(M'; \mathbb{R})$ if and only if

$$A \cdot [\gamma_j] \times S^1 = 0.$$
for every $A \in H_2(M')$. This is true if and only if the image of the map $H_2(M') \to H_i(\partial N'(\partial U))$ in the above sequence contains only cycles of the form $\sum_j m_j \mu_j$. In light of the above description of $h_{\xi}^\Sigma$, this is true if and only if

\[ h(\lambda_1 + \ldots + \lambda_N) \not\in \ker A^m \]

for all $k \neq 0$. This completes the proof of Theorem 7.

The proof of Theorem 7 proceeds similarly: Assume $W = ([0, 1] \times M) \cup \overline{\partial U}$ is a $\Sigma$-decoupling cobordism, with $\overline{\partial U} = \Sigma \times \Delta$ attached along a neighborhood $\{N'I\}(Z_0)$ of $T_1 \cup \ldots \cup T_N$, where $\partial W = M' \cup (-M')$, $\omega = \omega_{M'}$ and $\omega' = \omega^{\overline{\partial U}}$. We again assume that $\int_{T_j} \omega$ is arbitrarily large, and that $\Omega$ takes the form of (3.21), and we also impose the extra condition

\[ \int_{T_j} \Omega = 0 \]

for every component $T_j \subset Z_0$.

In this case we can find a real 1-cycle $h$ in $M \setminus N'\{Z_0\}$ that represents $PD(\langle \Omega \rangle) \in H_1(M'; \mathbb{R})$. Without changing the cohomology class or the symplectic properties of $\omega$, we can then also assume that $\Omega_0$ is supported in a tubular neighborhood of the cycle $h$.

Recall from (10) that each oriented torus $T_j \subset Z_0$ comes with a distinguished homology basis $\{\mu_j, \lambda_j\} \subset H_2(T_j)$, where $\lambda_j$ is a boundary component of a properly embedded $\lambda_j$-perturbed $\lambda_j$ by a Legendrian loop in $T_j$. This also gives rise to bases $\{\mu_j^\Sigma, \lambda_j^\Sigma\}$ of $H_i(\partial N'(\Sigma))$, where the orientation of $\mu_j^\Sigma$ is reversed compared with $\mu_j$. For $W = ([0, 1] \times M) \cup \overline{\partial U}$ we have the Mayer-Vietoris sequence

\[
\cdots \to H_2(M') \oplus H_2(\overline{\partial U}) \to H_2(W) \to H_1(N'\{Z_0\}) \to H_1(M') \oplus H_1(\overline{\partial U}) \to \cdots
\]

and resulting isomorphism

\[
H_2(W) \cong \ker \left( H_1(M') \oplus H_1(\overline{\partial U}) \to H_1(N'\{Z_0\}) \to H_1(M') \oplus H_1(\overline{\partial U}) \to \cdots \right) \cong \ker \left( H_1(N'\{Z_0\}) \to H_1(M') \oplus H_1(\overline{\partial U}) \right).
\]

(3.21)

Denote the generator of $H_1(\Delta) = \mathbb{Z}$ by $[S^1]$, which can also naturally be regarded as a primitive class in $H_2(\overline{\partial U}) = H_2(\Sigma') \cong H_2(\Sigma)$. Then writing the inclusions as $i^0 : N'\{Z_0\} \to M$ and $i^1 : N'\{Z_0\} \to \overline{\partial U}$, we have $i^0_*(\lambda_j) = [\Sigma] \times \{0\}$ and $i^1_*(\mu_j) = [S^1]$, hence $\ker h$ consists of all classes of the form

\[ k \sum_{j=1}^N \lambda_j + \sum_{j=m}^N m_j \mu_j \]

for $k, m_1, \ldots, m_N \in \mathbb{Z}$ with $\sum m_j = 0$. For any $\Gamma \in H_1(Z_0)$ of this form which is also nullhomologous in $M$, we can form a cycle $\partial \Gamma \subset H_0(W)$ as follows. First choose a 2-chain $C_M$ in $[0, 1] \times M$ with $\partial C_M = \Gamma$. Choose also a 1-chain $\varepsilon \in \Sigma$ with boundary in $\partial \Sigma$ such that the 2-chain $\varepsilon \times \{0\} \times S^1$ in $\overline{\partial U}$ has boundary

\[ \partial (\varepsilon \times \{0\} \times S^1) = - \sum_{j=m}^N m_j \mu_j, \]

which is always possible since $\sum m_j = 0$. We can represent $\varepsilon$ by a properly immersed submanifold in $\Sigma$ so that by Remark 8.11, \[ \int_{\varepsilon} \omega = 0. \] Now extend $\varepsilon \times \{0\} \times S^1$ to a 2-chain in $W$ with boundary in $[0, 1] \times M$ by attaching cylinders over the appropriate coves of Legendrian representatives of $\mu_j$. Since these cylinders are Lagrangian, this construction yields an immersed submanifold $L \subset W$ which satisfies

\[ \int_{L} \omega = 0 \]

and $DL \subset \{0\} \times M$, with $|DL| = - \sum_{j=m}^N m_j \mu_j$. We define $A_M \in H_2(W)$ by

\[ A_M = C_M + DL + h(\xi) = \mathbb{R}. \]

Proposition 3.17. There is a number $c > 0$ such that $PD(\langle \omega \rangle) = [0, 1] \times [h] + cK_{\Sigma}$, $e_{\Sigma}$.

Proof. The goal is again to prove

\[
\int_A \Omega = A \cdot \left( [0, 1] \times [h] + cK_{\Sigma} \right)
\]

for every $A \in H_2(W)$, and it is again immediate if $A \in \text{im}(H_2(M) \to H_2(W))$. It is also clear for $A \in \text{im}(H_2(\overline{\partial U}) \to H_2(W))$, as $H_2(\overline{\partial U})$ is generated by classes of the form $[\ell] \times [S^1]$ for $\ell \in H_1(\Sigma)$, hence both sides of (3.28) vanish (see Remark 8.11).

The rest of $H_2(W)$ is generated by classes of the form $A_M$ defined in (3.25), for which

\[
\int_{A_M} \omega = \int_{\overline{\partial U}} \Omega + h \int_{\overline{\partial U}} \omega
\]

in light of (3.29). Similarly, $L$ does not intersect either $[0, 1] \times h$ or $K_{\Sigma}$, thus

\[ A_M \cdot \left( [0, 1] \times [h] + cK_{\Sigma} \right) = C_M \cdot [h] + bh, \]

and (3.28) is thus satisfied if and only if

\[
\int_{\overline{\partial U}} \Omega - C_M \cdot [h] = \left( \int_{\overline{\partial U}} \Omega - C_M \cdot [h] \right) - e_{\Sigma}
\]

which is positive if $\int_{\overline{\partial U}} \omega$ is made sufficiently large. To see that this formula for $c$ doesn’t depend on any choices, observe that if $\Omega$ is exact, then $h = 0$ and $\Omega = C d \alpha$, so

\[
\int_{\overline{\partial U}} \Omega - C_M \cdot [h] = C \int_{\overline{\partial U}} \lambda
\]

is proportional to $k$, as the integral of $\lambda$ vanishes on all the meridians $\mu_j$. When $\Omega$ is not exact but equals $C d \alpha + \Omega_0$ with $\Omega_0$ supported in a tubular neighborhood of $h$, we can find a real homology class $B \in H_2(M; \mathbb{R})$ with $B \cdot [h] = C_M \cdot [h]$ and thus define a real 2-chain

\[ C_M := C_M - B \]

with $\partial C_M = -\partial B$ and $C_M \cdot [h] = 0$. Then up to relative homology, $C_M$ can be represented by a real linear combination of immersed surfaces that have no geometric intersection with $h$, hence $\int_{\overline{\partial U}} \Omega = 0 = C_M \cdot [h]$.

\[ \int_{\overline{\partial U}} C_M \cdot [h] = \int_{\overline{\partial U}} \Omega = C \int_{\overline{\partial U}} d \alpha - C \int_{\overline{\partial U}} \lambda, \]

and this is again proportional to $k$. \[ \square \]
If $\emptyset \times M \subset (W, \omega)$ is concave, then writing $h = 0$ and $\omega = d\lambda$ gives

$$
\int_{\mathcal{M}} \omega = \int_{\mathcal{M} \setminus \partial \mathcal{M}} \omega + \int_{\partial \mathcal{M}} \omega
$$

for any cycle $\Gamma = k_1(\lambda_1) + \ldots + k_N(\lambda_N) + \sum j m_j \mu_j \in H_1(\mathcal{Z}_0)$ with $\sum j m_j = 0$ that is nullhomologous in $\mathcal{M}$. Since $f_k \lambda$ is also positively proportional to $k$, this proves that $\omega$ is exact if and only if there is no such nullhomologous cycle $\Gamma$ with $k > 0$. Moreover, $PD(\mathcal{T}) = -\partial \mathcal{K}_2^{\mathcal{M}} \subset H_1(\mathcal{M} \setminus \partial \mathcal{M}; \mathbb{R})$ for some $c > 0$, so it remains to characterize the situations where this homology class vanishes. Write $\mathcal{M}' = (\mathcal{M} \setminus \partial \mathcal{M}(\mathcal{Z}_0)) \cup (-S_\bot \times S^1) \cup (S_\bot \times S^1)$ and consider the resulting Mayer-Vietoris sequence:

$$
\ldots \rightarrow H_2(\mathcal{M}') \rightarrow H_1(\partial \mathcal{M}(\mathcal{Z}_0)) \rightarrow H_1(\mathcal{M} \setminus \partial \mathcal{M}) \oplus H_1(\mathcal{Z}_0) \rightarrow \ldots,
$$

where $H_1(\partial \mathcal{M}(\mathcal{Z}_0))$ is freely generated by the $4N$ cycles $\beta^i \alpha^j_j$. Denote the inclusion $\mathcal{M} \rightarrow \mathcal{M}$ and $\mathcal{M} \rightarrow \mathcal{M}$ by $\partial \mathcal{N}^{\mathcal{M}}(\mathcal{Z}_0)$, and the latter map $\partial \mathcal{N}^{\mathcal{M}'}(\mathcal{Z}_0)$ into $S_\bot \times S^1$. Then

$$
\mathcal{K}_2^{\mathcal{M}} = [\partial \mathcal{N}(\mathcal{Z}_0)] \in H_1(\mathbb{Z}_0), \quad \mathcal{K}_2^{\mathcal{M}'} = [\partial \mathcal{N}^{\mathcal{M}'}(\mathcal{Z}_0)] \in H_1(S_\bot \times S^1),
$$

and

$$
\mathcal{K}_2^{\mathcal{M}'} = \mathcal{K}_2^{\mathcal{M}},
$$

with $i, j$ defined in (3.25). This is true for all $\mathcal{M}$ which are concave. The key step is to compute $\mathcal{K}_2^{\mathcal{M}'}$ for each $\mathcal{M}$.

3.7. Proofs of the results from §3

Proofs of Theorems 1 and 2. To prove Theorem 1, suppose $(\mathcal{M}, \xi)$ contains an $\Omega$-separating planar $k$-torus domain $M_k$ for some closed 2-form $\omega$ with $H_1(\mathcal{M}, \omega) > 0$ and an integer $k \geq 1$. Then $f_k \omega = 0$ for every interface torus $T$ in $M_k$ that lies in the planar piece, so we are free to remove any such torus by attaching a $3$-decoupling cobordism whose symplectic structure matches $\Omega$ at $M$. By Proposition 1, we can find a binding component $\gamma$ or interface torus $T$ such that $[\gamma] = [\partial \gamma]$. Then $\mathcal{M}'$ contains a planar $k$-torus domain of order either $k - 1$ or $k - 2$. Writing $\omega' := f_{k-1} \omega$, the latter is also $\Omega'$-separating since each of the remaining interface tori, which lie outside the region of surgery, $\omega'$ is still homologous to the original $\omega$. The process can therefore be repeated until the manifold at the top has planar $k$-torus, meaning it is overtwisted.

Theorem 1 is essentially the special case of Theorem 2 for which we assume $\Omega$ to be exact to start with, except that the above argument actually gives a weak symplectic cobordism $[\mathcal{M}, \omega]$, which we can assume $\mathcal{M} \subset (W, \omega)$ is concave and $\mathcal{M} \subset (W, \omega)$ is not necessarily convex, but $\mathcal{U} \mathcal{A} \mathcal{V} \mathcal{B} \mathcal{C}$. There can now be turned into a strong cobordism by the following trick which was suggested to me by David Gay: first, observe that if $\mathcal{M} \subset (W, \omega)$ is a rational homology sphere, then $\mathcal{U} \mathcal{B} \mathcal{C} \mathcal{D} \mathcal{E} \mathcal{F} \mathcal{G} \mathcal{H} \mathcal{I} \mathcal{J} \mathcal{K} \mathcal{L} \mathcal{M} \mathcal{N} \mathcal{O} \mathcal{P} \mathcal{Q} \mathcal{R} \mathcal{S} \mathcal{T} \mathcal{U} \mathcal{V} \mathcal{W} \mathcal{X} \mathcal{Y} \mathcal{Z}$

Lemma 3.18. Suppose $\mathcal{M}$ is a closed oriented 3-manifold, $K \subset \mathcal{M}$ is a knot with $[K] \neq 0$ in $H_1(\mathcal{M}; \mathbb{Q})$ and $\mathcal{M}$ is the result of performing Dehn surgery along $K$ with framing. Then:

$\dim H_1(\mathcal{M}; \mathbb{Q}) = 0$.

Proof. As preparation, suppose $K$ is any knot in a closed oriented 3-manifold $\mathcal{M}$, a neighborhood of a 3-manifold $\mathcal{M}$, $\mathcal{M}$ is the result of performing Dehn surgery along $K$ with framing. Then:

$\dim H_1(\mathcal{M}; \mathbb{Q}) = 0$.

Since any 1-cycle $M$ can be decomposed into $\mathbb{N}$, the map $H_1(\mathcal{M}; \mathbb{Q}) \to H_0(\mathbb{N}; \mathbb{Q})$ in this sequence is trivial, thus exactness implies $H_1(\mathcal{M}; \mathbb{Q}) \simeq H_1(\mathbb{N}; \mathbb{Q}) \oplus H_1(\mathcal{M} \setminus \mathcal{K}; \mathbb{Q})$.
The map $\Phi : H_2(\mathbb{K}; \mathbb{Q}) \to H_2(\mathbb{K}; \mathbb{Q}) \otimes H_1(\mathbb{M} \setminus K; \mathbb{Q})$ is nontrivial since $\lambda$ maps to the generator of $H_1(\mathbb{K}; \mathbb{Q}) \cong \mathbb{Q}$. Since $\mu$ maps to 0 in $H_1(\mathbb{K}; \mathbb{Q})$, in $\Phi$ it is 1-dimensional if and only if $\iota \mu - \mu \in H_1(\mathbb{M} \setminus K; \mathbb{Q})$; and, if this is 2-dimensional, this proves the claim.

Now assume $|K| \geq 0 \in H_1(\mathbb{K}; \mathbb{Q})$, and we are given a framing such that $\lambda$ is the preferred longitude. This implies immediately that $\iota \lambda \neq 0 \in H_1(\mathbb{M} \setminus K; \mathbb{Q})$. Likewise, $\iota \mu \neq 0 \in H_1(\mathbb{M} \setminus K; \mathbb{Q})$.

To see this, note that the nondegeneracy of the intersection form, there exists a 2-cycle $C$ in $\mathbb{M}$ such that $|C| \cdot |K| = 1$, hence the restriction of $C$ to the complement of $K$ defines a chain whose boundary is $\mu$; alternatively, one can also derive this from the exact sequence above by considering the image of $C$ under $H_2(\mathbb{M}) \to H_2(\mathbb{M} \setminus K; \mathbb{Q})$.

We therefore have $\dim H_2(\mathbb{M}; \mathbb{Q}) = \dim H_2(\mathbb{M} \setminus K; \mathbb{Q})$ by the claim above. If $M$ is now defined by gluing another solid torus into $\mathbb{M} \setminus K$ such that $\lambda$ becomes the meridian, then the claim is again applicable and implies $\dim H_2(\mathbb{M}'; \mathbb{Q}) = \dim H_2(\mathbb{M} \setminus K; \mathbb{Q}) = 1$.

**Proof of Theorem 1** Suppose $(\mathbb{M}, \xi)$ contains an $\Omega$-separating partially planar domain $\mathbb{M}_0 \subset \mathbb{M}$ with planar piece $\mathbb{M}_0 \subset \mathbb{M}_0$, where $\Omega$ is a closed 2-form on $\mathbb{M}$ satisfying $\Omega \cdot \Lambda > 0$. Then for every binding circle or interface torus in $\mathbb{M}_0$, we can attach $\Sigma$-capping or $\Sigma$-decapping cobordisms to produce a symplectic manifold $(\mathbb{M}, \omega)$ with $\partial \mathbb{M} = \mathbb{M}_0 \cup (-\mathbb{M})$, $\omega_{|\mathbb{M}_0} = \Omega$ and

$$M' = M_0 \cup M_{\text{bdry}},$$

where $M_{\text{bdry}}$ carries a contact structure $\xi'$ with $\omega_{|\mathbb{M}_0} > 0$. The desired cap is then obtained from $(\mathbb{M}, \omega)$ after capping $M_{\text{bdry}}$ via $\mathbb{M}_0$ or $\mathbb{M}_{\text{bdry}}$.

**Proof of Theorem 2** Note that since $H_2(\mathbb{M}_0, S^2) = \dim(S^2, \xi_0)$ is concave at $\mathbb{M}$ when $\mathbb{M}$ is deformed to a strong cobordism, so it suffices to prove that $(S^2, \xi)$ can be obtained from $(\mathbb{M}, \xi)$ by a finite sequence of (generally weak) capping and decapping cobordisms.

Suppose $\mathbb{M}_0 \subset \mathbb{M}$ is a partially planar domain. If it is also a planar torus domain then the result already follows from Corollary 1 as desired. Assume $\mathbb{M}_0$ has one irreducible subdomain $\Omega$ with nonempty binding; we can remove binding components by $\Sigma$-capping cobordisms and interface tori by $\Sigma$-decapping cobordisms until the planar piece has exactly one binding component left and no interface or boundary, which means it is the tight $S^2$. The desired cobordism can then be obtained by cascading any additional components that may remain at the end of this process.

If $\mathbb{M}_0$ has more than one irreducible subdomain but does not have planar torus, then it must be symmetric (by Definition 2). This means that $M = M_0$, the binding and boundary are empty and the interface tori divide $\mathbb{M}$ into exactly two irreducible subdomains that have diffeomorphic planar pages. Then we can remove interface tori by $\Sigma$-decapping cobordisms until exactly one remains, and the resulting contact manifold is the tight $S^2 \times S^1$. The latter is cobordant to $S^2$ by a $\Sigma$-capping cobordism that removes one binding component from the supporting open book with cylindrical pages and trivial monodromy.

Theorem 3 essentially follows from the same argument since every planar open book is also a fully twisted partially planar domain. We only need to add that the topological construction of the cobordism by attaching $\Sigma$-handles along binding components does not depend on the choice of 2-form on $\mathbb{M}$, which after the deformation carried out in 12 always looks the same on a large tubular neighborhood of the binding.
An almost complex structure $J$ on $(W^m, \omega)$ is now \textit{admissible} if it is $\omega$-compatible on $W$ and is compatible with the stable Hamiltonian structures on both cylindrical ends, meaning it is $\Phi$-invariant, restricts to a complex structure on the respective distributions $\mathfrak{g}_0$ and $\mathfrak{g}_s$, defining the correct orientations, and maps the unit vector $\partial_\nu$ in the $3$-direction to the vector field $X_0$ or $X'_0$.

It was shown in [Wen] that for a suitable choice of almost complex structure $J_0$ on $\mathbb{R} \times M$ compatible with $(\mathfrak{g}_0, \mathfrak{g}_s)$, the pages of the blown up summed open book in $M_0$ admit lifts to embedded $J_0$-holomorphic curves in $\mathbb{R} \times M$ which match the fibers of the mapping torus $X_0$ outside of the neighborhoods $\mathcal{N}_J$ of $B \cup T \cup M_0$, have positive cylindrical ends approaching closed orbits of $X_0$ in $B \cup T \cup M_0$, and satisfy a suitable finite energy condition. We can now define an admissible almost complex structure on $(W^m, \omega)$ which matches $J_0$ outside of $U_{\omega}^2$ and is $\omega$-compatible on $\mathbb{R}^2$. The $J_0$-holomorphic curves in $(\mathbb{R} \times M) \setminus U_{\omega}^2$ can be extended into $\mathbb{R}^2$ as symplectic surfaces that are diffeomorphic to $\Sigma$ and foliate $\mathbb{R}^2$, thus we can extend $J_0$ into the handle so that it is $\omega$-compatible and these surfaces become $J_0$-holomorphic. In doing this, we can also make the natural completion of the core

\[ \bar{K}_{\Sigma, \omega} := \bar{K}_{\Sigma} \cup \partial B_{\infty} \times (-\infty,0] \times \bar{B}_0 \]

$J_0$-holomorphic, as well as its transitions under the $s^2$-action by translating the local $\phi$-coordinates, and the completion of the co-core

\[ \bar{K}_{\Sigma, \omega} := \bar{K}_{\Sigma} \cup \partial B_{\infty} \times [1,\infty) \times \partial \bar{\mathcal{K}}_{\Sigma} \].

The result is a foliation of $W^m$ (or at least the region outside of $\mathbb{R} \times (M \setminus M_0)$) by finite energy $J_0$-holomorphic curves. We summarize this construction as follows (see Figure 7).

**Proposition 3.19.** One can choose an admissible almost complex structure $J_0$ on the completion $(W^m, \omega)$ of a $\Sigma$-decaying codimension $(W, \omega)$, such that there exists a foliation $\mathcal{F}$ by embedded $J_0$-holomorphic curves with the following properties:

1. In each cylindrical end, the leaves of $\mathcal{F}$ match the holomorphic lifts of the pages of $\pi$ and $\pi'$. constructed in [Wen].
2. The completed core $\bar{K}_{\Sigma, \omega}$ and all its $s^2$-translations are leaves of $\mathcal{F}$.
3. The trivial cylinders over periodic orbits in $B, \partial M_0$ and $I \setminus \mathcal{D}_0$ are all leaves of $\mathcal{F}$.
4. All other leaves of $\mathcal{F}$ have only positive cylindrical ends asymptotic to orbits in $B \cup (I \setminus \mathcal{D}_0) \cup \partial M_0$, and they are homotopic in the moduli space to the holomorphic lifts of the pages of $\pi'$ over $[1,\infty) \times M'$.
5. The completed co-core $\bar{K}_{\Sigma, \omega}$ is also $J_0$-holomorphic and intersects the leaves of $\mathcal{F}$ transversely.

In considering the behavior of holomorphic curve invariants under symplectic cobordisms, a special role is typically played by curves that have no positive ends—such curves can only exist in non-exact cobordisms. One useful application of the foliation constructed above is that we can now characterize all such curves precisely:

**Proposition 3.20.** Suppose $\Sigma: \Sigma \to W^m$ is a finite energy $J_0$-holomorphic curve that is connected, somewhere injective and has no positive ends. Then $u$ is a leaf of $\mathcal{F}$, specifically it is an $s^2$-translation of the core $\bar{K}_{\Sigma, \omega}$.

**Proof.** There are no curves without positive ends outside the region of surgery since here the symplectic form is exact, thus we may assume $u$ intersects both the handle and its complement.

If $u$ is a leaf of $\mathcal{F}$ then it must be an $s^2$-translation of the core, as all other leaves have positive ends. If it is not a leaf of $\mathcal{F}$ then it has a positive intersection with some leaf $v$, and without loss of generality we may suppose that $v$ has only positive ends. Then $v$ is homotopic in the moduli space to a holomorphic lift of a page $\pi'$, which we may assume lies in the region $[c,\infty) \times M'$ for an arbitrarily large number $c > 0$, and thus cannot intersect $u$. This is a contradiction, due to positivity of intersections.

To apply these constructions to ECH, or to Symplectic Field Theory for that matter, one must perturb $\mathfrak{g}_0$ and $\mathfrak{g}'_0$ to contact structures and perturb $J_0$ with them. The $J_0$-holomorphic leaves of $\mathcal{F}$ will not generally behave well under this perturbation: a leaf with only positive ends for instance, if it has genus $g$, will have Fredholm index $2 - 2g$ and thus disappears under any generic perturbation of the data unless $g = 0$. Proposition 3.20 however, should still hold under a sufficiently small perturbation, because for any sequence $J_0$ of perturbed almost complex structures converging to $J_0$, a sequence of $J_0$-holomorphic curves should converge in the sense of [Hov] to a $J_0$-holomorphic building, and Proposition 3.24 determines what this building can look like. This is a variation on the uniqueness argument used in [Wen] to prove vanishing of the ECH contact invariant: higher genus holomorphic curves do not generically exist, but they remain useful for proving that no other curves can exist either.

**References**


M. Hutchings, *Embedded contact homology as a (symplectic) field theory*, in preparation.


