Chapter 4

Natural Constructions on Vector Bundles

Contents

For the entirety of this chapter, we assume $\pi : E \to M$ is a smooth vector bundle. We have already seen that any extra structure attached to a bundle determines a preferred class of connections. We now examine a variety of other situations in which a given vector bundle may inherit a natural class of connections—or a unique preferred connection—from some external structure. A particularly important case is considered in §4.2 and §4.3, where E is the tangent bundle of M, and we find that there is a preferred metric connection for any Euclidean structure on TM. This is the fundamental fact underlying Riemannian geometry.

4.1 Direct sums, tensor products and bundles of linear maps

Suppose $\pi_E : E \to M$ and $\pi_F : F \to M$ are two vector bundles of rank m and ℓ respectively, and assume they are either both real ($\mathbb{F} = \mathbb{R}$) or both complex ($\mathbb{F} = \mathbb{C}$). In this section we address the following question: given connections on E and F, what natural connections are induced on the other bundles that can be constructed out of E and F? We answer this by defining parallel transport in the most natural way for each construction and observing the consequences.

Direct sums

To start with, the direct sum $E \oplus F \to M$ inherits a natural connection such that the parallel transport of $(v, w) \in E \oplus F$ along a path $\gamma(t) \in M$ takes the form

$$P^t_\gamma(v, w) = (P^t_\gamma(v), P^t_\gamma(w)).$$

We leave it as an easy exercise to the reader to verify that the covariant derivative is then

$$\nabla_X(v, w) = (\nabla_X v, \nabla_X w)$$

for $v \in \Gamma(E)$, $w \in \Gamma(F)$ and $X \in TM$.

If E and F have extra structure, this structure is inherited by $E \oplus F$. For example, bundle metrics on E and F define a natural bundle metric on $E \oplus F$: this has the property that if $(e_1, \dots, e_m)$ and $(f_1, \dots, f_\ell)$ are orthogonal frames on E and F respectively, then $(e_1, \dots, e_m, f_1, \dots, f_\ell)$ is an orthogonal frame on $E \oplus F$. In fact, E and F need not have the same type of structure: suppose they have structure groups $G \subset \mathrm{GL}(m, \mathbb{F})$ and $H \subset \mathrm{GL}(\ell, \mathbb{F})$ respectively. The product $G \times H$ is a subgroup of $\mathrm{GL}(m, \mathbb{F}) \times \mathrm{GL}(\ell, \mathbb{F})$, which admits a natural inclusion into $\mathrm{GL}(m + \ell, \mathbb{F})$:

$$\mathrm{GL}(m, \mathbb{F}) \times \mathrm{GL}(\ell, \mathbb{F}) \hookrightarrow \mathrm{GL}(m + \ell, \mathbb{F}) : (\mathbf{A}, \mathbf{B}) \mapsto \begin{pmatrix} \mathbf{A} & \\ & \mathbf{B} \end{pmatrix}. \tag{4.1}$$

The combination of G-compatible and H-compatible trivializations on E and F respectively then gives $E \oplus F$ a $(G \times H)$-structure. If the connections on E and F are compatible with their corresponding structures, it's easy to see that our connection on $E \oplus F$ is $(G \times H)$-compatible.

Example 4.1 (Metrics on direct sums). Let us see how the general framework just described applies to the previously mentioned construction of a bundle metric on $E \oplus F$ when E and F are both Euclidean bundles. In this case, the structure groups of E and F are $\mathrm{O}(m)$ and $\mathrm{O}(\ell)$ respectively, so the direct sum has structure group $\mathrm{O}(m) \times \mathrm{O}(\ell)$, which is included naturally in $\mathrm{O}(m+\ell)$ by (4.1). Thus $E\oplus F$ inherits an $\mathrm{O}(m+\ell)$-structure, that is, a bundle metric. To see that it matches the metric we defined earlier, we only need observe that by definition, any pair of orthogonal frames for E_x and F_x defines an $\mathrm{O}(m)\times\mathrm{O}(\ell)$-compatible frame for $E_x\oplus F_x$, which is therefore also an orthogonal frame.

Tensor products

Things are slightly more interesting for the tensor product bundle $E\otimes F$. Parallel transport is defined naturally by the condition

$$P_\gamma^t(v\otimes w) = P_\gamma^t(v)\otimes P_\gamma^t(w).$$

Indeed, since every element of $E\otimes F$ is a sum of such products, this defines P_γ^t on $E\otimes F$ uniquely via linearity. Computation of the covariant derivative makes use of the following general fact: bilinear operations give rise to product rules.

Exercise 4.2. Suppose V, W and X are finite dimensional vector spaces, $\beta : V\times W\to X$ is a bilinear map and $v(t)\in V$, $w(t)\in W$ are smooth paths. Then

$$\frac{d}{dt}\beta(v(t),w(t)) = \beta(\dot v(t),w(t)) + \beta(v(t),\dot w(t)).$$

It follows that the covariant derivative satisfies a Leibnitz rule for the tensor product: if $\dot\gamma(0) = X\in T_xM$,

$$\begin{aligned}\nabla_X(v\otimes w) &= \frac{d}{dt}\left(P_\gamma^t\right)^{-1}\left[v(\gamma(t))\otimes w(\gamma(t))\right]\Big|_{t=0}\\ &= \frac{d}{dt}\left[(P_\gamma^t)^{-1}(v(\gamma(t)))\otimes(P_\gamma^t)^{-1}(w(\gamma(t)))\right]_{t=0}\\ &= \nabla_X v\otimes w + v\otimes\nabla_X w.\end{aligned}$$

We express this more succinctly with the formula

$$\nabla(v\otimes w) = \nabla v\otimes w + v\otimes\nabla w.$$

Bundles of linear maps

For the bundle $\mathrm{Hom}(E,F)\to M$, it is natural to define parallel transport so that if $A\in\mathrm{Hom}(E_{\gamma(0)},F_{\gamma(0)})$ and $v\in E_{\gamma(0)}$, then

$$P_\gamma^t(Av) = P_\gamma^t(A)P_\gamma^t(v).$$

Applying Exercise 4.2 to the bilinear pairing $\mathrm{Hom}(E,F)\oplus E\to F$: $(A,v)\mapsto Av$, we find that the covariant derivatives on these three bundles are related by

$$\nabla_X(Av) = (\nabla_X A)v + A(\nabla_X v) \tag{4.2}$$

for $A\in\Gamma(\mathrm{Hom}(E,F))$ and $v\in\Gamma(E)$. In particular for the dual bundle $E^* = \mathrm{Hom}(E, M\times\mathbb{F})$, one chooses the trivial connection on $M\times\mathbb{F}$ (i.e. $\nabla = d$) so that for $\alpha\in\Gamma(E^*)$ and $v\in\Gamma(E)$, (4.2) becomes

$$L_X(\alpha(v)) = (\nabla_X\alpha)(v) + \alpha(\nabla_X v). \tag{4.3}$$

Observe that the left hand side of this expression has no dependence on the connection—evidently this dependence for E and E^* on the right hand side cancels out.

Tensor bundles and contractions

By the above constructions, a choice of any connection on a vector bundle $E\to M$ induces natural connections on the tensor bundles

$$E_\ell^k = \left(\otimes^\ell E^*\right)\otimes\left(\otimes^k E\right).$$

Recall that one can interpret the fibers $(E_\ell^k)_p$ as spaces of multilinear maps

$$\underbrace{E_p\times\ldots\times E_p}_{\ell}\times\underbrace{E_p^*\times\ldots\times E_p^*}_{k}\to\mathbb{F},$$

and by convention E_0^0 is the trivial line bundle $M\times\mathbb{F}$; for this it is natural to choose the trivial connection $\nabla = d$. Then the convariant derivative satisfies a Leibnitz rule on the tensor algebra:

$$\nabla_X(S\otimes T) = \nabla_X S\otimes T + S\otimes\nabla_X T.$$

It also commutes with *contractions*: recall that there is a unique linear bundle map

$$\mathrm{tr} : E_1^1\to E_0^0$$

with the property that $\mathrm{tr}(\alpha\otimes v) = \alpha(v)$ for any $v\in E$ and α in the corresponding fiber of E^*. Writing $A\in E_1^1$ in components $A^i{}_j$ with respect to a chosen frame, $\mathrm{tr}(A)$ is literally the trace of the matrix with entries $A^i{}_j$ (see Appendix A). Then combining the Leibnitz rule for the tensor product with (4.3), we see that

$$L_X\,\mathrm{tr}(A) = \mathrm{tr}\left(\nabla_X A\right)$$

for any $A \in \Gamma(E_1^1)$. More generally for any $p \in 1, \dots, k+1$ and $q \in 1, \dots, \ell+1$, one can define a contraction operation $\mathrm{tr} : E_{\ell+1}^{k+1} \to E_\ell^k$ by the condition

$$\mathrm{tr}(\alpha^1 \otimes \dots \otimes \alpha^{\ell+1} \otimes v_1 \otimes \dots \otimes v_{k+1}) = \alpha^q(v_p) \cdot \alpha^1 \otimes \dots \otimes \widehat{\alpha^q} \otimes \dots \otimes \alpha^{\ell+1} \otimes v_1 \otimes \dots \otimes \widehat{v_p} \otimes \dots \otimes v_{k+1},$$

where the hat notation is used to indicate the *lack* of the corresponding term. A similar argument then shows that for any such operation,

$$\nabla_X \mathrm{tr}(T) = \mathrm{tr}\,(\nabla_X T) \tag{4.4}$$

for all $T \in \Gamma(E_{\ell+1}^{k+1})$.

Exercise 4.3. Verify (4.4)

Another perspective on compatibility

If $E \to M$ is a vector bundle with structure group G, we have thus far defined G-compatibility for a connection purely in terms of parallel transport, which is not always the most convenient description. We shall now see, at least in certain important special cases, how this definition can be framed in terms of covariant derivatives.

The case of greatest general interest is when $G = \mathrm{O}(m)$ or $\mathrm{U}(m)$, so E is equipped with a bundle metric $g(\ ,\ ) \in \Gamma(E_2^0)$, and compatibility of a connection means precisely that parallel transport always preserves this inner product on the fibers. More generally, suppose there is a covariant tensor field $T \in \Gamma(E_k^0)$ with the property that a family of parallel transport isomorphisms $P_\gamma^t : E_{\gamma(0)} \to E_{\gamma(t)}$ is G-compatible if and only if

$$T(P_\gamma^t(v_1), \dots, P_\gamma^t(v_k)) = T(v_1, \dots, v_k)$$

for all $v_1, \dots, v_k \in E_{\gamma(0)}$. Special cases of this situation include not only bundle metrics but also volume forms and symplectic structures. By the above constructions, a connection on E induces a connection on E_k^0.

Proposition 4.4. *If $E \to M$ is a vector bundle with G-structure defined by a fixed covariant tensor field $T \in \Gamma(E_k^0)$ as described above, and ∇ is a connection on E, then the following statements are equivalent:*

(i) ∇ *is G-compatible*

(ii) $\nabla T \equiv 0$

(iii) For any sections $v_1, \dots, v_k \in \Gamma(E)$ and vector $X \in TM$,

$$L_X\,(T(v_1, \dots, v_k)) = T(\nabla_X v_1, v_2, \dots, v_k) + T(v_1, \nabla_X v_2, \dots, v_k) + \dots + T(v_1, \dots, v_{k-1}, \nabla_X v_k).$$

Proof. It follows from the various Leibnitz rules that (ii) and (iii) are equivalent. We can show the equivalence of (i) and (iii) by a direct computation: assume first that ∇ is G-compatible, and choose a path $\gamma(t) \in M$ with $\gamma(0) = p$, $\dot\gamma(0) = X \in T_pM$. Then

$$\begin{aligned} L_X\,(T(v_1, \dots, v_k)) &= \frac{d}{dt} T\,(v_1(\gamma(t)), \dots, v_k(\gamma(t)))\Big|_{t=0} \\ &= \frac{d}{dt} T\left((P_\gamma^t)^{-1}(v_1(\gamma(t))), \dots, (P_\gamma^t)^{-1}(v_k(\gamma(t)))\right)\Big|_{t=0} \\ &= T\left(\frac{d}{dt}(P_\gamma^t)^{-1}(v_1(\gamma(t)))\Big|_{t=0}, \dots, v_k(p)\right) + \dots \\ &\qquad + T\left(v_1(p), \dots, \frac{d}{dt}(P_\gamma^t)^{-1}(v_k(\gamma(t)))\Big|_{t=0}\right) \\ &= T\,(\nabla_X v_1, \dots, v_k(p)) + \dots + T\,(v_1(p), \dots, \nabla_X v_k)\,, \end{aligned}$$

where we've used the obvious generalization of Exercise 4.2 for multilinear maps. Conversely if (iii) holds, then it follows from this calculation that for any $v_1, \dots, v_k \in E_p$,

$$\frac{d}{dt} T\left(P_\gamma^t(v_1), \dots, P_\gamma^t(v_k)\right)\Big|_{t=0} = \frac{d}{dt} T(v_1, \dots, v_k)\Big|_{t=0} = 0.$$

Since there's nothing intrinsically special about the condition $t = 0$, we conclude that $T\left(P_\gamma^t(v_1), \dots, P_\gamma^t(v_k)\right)$ is independent of t, and the connection is therefore G-compatible. $\square$

4.2 Tangent bundles

4.2.1 Torsion and symmetric connections

For the remainder of this chapter we consider connections on the tangent bundle TM of a smooth n-dimensional manifold M. In this setting there turns out to be a special class of connections, resulting from the simultaneous interpretation of elements $X \in TM$ as vectors in the bundle $TM \to M$ and as velocity vectors of smooth paths in M. In particular, given a connection on TM, one can ask the following rather imprecise question:

> *If $\gamma(t) \in M$ is a smooth path and $X \in T_{\gamma(0)}M$, how much does X* twist *around γ as it moves by parallel translation?*

For general vector bundles this question has no meaning: one can certainly choose a trivialization and define some notion of twisting with respect to this choice, but the answer may change if a different trivialization is chosen. For tangent bundles it turns out that more can be said, and there is a special class of connections for which the twisting is in some sense minimal. We will now make this precise.

Assuming $\dim M \geq 2$, consider a smooth embedding

$$\alpha : (-\epsilon, \epsilon) \times (-\epsilon, \epsilon) \to M,$$

which traces out a surface in M. We wish to measure the degree to which parallel vector fields along $\gamma(s) := \alpha(s, 0)$ twist. Note that without changing $\gamma(s)$, the map $\alpha(s, t)$ can be chosen so that the vector field $X(t) := \partial_s \alpha(0, t)$ along the path $t \mapsto \alpha(0, t)$ is parallel, i.e. $\nabla_t X \equiv 0$. Now we ask: defining the vector field $Y(s) := \partial_t \alpha(s, 0)$ along $\gamma(s)$, can we also choose $\alpha(s, t)$ so that $Y(s)$ is parallel? If so then

$$\nabla_s \partial_t \alpha(0, 0) = \nabla_t \partial_s \alpha(0, 0),$$

a relation that looks reasonable enough, but as we'll see in a moment, it's not always possible. The failure of the commutation formula $\nabla_s \partial_t \alpha = \nabla_t \partial_s \alpha$ can be interpreted as a measure of twisting under parallel transport.

With this motivation, define an antisymmetric bilinear map $T : \mathrm{Vec}(M) \times \mathrm{Vec}(M) \to \mathrm{Vec}(M)$ by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

Exercise 4.5. Show that the map T defined above is C^∞-linear in both variables, i.e. for any $f \in C^\infty(M)$, $T(fX, Y) = fT(X, Y)$ and $T(X, fY) = fT(X, Y)$.

The result of Exercise 4.5 implies that T defines a tensor field of type $(1, 2)$: we call it the *torsion tensor* associated to the connection. If T vanishes identically, then referring again to the embedding $\alpha(s, t)$ above, it's easy to see that $\nabla_s \partial_t \alpha \equiv \nabla_t \partial_s \alpha$: this follows because one can extend $X = \partial_s \alpha$ and $Y = \partial_t \alpha$ to commuting vector fields in a neighborhood of $\alpha(s, t)$, so that

$$T(X, Y) = \nabla_s \partial_t \alpha - \nabla_t \partial_s \alpha \equiv 0.$$

In fact, this remains true without assuming that α is an embedding; it could in general be any smooth map (we leave the details as an exercise to the reader). This "commuting partials" relation will often come in useful in computations.

Definition 4.6. A connection on $TM \to M$ is called *symmetric* (or equivalently *torsion free*) if its torsion tensor vanishes identically.

We will see shortly that symmetric connections always exist, and in fact there is a unique symmetric connection compatible with any bundle metric on TM (see §4.3.1).

Given local coordinates $(x^1, \dots, x^n)$, Equation (3.13) gives an expression for the components of T in terms of the Christoffel symbols:

$$T^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj}. \tag{4.5}$$

Thus a connection is symmetric if and only if the Christoffel symbols Γ^i_{jk} are always symmetric under interchange of the two lower indices.

Exercise 4.7. Check that the components T^i_{jk} satisfy the appropriate transformation formula for a tensor under coordinate transformations (see Exercise A.11 in Appendix A and Exercise 3.17 in Chapter 3).

By the constructions of §4.1, a choice of connection on $TM \to M$ defines connections on each of the tensor bundles $T^k_\ell M \to M$; in particular, we can now take covariant derivatives of tensor fields and differential forms. We now give two applications of this that require symmetry for the connection on TM. The first is a new formula for the exterior derivative $d : \Omega^k(M) \to \Omega^{k+1}(M)$, which we state below for $k = 1$, referring to [GHL04] for the general case.

Proposition 4.8. *If ∇ is a symmetric connection on $TM \to M$, then for any 1-form λ and vector fields X and Y on M,*

$$d\lambda(X, Y) = (\nabla_X \lambda)(Y) - (\nabla_Y \lambda)(X).$$

Proof. We use the formula $d\lambda(X, Y) = L_X(\lambda(Y)) - L_Y(\lambda(X)) - \lambda([X, Y])$ together with the Leibniz rule for the covariant derivative, thus

$$\begin{aligned} d\lambda(X, Y) &= L_X(\lambda(Y)) - L_Y(\lambda(X)) - \lambda([X, Y]) \\ &= (\nabla_X \lambda)(Y) + \lambda(\nabla_X Y) - (\nabla_Y \lambda)(X) - \lambda(\nabla_Y X) - \lambda([X, Y]) \\ &= (\nabla_X \lambda)(Y) - (\nabla_Y \lambda)(X) + \lambda(T(X, Y)), \end{aligned}$$

which implies the stated formula if $T(X, Y) = 0$. □

We state the second application as an exercise. Recall that for any covariant tensor field $S \in \Gamma(T^0_k M)$ and vector field $X \in \mathrm{Vec}(M)$, the *Lie derivative* of S with respect to X is the tensor field $L_X S \in \Gamma(T^0_k M)$ defined by

$$L_X S = \left. \frac{d}{dt} (\varphi^t_X)^* S \right|_{t=0},$$

where φ^t_X denotes the flow of X.

Exercise 4.9. Show that if ∇ is a symmetric connection on $TM \to M$, then for any $S \in \Gamma(T^0_k M)$ and $X \in \mathrm{Vec}(M)$, the Lie derivative $L_X S$ satisfies the formula

$$\begin{aligned} L_X S(Y_1, \dots, Y_k) &= (\nabla_X S)(Y_1, \dots, Y_k) + S(\nabla_{Y_1} X, Y_2, \dots, Y_k) \\ &\quad + S(Y_1, \nabla_{Y_2} X, \dots, Y_k) + \dots + S(Y_1, \dots, Y_{k-1}, \nabla_{Y_k} X). \end{aligned}$$

4.2.2 Geodesics

A connection on $TM \to M$ defines a special class of smooth curves $\gamma(t) \in M$ which generalize the notion of a "straight path" in Euclidean space. Observe that a path $\gamma(t) \in \mathbb{R}^n$ is a straight line if its velocity $\dot{\gamma}(t)$ is constant. Replacing $\mathbb{R}^n$ with an n-manifold M, it no longer makes sense to say that $\dot{\gamma}(t)$ is constant unless we first choose a connection, since $\dot{\gamma}(t)$ generally belongs to a different tangent space $T_{\gamma(t)}M$ at different times t. Once a connection is chosen, the natural generalization is clear: a path $\gamma : (a, b) \to M$ is called a *geodesic* if its velocity $\dot{\gamma}(t)$ is a parallel vector field along γ, in other words

$$\nabla_t \dot{\gamma} \equiv 0.$$

This is the *geodesic equation*; in local coordinates $(x^1, \ldots, x^n)$, we write $\gamma(t) = (x^1(t), \ldots, x^n(t))$ and express the equation as

$$\ddot{x}^i + \Gamma^i_{jk}(x^1, \ldots, x^n)\dot{x}^i\dot{x}^j = 0.$$

The geodesic equation is therefore a system of n second-order nonlinear ordinary differential equations, and as such has a unique solution

$$\gamma : (t_0 - \epsilon, t_0 + \epsilon) \to M$$

for some $\epsilon > 0$ and any choice of initial position $\gamma(t_0) \in M$ and velocity $\dot{\gamma}(t_0) \in T_{\gamma(t_0)}M$.

Geodesics are most important in Riemannian geometry, where they serve as length-minimizing paths between nearby points, as we will show in the next section. Before discussing their geometric significance further, we shall establish the existence of symmetric connections by way of answering the following question:

> *Given a connection, how many other connections are there that have the same geodesics?*

To be precise, we say that two connections ∇ and $\widetilde{\nabla}$ have the same geodesics if solutions to $\nabla_t \dot{\gamma} = 0$ are also solutions to $\widetilde{\nabla}_t \dot{\gamma} = 0$ and vice versa.

Proposition 4.10. *Given a connection ∇ on $TM \to M$, all other connections with the same geodesics are of the form*

$$\widetilde{\nabla}_X Y = \nabla_X Y + A(X, Y)$$

where $A : TM \oplus TM \to TM$ is an arbirary antisymmetric bilinear bundle map, and there is a unique symmetric connection of this form.

Proof. Given ∇ and an antisymmetric bilinear bundle map $A : TM \oplus TM \to TM$, it's easy to check that $\widetilde{\nabla}_X Y = \nabla_X Y + A(X, Y)$ satisfies the appropriate Leibnitz rule and therefore defines a connection. Moreover for any smooth path $\gamma(t) \in M$,

$$\widetilde{\nabla}_t \dot{\gamma} = \nabla_t \dot{\gamma} + A(\dot{\gamma}, \dot{\gamma}) = \nabla_t \dot{\gamma}, \tag{4.6}$$

so the two connections clearly have the same geodesics. Conversely, if ∇ and $\widetilde{\nabla}$ are any two connections, they are related by the formula above for some bilinear bundle map $A : TM \oplus TM \to TM$ (not necessarily antisymmetric). Suppose now that they have the same geodesics. Then for any $X \in TM$, we can choose a geodesic $\gamma(t)$ with $\dot{\gamma}(0) = X$ and use (4.6) to find $A(X, X) = 0$. It follows now by evaluating the expression $A(X + Y, X + Y)$ that A is antisymmetric. In this case the torsions for ∇ and $\widetilde{\nabla}$ are related by

$$\begin{aligned}\widetilde{T}(X, Y) - T(X, Y) &= \widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X - [X, Y] - (\nabla_X Y - \nabla_Y X - [X, Y]) \\ &= A(X, Y) - A(Y, X) = 2A(X, Y).\end{aligned}$$

Thus the unique symmetric connection with the same geodesics is found by setting

$$\widetilde{\nabla}_X Y = \nabla_X Y - \frac{1}{2}T(X, Y).$$

□

In light of this result (as well as Theorem 4.13 below), it is common to restrict attention to symmetric connections when the bundle is a tangent bundle. This has a number of advantages (cf. Prop. 4.8 and Exercise 4.9 above) and is usually the most natural thing to do, though not always: e.g. if $TM \to M$ has a complex structure (this is called an *almost complex structure* on M), there may not exist a connection that is both compatible with this structure and symmetric. Therefore we will not universally assume that all connections under consideration on TM are symmetric, but will always specify when this assumption is being made.

Definition 4.11. For any point $p \in M$ and a suitably small neighborhood $0 \in \mathcal{U}_p \subset T_pM$, define the *exponential map*

$$\exp_p : \mathcal{U}_p \to M$$

by $\exp_p(X) = \gamma(1)$, where $\gamma(t)$ is the unique geodesic through p with $\dot{\gamma}(0) = X$. When there is no ambiguity we denote simply $\exp(X) := \exp_p(X)$.

The local existence of geodesics guarantees that $\exp_p$ is well defined if the neighborhood $\mathcal{U}_p$ is sufficiently small; in fact in most examples of interest, $\exp_p$ will be globally defined as an immersion $T_pM \to M$.

Exercise 4.12. Show that for any $p \in M$, $X \in T_pM$, the curve $\gamma(t) = \exp(tX)$ defined for t in some neighborhood of 0 is the unique geodesic with $\dot\gamma(0) = X$.

The choice of notation and terminology reflects the close analogy between $\exp_p : T_pM \to M$ and the corresponding concept for a Lie group G and its Lie algebra $\mathfrak{g}$: in that case $X \in \mathfrak{g} = T_eG$ defines not a geodesic but rather a Lie group homomorphism $\mathbb{R} \to G : t \mapsto \exp(tX) \in G$ (cf. Appendix B).

4.3 Riemannian manifolds

4.3.1 The Levi-Civita connection

Recall from §2.4.2 that a *Riemannian metric* on a manifold M is a bundle metric on the tangent bundle $TM \to M$. This is defined by a symmetric, positive definite tensor field of type $(0, 2)$ traditionally denoted by $g \in \Gamma(T^0_2M)$, and can be used to define the lengths of tangent vectors $X \in T_pM$ by

$$|X| = \sqrt{g(X,X)},$$

as well as angles between them according to the formula

$$g(X,Y) = |X||Y| \cos\theta$$

for $X, Y \in T_pM$. The length of a smooth path $\gamma : [t_0,t_1] \to M$ is then defined to be

$$\text{length}(\gamma) = \int_{t_0}^{t_1} |\dot\gamma(t)|\, dt.$$

The pair (M, g) is called a *Riemannian manifold*.

We will show in this section that a Riemannian metric defines a unique special connection on TM, for which the geodesics take on geometric significance as length-minimizing paths. Since g is a bundle metric on TM, it is natural to consider only connections that are compatible with this structure. This is a serious restriction, but as we will see presently, not enough to achieve uniqueness.

Let us try to motivate heuristically how the natural connection on a Riemannian manifold is defined. As mentioned above, the geodesics in this context can be characterized by the property of minimizing length. Specifically, one could in principle take the following as the definition of a geodesic:

> *A geodesic $\gamma(t) \in M$ is a smooth curve such that $|\dot\gamma(t)|$ is constant and such that for any two sufficiently close values $a < b$ of the parameter, $\gamma|_{[a,b]} : [a,b] \to M$ minimizes the length among all smooth paths from $\gamma(a)$ to $\gamma(b)$.*

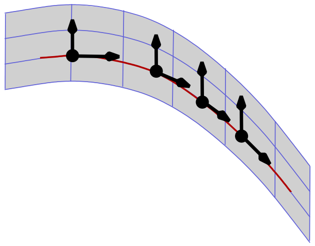

Figure 4.1: Parallel transport preserving the inner product along a geodesic in a surface.

As a definition this is sound, though proving it would take some work which we'll forego for the moment. Just assume for now that there exists a connection that's compatible with the bundle metric and whose geodesics have this property. Is this connection unique? If $\dim M = 2$ the answer is clearly yes: as shown in Figure 4.1, there are unique parallel transport isomorphisms that preserve both the velocity vector of the geodesic and a unit vector orthogonal to this.

The situation is less clear however if $\dim M > 2$. There are then infinitely many admissible ways to translate an orthonormal basis along a geodesic. If $\dim M = 3$ for instance, one can easily picture two of the basis vectors twisting around the path in arbitrary ways. But this mention of "twisting" suggests the remedy: we add the requirement that the connection should be symmetric, so that twisting is minimized. The next result shows that this is precisely the right thing to do; it is sometimes called the "fundamental lemma of Riemannian geometry."

Theorem 4.13. *On any Riemannian manifold (M, g), there exists a unique connection on TM that is both symmetric and compatible with g.*

Proof. We first show uniqueness: assuming ∇ is such a connection, compatibility with g implies that for any vector fields X, Y and Z, we have the three relations

$$\begin{aligned} L_X\,(g(Y,Z)) &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z),\\ L_Y\,(g(Z,X)) &= g(\nabla_Y Z, X) + g(Z, \nabla_Y X),\\ L_Z\,(g(X,Y)) &= g(\nabla_Z X, Y) + g(X, \nabla_Z Y).\end{aligned}$$

Adding the first two, subtracting the third and using the assumption

$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] \equiv 0$, we find

$$\begin{aligned} &L_X\,(g(Y,Z)) + L_Y\,(g(Z,X)) - L_Z\,(g(X,Y)) \\ &\quad = g(\nabla_X Y + \nabla_Y X, Z) + g(Y, \nabla_X Z - \nabla_Z X) + g(X, \nabla_Y Z - \nabla_Z Y) \\ &\quad = g(2\nabla_X Y, Z) - g([X,Y],Z) + g([X,Z],Y) + g([Y,Z],X), \end{aligned}$$

thus

$$\begin{aligned} g(\nabla_X Y, Z) = \frac{1}{2}\Big[&L_X\,(g(Y,Z)) + L_Y\,(g(Z,X)) - L_Z\,(g(X,Y)) \\ &+ g([X,Y],Z) - g([X,Z],Y) - g([Y,Z],X)\Big]. \end{aligned} \tag{4.7}$$

A straightforward (though slightly tedious) calculation shows that the right hand side of this expression is C^∞-linear with respect to X and Z.

Observe now that at any point $p \in M$, the inner product $g(\ ,\ )$ on T_pM defines an isomorphism

$$\flat : T_pM \to T_p^*M : X \mapsto X^\flat$$

by $X^\flat(Y) = g(X,Y)$. It's clear that this is a linear map; the fact that it's an isomorphism follows from the positivity of g (see Appendix A, §A.4). Thus for any $X, Y \in \mathrm{Vec}(M)$ and $p \in M$, (4.7) gives a formula for $(\nabla_{X(p)}Y)^\flat$, proving the uniqueness of $\nabla_{X(p)}Y$. Moreover, in light of the above remark on C^∞-linearity, this formula can be taken as a definition of the covariant derivative $\nabla_X Y$ for any $X \in T_pM$ and $Y \in \mathrm{Vec}(M)$; one checks by another calculation that the resulting object satisfies the Leibnitz rule and therefore defines a connection. □

Definition 4.14. The connection constructed in Theorem 4.13 is called the *Levi-Civita connection* on (M,g).

Henceforward, whenever a manifold M has a Riemannian metric g, we will assume that all calculations requiring a connection on $TM \to M$ use the Levi-Civita connection.

4.3.2 Geodesics and arc length

We shall now explore the relationship between the geodesics of the Levi-Civita connection and the problem of finding paths of minimal length between fixed points. This uses some basic concepts from the *calculus of variations*, which deals with optimization problems on infinite dimensional spaces. Fix two points $p, q \in M$ and real numbers $a < b$. We denote by

$$C^\infty([a,b], M; p, q)$$

the space of all smooth paths $\gamma : [a,b] \to M$ such that $\gamma(a) = p$ and $\gamma(b) = q$. We then use the metric g to define on this space two maps to the real numbers, the *length functional*

$$\ell_a^b(\gamma) = \int_a^b |\dot\gamma(t)|\ dt$$

and the *energy functional*

$$E_a^b(\gamma) = \int_a^b |\dot\gamma(t)|^2\ dt.$$

The geometric meaning of the first is clear: $\ell_a^b(\gamma)$ is the length of the path traced out by γ. As such, it depends only on the image, and is thus invariant under reparametrizations, i.e. for any diffeomorphism $\varphi : [a,b] \to [a',b']$ and smooth path $\gamma \in C^\infty([a',b'], M; p, q)$, we have

$$\ell_a^b(\gamma \circ \varphi) = \ell_{a'}^{b'}(\gamma).$$

It is less obvious what geometric meaning the energy functional may have, but we will find it convenient as a computational tool in order to understand the length functional better.

We wish to view $C^\infty([a,b], M; p, q)$ informally as an infinite dimensional manifold, and E_a^b and ℓ_a^b as "smooth functions" on this manifold which can be differentiated. The word *functional* is generally used to describe real valued functions on infinite dimensional spaces such as $C^\infty([a,b], M; p, q)$. Given a functional

$$F : C^\infty([a,b], M; p, q) \to \mathbb{R},$$

the goal of the calculus of variations is then to find necessary conditions on a path $\gamma \in C^\infty([a,b], M; p, q)$ so that $F(\gamma)$ may attain a minimal or maximal value among all paths $\gamma_\epsilon \in C^\infty([a,b], M; p, q)$ close to γ; this condition will take the form of a differential equation that γ must satisfy. To make this precise, we say that a *smooth 1-parameter family of paths from p to q* is a collection $\gamma_s \in C^\infty([a,b], M; p, q)$ for $s \in (-\epsilon, \epsilon)$ such that the map $(s,t) \mapsto \gamma_s(t)$ is smooth. We think of this as a smooth path in $C^\infty([a,b], M; p, q)$ through γ_0. The "velocity vector" of this path at $s = 0$ is then given by the partial derivatives $\partial_s \gamma_s(t)|_{s=0} \in T_{\gamma_0(t)}M$ for all t, which defines a section

$$\eta := \partial_s \gamma_s|_{s=0} \in \Gamma(\gamma_0^* TM)$$

such that $\eta(a) = 0$ and $\eta(b) = 0$. We therefore think of the vector space

$$\{\eta \in \Gamma(\gamma^* TM) \mid \eta(a) = 0 \text{ and } \eta(b) = 0\}$$

as the "tangent space" to $C^\infty([a,b], M; p, q)$ at γ. It is now clear how one should define a "directional derivative" of F in a direction defined by a section of $\gamma^* TM$. This motivates the following definition, which generalizes the notion of a critical point.

Definition 4.15. The path $\gamma \in C^\infty([a,b],M;p,q)$ is called *stationary* for the functional $F : C^\infty([a,b],M;p,q) \to \mathbb{R}$ if for every smooth 1-parameter family $\gamma_s \in C^\infty([a,b],M;p,q)$ with $\gamma_0 = \gamma$,

$$\left.\frac{d}{ds}F(\gamma_s)\right|_{s=0} = 0. \tag{4.8}$$

Note that for an arbitrary functional, it is not *a priori* clear that the derivatives in (4.8) will always exist. This is however true in many cases of interest, and in such a situation, it's easy to see that (4.8) is a necessary condition for F to attain an extremal value at γ.

Proposition 4.16. *The energy functional E_a^b is stationary at γ if and only if γ is a geodesic with respect to the Levi-Civita connection.*

Proof. Pick any smooth 1-parameter family $\gamma_s \in C^\infty([a,b],M;p,q)$ with $\gamma_0 = \gamma$ and denote $\eta = \partial_s\gamma_s|_{s=0} \in \Gamma(\gamma^*TM)$. Differentiating under the integral sign and using the properties of the Levi-Civita connection,

$$\begin{aligned}\left.\frac{d}{ds}E_a^b(\gamma_s)\right|_{s=0} &= \int_a^b \left.\frac{\partial}{\partial s} g\big(\partial_t\gamma_s(t), \partial_t\gamma_s(t)\big)\right|_{s=0} dt \\ &= \int_a^b \Big(g\big(\left.\nabla_s\partial_t\gamma_s(t)\right|_{s=0}, \dot\gamma(t)\big) + g\big(\dot\gamma(t), \left.\nabla_s\partial_t\gamma_s(t)\right|_{s=0}\big)\Big)\, dt \\ &= 2\int_a^b g\big(\dot\gamma(t), \nabla_t\, \partial_s\gamma_s(t)|_{s=0}\big)\, dt = 2\int_a^b g\big(\dot\gamma(t), \nabla_t\eta(t)\big)\, dt.\end{aligned}$$

We now perform a geometric version of integration by parts, using the fact that $\eta(t)$ vanishes at its end points. It follows indeed from the fundamental theorem of calculus that

$$0 = \int_a^b \frac{d}{dt} g\big(\dot\gamma(t), \eta(t)\big)\, dt = \int_a^b g\big(\nabla_t\dot\gamma(t), \eta(t)\big)\, dt + \int_a^b g\big(\dot\gamma(t), \nabla_t\eta(t)\big)\, dt,$$

thus

$$\left.\frac{d}{ds}E_a^b(\gamma_s)\right|_{s=0} = -2\int_a^b g\big(\nabla_t\dot\gamma(t), \eta(t)\big)\, dt.$$

Since choosing arbitrary 1-parameter families γ_s leads to arbitrary sections $\eta \in \Gamma(\gamma^*TM)$ with $\eta(a) = \eta(b) = 0$, this expression will vanish for all such choices if and only if $\nabla_t\dot\gamma \equiv 0$, which means γ is a geodesic. □

To see what this tells us about the length functional, we take advantage of parametrization invariance. Assume $\gamma_s \in C^\infty([a,b],M;p,q)$ is a smooth 1-parameter family of paths which are all immersed. Then we claim that there exists a unique smooth 1-parameter family $\beta_s \in C^\infty([a,b],M;p,q)$ with the following two properties for each s:

1. β_s is a reparametrization of γ_s,
2. $\left|\dot\beta_s(t)\right|$ is constant with respect to t.

Indeed, a straightforward calculation shows $\beta_s(t) = \gamma_s(\varphi_s(t))$, where for each s, $\varphi_s : [a,b] \to [a,b]$ is found by solving the initial value problem

$$\begin{aligned}\frac{d\varphi_s}{dt} &= \frac{\ell_b^a(\gamma_s)}{(b-a)\dot\gamma_s(\varphi_s)}, \\ \varphi_s(a) &= a.\end{aligned}$$

We say that the paths $\beta_s \in C^\infty([a,b],M;p,q)$ have *constant speed.* Let $v_s = |\dot\beta_s(t)|$. Then since length is independent of parametrization,

$$\begin{aligned}\left.\frac{d}{ds}\ell_a^b(\gamma_s)\right|_{s=0} = \left.\frac{d}{ds}\ell_a^b(\beta_s)\right|_{s=0} &= \int_a^b \left.\frac{\partial}{\partial s}\sqrt{g\left(\dot\beta_s(t), \dot\beta_s(t)\right)}\right|_{s=0} dt \\ &= \int_a^b \frac{1}{2\sqrt{g\left(\dot\beta_0(t), \dot\beta_0(t)\right)}} \left.\frac{\partial}{\partial s} g\left(\dot\beta_s(t), \dot\beta_s(t)\right)\right|_{s=0} dt \\ &= \frac{1}{2v_0}\left.\frac{d}{ds}E_a^b(\beta_s)\right|_{s=0}.\end{aligned}$$

Thus if γ is stationary for ℓ_a^b, then it has a reparametrization with constant speed that is stationary for E_a^b, and is therefore a geodesic. Conversely, every geodesic is stationary for ℓ_a^b, and also has constant speed since $\nabla_t\dot\gamma = 0$ implies

$$\frac{d}{dt}|\dot\gamma(t)|^2 = 2g(\nabla_t\dot\gamma(t), \dot\gamma(t)) = 0.$$

This proves:

Corollary 4.17. *An immersed path $\gamma \in C^\infty([a,b],M;p,q)$ is a geodesic if and only if it both is stationary for ℓ_a^b and has constant speed.*

We conclude that any path $\gamma \in C^\infty([a,b],M;p,q)$ which minimizes the length $\ell_a^b(\gamma)$ among all nearby paths from p to q can be parametrized by a geodesic. One must be careful in stating the converse to this, for it is not always true that a geodesic gives the shortest path between two points. This is however true in a *local* sense, as shown in the next result; we refer to [Spi99] for the proof.

Proposition 4.18. *Suppose (M,g) is a Riemannian manifold. Then every point $p \in M$ has a neighborhood $p \in \mathcal{U}_p \subset M$ such that for every $q \in \mathcal{U}_p$, there is a unique geodesic $\gamma_q : [0,1] \to \mathcal{U}_p$ with $\gamma_q(0) = p$ and $\gamma_q(1) = q$. This geodesic is embedded and parametrizes the shortest path from p to q.*

4.3.3 Computations

It's useful for computational purposes to have a formula for the Levi-Civita connection in coordinates: in particular, given a chart $(x^1, \dots, x^n) : \mathcal{U} \to \mathbb{R}^n$, we can write the Christoffel symbols Γ^i_{jk} explicitly in terms of the components

$$g_{ij}(p) = g(\partial_i|_p, \partial_j|_p) \in \mathbb{R}$$

of the metric. At any point $p \in \mathcal{U}$, denote by $g^{ij}(p)$ the entries of the n-by-n matrix inverse to $g_{ij}(p)$, so using the Einstein summation convention,

$$g^{ij} g_{jk} = \delta^i_k,$$

where the *Kronecker* δ is defined by $\delta^i_k = 1$ if $i = k$ and otherwise 0.[1] Recall now that the Christoffel symbols $\Gamma^i_{jk} : \mathcal{U} \to \mathbb{R}$ are defined so that for any vector field $X = X^j \partial_j \in \mathrm{Vec}(\mathcal{U})$, $\nabla_j X^i = \partial_j X^i + \Gamma^i_{jk} X^k$. Setting $X = \partial_k$, this implies

$$\Gamma^i_{jk} = (\nabla_j \partial_k)^i.$$

Using the assumption that ∇ is compatible with the metric, we find

$$\begin{aligned} \partial_i g_{jk} &= \partial_i \left(g(\partial_j, \partial_k) \right) = g(\nabla_i \partial_j, \partial_k) + g(\partial_j, \nabla_i \partial_k) \\ &= g_{\ell m} (\nabla_i \partial_j)^\ell \delta^m_k + g_{\ell m} \delta^\ell_j (\nabla_i \partial_k)^m \\ &= g_{\ell k} \Gamma^\ell_{ij} + g_{j\ell} \Gamma^\ell_{ik} \end{aligned} \tag{4.9}$$

Exercise 4.19. Use (4.9) to derive the following formula for the Christoffel symbols of the Levi-Civita connection:

$$\Gamma^k_{ij} = \frac{1}{2} g^{k\ell} \left(\partial_i g_{j\ell} + \partial_j g_{\ell i} - \partial_\ell g_{ij} \right). \tag{4.10}$$

Hint: write down three copies of (4.9) with cyclic permutations of the indices i, j and k. Add the first two and subtract the third (notice the resemblance to the proof of Theorem 4.13). Remember that symmetry implies $\Gamma^k_{ij} = \Gamma^k_{ji}$.

Remark 4.20. With a bit more effort one could turn Exercise 4.19 into an alternative proof of Theorem 4.13: in fact deriving (4.10) from the assumptions that ∇ should be symmetric and compatible with g already proves the uniqueness of ∇. To show that this formula really does define a connection, it remains to check that the right hand side transforms properly under coordinate changes, namely according to the formula in Exercise 3.17.

We are now finally in a position to give some examples of Riemannian manifolds and compute their geodesics.

[1]For an explanation as to why one might sensibly denote the inverse to the matrix g_{ij} by the same symbol with raised indices, see Appendix A, §A.4.

Exercise 4.21. Show that if $M = \mathbb{R}^n$ and g is the standard inner product (also known as the *flat metric* on $\mathbb{R}^n$), then in the obvious coordinate system, all the Christoffel symbols vanish, and the geodesics are all straight lines.

Example 4.22. The *hyperbolic half-plane* $(\mathbb{H}, h)$ is the open set

$$\mathbb{H} = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$$

with Riemannian metric

$$h(X, Y) = \frac{1}{y^2} \langle X, Y \rangle_{\mathbb{R}^2}$$

for $X, Y \in T_{(x,y)}\mathbb{H} = \mathbb{R}^2$, where $\langle\ ,\ \rangle_{\mathbb{R}^2}$ denotes the standard inner product on $\mathbb{R}^2$. As we will see in Chapter 5, this is an example of a surface with constant negative curvature; it plays an important role in complex analysis, particularly the theory of Riemann surfaces (see, for example [SS92]).

Exercise 4.23. Using the obvious global coordinates on $\mathbb{H}$, derive the Christoffel symbols and show that the geodesic equation can be written as

$$\ddot{x} - \frac{2}{y} \dot{x} \dot{y} = 0,$$
$$\ddot{y} + \frac{1}{y} (\dot{x}^2 - \dot{y}^2) = 0$$

for a smooth path $\gamma(t) = (x(t), y(t))$.

Exercise 4.24. Show that $(\mathbb{H}, h)$ has geodesics of the form

$$\gamma(t) = (c, y(t)),$$

for any constant $c \in \mathbb{R}$ and appropriately chosen functions $y(t) > 0$, and that it also has geodesics of the form

$$\gamma(t) = (r \cos\theta(t) + c, r \sin\theta(t))$$

for any constants $c \in \mathbb{R}$, $r > 0$ and appropriately chosen functions $\theta(t) \in (0, \pi)$. In fact, these are *all* the geodesics on $(\mathbb{H}, h)$: they consist of all vertical lines and circles that meet the x-axis orthogonally.

Exercise 4.25. Show that any two points in $(\mathbb{H}, h)$ are connected by a unique geodesic segment, and compute the length of this segment.

4.3.4 Riemannian submanifolds and surfaces in $\mathbb{R}^3$

Consider $(M, g) = (\mathbb{R}^3, g_{\mathrm{E}})$, where g_{E} is defined by the standard Euclidean inner product on $\mathbb{R}^3$, and suppose $\Sigma \subset \mathbb{R}^3$ is an embedded surface. This inherits a Riemannian metric $j^* g_{\mathrm{E}}$ via the inclusion map $j : \Sigma \hookrightarrow \mathbb{R}^3$, which is the same thing as defining the inner product of X and Y in $T_p\Sigma$ to be $\langle X, Y\rangle_{\mathbb{R}^3}$. As we've seen, it's easy enough to compute the geodesics in $(\mathbb{R}^3, g_{\mathrm{E}})$: these are straight lines with constant velocity. We now ask how this information can be used to identify the geodesics on the surface $(\Sigma, j^* g_{\mathrm{E}})$.

The question can be framed somewhat more generally without making it any harder. Suppose indeed that (M, g) is an arbitrary Riemannian manifold and $\Sigma \subset M$ is a submanifold, to which we assign the pullback metric j^*g using the inclusion map $j : \Sigma \hookrightarrow M$. There is a linear bundle map

$$\pi_\Sigma : TM|_\Sigma \to T\Sigma$$

which for each $p \in \Sigma$ acts as the projection from T_pM to $T_p\Sigma$ along the orthogonal complement of $T_p\Sigma$.

Proposition 4.26. *If ∇ is the Levi-Civita connection for (M, g), the Levi-Civita connection $\widetilde{\nabla}$ for (Σ, j^*g) is defined by*

$$\widetilde{\nabla}_X Y = \pi_\Sigma(\nabla_X Y)$$

for $X \in T\Sigma$ and $Y \in \mathrm{Vec}(\Sigma)$.

Proof. We check first that $\widetilde{\nabla}$ satisfies the Leibnitz rule: for $f \in C^\infty(\Sigma)$,

$$\begin{aligned}\widetilde{\nabla}_X(fY) &= \pi_\Sigma\left(\nabla_X(fY)\right) = \pi_\Sigma\left((L_X f)Y + f\nabla_X Y\right)\\ &= (L_X f)\pi_\Sigma(Y) + f\pi_\Sigma(\nabla_X Y) = (L_X f)Y + f\widetilde{\nabla}_X Y,\end{aligned}$$

thus $\widetilde{\nabla}$ defines a connection on $T\Sigma \to \Sigma$. To see that it is compatible with the metric j^*g, pick vector fields $X, Y, Z \in \mathrm{Vec}(\Sigma)$: then extending these to vector fields on M and using the fact that ∇ is compatible with g,

$$\begin{aligned}L_X\left(j^*g(Y, Z)\right) &= L_X\left(g(Y, Z)\right)\\ &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z)\\ &= g(\pi_\Sigma(\nabla_X Y), Z) + g(Y, \pi_\Sigma(\nabla_X Z))\\ &= j^*g(\widetilde{\nabla}_X Y, Z) + j^*g(Y, \widetilde{\nabla}_X Z).\end{aligned}$$

Here we've used the fact that for any $v \in T_pM$, the difference $v - \pi_\Sigma(v)$ is orthogonal to $T_p\Sigma$, thus if $w \in T_p\Sigma$, $g(v, w) = g(\pi_\Sigma v, w)$. Finally to see that $\widetilde{\nabla}$ is symmetric, we evaluate the corresponding torsion tensor $\widetilde{T}$ on a

pair of vector fields $X, Y \in \mathrm{Vec}(\Sigma)$, noting that the bracket $[X, Y]$ is also tangent to Σ, thus

$$\begin{aligned}\widetilde{T}(X, Y) &= \widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X - [X, Y]\\ &= \pi_\Sigma\left(\nabla_X Y - \nabla_Y X - [X, Y]\right) = \pi_\Sigma\left(T(X, Y)\right) = 0.\end{aligned}$$

The result now follows by the uniqueness of the Levi-Civita connection. □

Corollary 4.27. *Let ∇ denote the Levi-Civita connection on (M, g) and suppose $\Sigma \subset M$ is a submanifold with induced metric j^*g. Then a smooth path $\gamma(t) \in \Sigma$ is a geodesic with respect to j^*g if and only if $\nabla_t\dot\gamma(t)$ is orthogonal to $T_{\gamma(t)}\Sigma$ for all t.*

Proof. Using $\widetilde{\nabla} = \pi_\Sigma \circ \nabla$, the geodesic equation on (Σ, j^*g) reads

$$0 = \widetilde{\nabla}_t\dot\gamma = \pi_\Sigma(\nabla_t\dot\gamma),$$

which is satisfied precisely when $\nabla_t\dot\gamma(t)$ belongs to the orthogonal complement of $T_{\gamma(t)}\Sigma$. □

It's quite easy to apply this result to the case of a surface in Euclidean 3-space: ∇ is now the trivial connection, thus the geodesic equation for $\gamma(t) \in \Sigma \subset \mathbb{R}^3$ reduces to the condition that $\ddot\gamma(t)$ be always orthogonal to Σ.

Exercise 4.28. Let $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ and define the usual spherical polar coordinates (ϕ, θ) by

$$\begin{aligned}x &= \cos\phi\ \cos\theta,\\ y &= \sin\phi\ \cos\theta,\\ z &= \sin\theta.\end{aligned}$$

Show that for any constants $a, b \in \mathbb{R}$, the path

$$(\phi(t), \theta(t)) = (a, bt)$$

is a geodesic. Use a symmetry argument to conclude that the geodesics on S^2 are precisely the *great circles*, i.e. circles which divide the sphere into two parts that are mirror images of each other.

References

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