Chapter 5

Curvature on Bundles

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5.1 Flat sections and connections

A connection on a fiber bundle $\pi : E \to M$ allows one to define when a section is "constant" along smooth paths $\gamma(t) \in M$; we call such sections *horizontal*, or in the case of a vector bundle, *parallel*. Since by definition horizontal sections always exist along any given path, the nontrivial implications of the following question may not be immediately obvious:

> *Given $p \in M$ and a sufficiently small neighborhood $p \in \mathcal{U} \subset M$, does E admit any section $s \in \Gamma(E|_{\mathcal{U}})$ for which $\nabla s \equiv 0$?*

A section whose covariant derivative vanishes identically is called a *flat* or *covariantly constant* section. It may seem counterintuitive that the answer could possibly be no—after all, one of the most obvious facts about smooth functions is that *constant functions exist*. This translates easily into a statement about sections of trivial bundles. Of course, all bundles are *locally* trivial, thus locally one can always find sections that are constant with respect to a trivialization. These sections are covariantly constant with respect to a connection determined by the trivialization. As we will see however, connections of this type are rather special: for generic connections, flat sections do not exist, even locally!

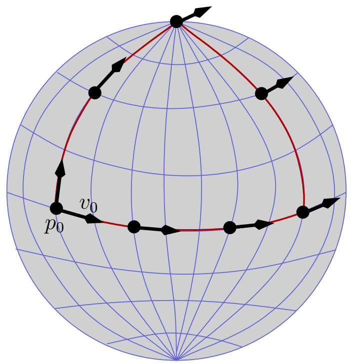

Figure 5.1: Parallel transport along a closed path on S^2.

Definition 5.1. A connection on the fiber bundle $\pi : E \to M$ is called *flat* if for every $x \in M$ and $p \in E_x$, there exists a neighborhood $x \in \mathcal{U} \subset M$ and a flat section $s : \mathcal{U} \to E$ with $s(x) = p$.

The flatness of a connection is closely related to the following question:

> *If $E \to M$ is a fiber bundle with a connection and $\gamma : [0,1] \to M$ is a smooth path with $\gamma(0) = \gamma(1)$, does the parallel transport $P^t_\gamma : E_{\gamma(0)} \to E_{\gamma(t)}$ satisfy $P^1_\gamma = \mathrm{Id}$?*

In other words, does parallel transport around a closed path bring every element of the fiber back to itself? Clearly this is the case if the path $\gamma(t)$ lies in an open set $\mathcal{U} \subset M$ on which flat sections $s : \mathcal{U} \to E$ exist, for then $P^t_\gamma(s(\gamma(0)) = s(\gamma(t))$.

We can now easily see an example of a connection that is not flat. Let S^2 be the unit sphere in $\mathbb{R}^3$, with Riemannian metric and corresponding Levi-Civita connection on $TS^2 \to S^2$ inherited from the standard Euclidean metric on $\mathbb{R}^3$. Construct a piecewise smooth closed path γ as follows: beginning at the equator, travel 90 degrees of longitude along the equator, then upward along a geodesic to the north pole, and down along a different geodesic back to the starting point (Figure 5.1). Parallel transport of any vector along this path has the effect of rotating the original vector by 90 degrees. This implies that the "triangle" bounded by this path does not admit any covariantly constant vector field.

5.2 Integrability and the Frobenius theorem

Our main goal in this chapter is to formulate precise conditions for identifying whether a connection is flat. Along the way, we can solve a related problem which is of independent interest and has nothing intrinsically to do with bundles: it leads to the theorem of Frobenius on integrable distributions.

The word *integrability* has a variety of meanings in different contexts: generally it refers to questions in which one is given some data of a linear nature, and would like to find some nonlinear data which produce the given linear data as a form of "derivative". The problem of finding antiderivatives of smooth functions on $\mathbb{R}$ is the simplest example: it is always solvable (at least in principle) and therefore not very interesting for the present discussion. A more interesting example is the generalization of this question to higher dimensions, which can be stated as follows:

> *Given a 1-form λ on an n-manifold M, under what conditions is λ* locally *the differential of a smooth function $f : M \to \mathbb{R}$?*

Our use of the word "locally" means that for every $p \in M$, we seek a neighborhood $p \in \mathcal{U} \subset M$ and smooth function $f : \mathcal{U} \to \mathbb{R}$ such that $df = \lambda|_{\mathcal{U}}$. The answer to this question is well known: the right condition is that λ must be a closed 1-form, $d\lambda = 0$. Indeed, the Poincaré lemma generalizes this result by saying that every closed differential k-form locally is the exterior derivative of some $(k-1)$-form.

Here are two more (closely related) integrability results from the theory of smooth manifolds which we shall find quite useful.

Theorem 5.2. *For any vector fields $X, Y \in \mathrm{Vec}(M)$, the flows φ_X^s and φ_Y^t commute for all $s, t \in \mathbb{R}$ if and only if $[X, Y] \equiv 0$.*

Corollary 5.3. *Suppose $X_1, \ldots, X_n \in \mathrm{Vec}(M)$ are vector fields such that $[X_i, X_j] \equiv 0$ for all $i, j \in \{1, \ldots, n\}$. Then for every $p \in M$ there exists a neighborhood $p \in \mathcal{U} \subset M$ and a coordinate chart $x = (x^1, \ldots, x^n) : \mathcal{U} \to \mathbb{R}^n$ such that $\frac{\partial}{\partial x^j} \equiv X_j|_{\mathcal{U}}$ for all j.*

We refer to [Spi99] for the proofs, though it will be important to recall why Corollary 5.3 follows from Theorem 5.2: the desired chart $x : \mathcal{U} \to \mathbb{R}^n$ is constructed as the inverse of a map of the form

$$f(t^1, \ldots, t^n) = \varphi_{X_1}^{t^1} \circ \ldots \circ \varphi_{X_n}^{t^n}(p),$$

which works precisely because the flows all commute.

The main integrability problem that we wish to address in this chapter concerns *distributions*: recall that if M is a smooth n-manifold, a k-dimensional distribution $\xi \subset TM$ is a smooth subbundle of rank k in the tangent bundle $TM \to M$, in other words, it assigns smoothly to each tangent space T_pM a k-dimensional subspace $\xi_p \subset T_pM$. We say that a vector field $X \in \mathrm{Vec}(M)$ is *tangent to* ξ if $X(p) \in \xi_p$ for all $p \in M$. In this case X is also a section of the vector bundle $\xi \to M$.

Definition 5.4. Given a distribution $\xi \subset TM$, a smooth submanifold $N \subset M$ is called an *integral submanifold* of ξ if for all $p \in N$, $T_pN \subset \xi_p$.

Definition 5.5. A k-dimensional distribution $\xi \subset TM$ is called *integrable* if for every $p \in M$ there exists a k-dimensional integral submanifold through p.[1]

It follows from the basic theory of ordinary differential equations that *every* 1-dimensional distribution is integrable: locally one can choose a nonzero vector field that spans the distribution, and the orbits of this vector field trace out integral submanifolds. More generally, any distribution admits 1-dimensional integral submanifolds, and one can further use the theory of ordinary differential equations to show that k-dimensional integral submanifolds of a k-dimensional distribution are unique if they exist. Existence is however not so clear when $k \geq 2$. For instance, a 2-dimensional distribution in $\mathbb{R}^3$ may "twist" in such a way as to make integral submanifolds impossible (Figure 5.2).

A simple necessary condition for $\xi \subset TM$ to be integrable comes from considering brackets of vector fields tangent to ξ. Indeed, suppose ξ is integrable, and for every $p \in M$, denote by $N_p \subset M$ the unique k-dimensional integral submanifold containing p. The key observation now is that a vector field $X \in \mathrm{Vec}(M)$ is tangent to ξ if and only if it is tangent to all the integral submanifolds, in which case it has well defined restrictions $X|_{N_p} \in \mathrm{Vec}(N_p)$. Thus if $X, Y \in \mathrm{Vec}(M)$ are both tangent to ξ, so is $[X, Y]$, as it must also have well defined restrictions $[X, Y]|_{N_p} \in \mathrm{Vec}(N_p)$ which match $[X|_{N_p}, Y|_{N_p}]$. We will see from examples that in general, the bracket of two vector fields tangent to ξ is not also tangent to ξ, which is clearly a necessary condition for integrability. The content of Theorem 5.16 below is that this condition is also sufficient.

We will approach the proof of this via a version that applies specifically to connections on fiber bundles, and is also highly relevant to the flatness question. Recalling the definition of the covariant derivative on a fiber bundle $E \to M$, we see that a section $s : \mathcal{U} \to E$ is flat if and only if the submanifold $s(\mathcal{U}) \subset E$ is everywhere tangent to the chosen horizontal

[1] Some authors give a different though equivalent definition for an integrable distribution: they reverse the roles of Definition 5.5 and Theorem 5.16 so that a distribution is said to be integrable if it satisfies the conditions stated in the theorem. Then their version of the Frobenius theorem states that this definition is equivalent to ours. It comes to the same thing in the end.

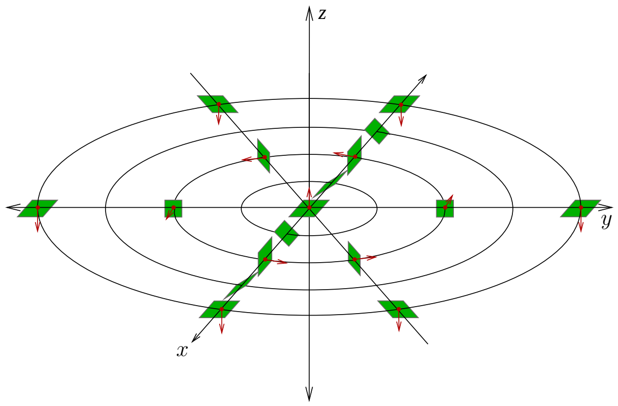

Figure 5.2: A non-integrable 2-dimensional distribution on $\mathbb{R}^3$.

subbundle $HE \subset TE$. This means it is an n-dimensional integral submanifold for an n-dimensional distribution on E, and allows us therefore to reformulate the definition of a flat connection as follows.

Proposition 5.6. *If $E \to M$ is a fiber bundle, then a connection is flat if and only if the corresponding horizontal distribution $HE \subset TE$ is integrable.*

Exercise 5.7. Show that if E is a fiber bundle over a 1-manifold M, then every connection on E is flat.

Exercise 5.8. Show that the trivial connection on a trivial bundle $E = M \times F$ is flat.

Let $\pi : E \to M$ be a fiber bundle with connection $HE \subset TE$ and denote by

$$K : TE \to VE, \qquad H : TE \to HE$$

the two projections defined by the splitting $TE = HE \oplus VE$. A vector field on E is called *horizontal* or *vertical* if it is everywhere tangent to HE or VE respectively. For $X \in \mathrm{Vec}(M)$, define the *horizontal lift* of X to be the unique horizontal vector field $X_h \in \mathrm{Vec}(E)$ such that $T\pi \circ X_h = X \circ \pi$, or equivalently,

$$X_h(p) = \mathrm{Hor}_p(X(x))$$

for each $x \in M$, $p \in E_x$.

Exercise 5.9. Show that for any $X \in \mathrm{Vec}(M)$ and $f \in C^\infty(M)$, $L_{X_h}(f \circ \pi) = (L_X f) \circ \pi$.

Lemma 5.10. *Suppose $\xi, \eta \in \mathrm{Vec}(E)$ are both horizontal and satisfy $L_\xi f \equiv L_\eta f$ for every function $f \in C^\infty(E)$ such that $df|_{VE} \equiv 0$. Then $\xi \equiv \eta$.*

Proof. If $\xi(p) \neq \eta(p)$ for some $p \in E$, assume without loss of generality that $\xi(p) \neq 0$. Since $\xi(p) \in H_pE$, we can then find a smooth real-valued function f defined near p which is constant in the vertical directions but satisfies $df(\xi(p)) \neq 0$ and $df(\eta(p)) = 0$, so $L_\xi f(p) \neq L_\eta f(p)$. □

Lemma 5.11. *For any $X, Y \in \mathrm{Vec}(M)$, $[X,Y]_h = H \circ [X_h, Y_h]$.*

Proof. Observe first that for any $\xi \in \mathrm{Vec}(E)$ and $f \in C^\infty(M)$,

$$L_\xi(f \circ \pi) = L_{H\circ\xi}(f \circ \pi) + L_{K\circ\xi}(f \circ \pi) = L_{H\circ\xi}(f \circ \pi)$$

since $d(f \circ \pi)|_{VE} \equiv 0$. Then for $X, Y \in \mathrm{Vec}(M)$, using Exercise 5.9,

$$\begin{aligned} L_{H\circ[X_h,Y_h]}(f \circ \pi) &= L_{[X_h,Y_h]}(f \circ \pi) \\ &= L_{X_h} L_{Y_h}(f \circ \pi) - L_{Y_h} L_{X_h}(f \circ \pi) \\ &= L_{X_h}((L_Y f) \circ \pi) - L_{Y_h}((L_X f) \circ \pi) \\ &= (L_X L_Y f) \circ \pi - (L_Y L_X f) \circ \pi = (L_{[X,Y]} f) \circ \pi. \end{aligned}$$

Likewise, again applying Exercise 5.9,

$$L_{[X,Y]_h}(f \circ \pi) = (L_{[X,Y]} f) \circ \pi = L_{H\circ[X_h,Y_h]}(f \circ \pi),$$

so the result follows from Lemma 5.10 □

Proposition 5.12. *The distribution $HE \subset TE$ is integrable if and only if for every pair of vector fields $X, Y \in \mathrm{Vec}(M)$, $[X_h, Y_h]$ is horizontal.*

Proof. If HE is integrable, then the lifts $X_h, Y_h \in \mathrm{Vec}(E)$ are tangent to the integral submanifolds, implying that $[X_h, Y_h]$ is as well. To prove the converse, for any $x \in M$, pick pointwise linearly independent vector fields $X_1, \ldots, X_n$ on a neighborhood $x \in \mathcal{U} \subset M$ such that $[X_i, X_j] = 0$ for each pair, and denote $\xi_j := (X_j)_h$. By assumption $[\xi_i, \xi_j]$ is horizontal, thus by Lemma 5.11,

$$[\xi_i, \xi_j] = H \circ [\xi_i, \xi_j] = [X_i, X_j]_h = 0.$$

Therefore for any $p \in E_x$, we can construct an integral submanifold through p via the commuting flows of ξ_i: it is parametrized by the map

$$f(t^1, \ldots, t^n) = \varphi^{t^1}_{\xi_1} \circ \ldots \circ \varphi^{t^n}_{\xi_n}(p) \tag{5.1}$$

for real numbers $t^1, \ldots, t^n$ sufficiently close to 0. □

Exercise 5.13. Verify that the map (5.1) parametrizes an embedded integral submanifold of HE.

Exercise 5.14. Show that the bilinear map $\Omega_K : \mathrm{Vec}(E) \times \mathrm{Vec}(E) \to \mathrm{Vec}(E)$ defined by

$$\Omega_K(\xi, \eta) = -K\left([H(\xi), H(\eta)]\right) \tag{5.2}$$

is C^∞-linear in both variables.

By the result of this exercise, (5.2) defines an antisymmetric bundle map $\Omega_K : TE \oplus TE \to VE$, called the *curvature* 2*-form* associated to the connection. Combining this with Prop. 5.12, we find that the vanishing of this 2-form characterizes flat connections:

Theorem 5.15. *A connection on the fiber bundle $\pi : E \to M$ is flat if and only if its curvature 2-form vanishes identically.*

Proof. If HE is integrable then the bracket of the two horizontal vector fields $H(\xi)$ and $H(\eta)$ is also horizontal, thus $\Omega_K(\xi, \eta) = 0$. Conversely if this is true for every $\xi, \eta \in \mathrm{Vec}(E)$, then it holds in particular for the horizontal lifts X_h and Y_h of $X, Y \in \mathrm{Vec}(M)$, implying that $[X_h, Y_h]$ is horizontal, and by Prop. 5.12, HE is integrable. □

We conclude this section with the promised integrability theorem for distributions in general—the result has nothing intrinsically to do with fiber bundles, but our proof rests on the fact that locally, both situations are the same.

Theorem 5.16 (Frobenius)**.** *A distribution $\xi \subset TM$ is integrable if and only if for every pair of vector fields $X, Y \in \mathrm{Vec}(M)$ tangent to ξ, $[X, Y]$ is also tangent to ξ.*

Proof. The question is fundamentally local, so we can assume without loss of generality that M is an open subset $\mathcal{U} \subset \mathbb{R}^n$, and arrange the k-dimensional distribution $\xi \subset T\mathbb{R}^n|_{\mathcal{U}}$ so that for each $x \in \mathcal{U}$, $\xi_p \subset \mathbb{R}^n$ is transverse to a fixed subspace $\mathbb{R}^{n-k} \subset \mathbb{R}^n$. We can then view ξ as a connection on a fiber bundle which has $\mathcal{U}$ as its total space and $\mathcal{U} \times \mathbb{R}^{n-k}$ as the vertical subbundle, and the result follows from Theorem 5.15. □

Exercise 5.17. Using cylindrical polar coordinates (ρ, ϕ, z) on $\mathbb{R}^3$, define the 1-form

$$\lambda = f(\rho)\, dz + g(\rho)\, d\phi, \tag{5.3}$$

where f and g are smooth real-valued functions such that $(f(\rho), g(\rho)) \neq (0, 0)$ for all ρ, and define a 2-dimensional distribution by $\xi = \ker \lambda \subset T\mathbb{R}^3$, i.e. for $p \in \mathbb{R}^3$, $\xi_p = \ker(\lambda_p)$. An example of such a distribution is shown in Figure 5.2. In this problem we develop a general scheme for determining whether distributions of this type are integrable. Indeed, if λ is any 1-form on $\mathbb{R}^3$ that is nowhere zero, show that the following are equivalent:

1. $\lambda \wedge d\lambda \equiv 0$.
2. The restriction of $d\lambda$ to a bilinear form on the bundle ξ is everywhere degenerate, i.e. at every $p \in \mathbb{R}^3$, there is a vector $X \in \xi_p$ such that $d\lambda(X, Y) = 0$ for all $Y \in \xi_p$.
3. ξ is integrable.

Conclude that the distribution defined as the kernel of (5.3) is integrable if and only if $f'(\rho)g(\rho) - f(\rho)g'(\rho) = 0$ for all ρ.

5.3 Curvature on a vector bundle

If $E \to M$ is a vector bundle and ∇ is a linear connection, it is natural to ask whether covariant derivative operators ∇_X and ∇_Y in different directions commute. Of course this is not even true in general for the Lie derivatives L_X and L_Y on $C^\infty(M)$, which one can view as the trivial connection on a trivial line bundle. Their lack of commutativity can however be measured via the identity

$$L_X L_Y - L_Y L_X = L_{[X,Y]},$$

and one might wonder whether it is true more generally that $\nabla_X \nabla_Y - \nabla_Y \nabla_X = \nabla_{[X,Y]}$ as operators on $\Gamma(E)$. The answer turns out to be no in general, but the failure of this identity can be measured precisely in terms of curvature.

Definition 5.18. Given a linear connection ∇ on a vector bundle $E \to M$, the *curvature tensor*[2] is the unique multilinear bundle map

$$R : TM \oplus TM \oplus E \to E : (X, Y, v) \mapsto R(X, Y)v$$

such that for all $X, Y \in \mathrm{Vec}(M)$ and $v \in \Gamma(E)$,

$$R(X, Y)v = (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})v.$$

Exercise 5.19. Show that $R(X, Y)v$ is C^∞-linear with respect to each of the three variables.

Exercise 5.20. Choosing coordinates $x = (x^1, \ldots, x^n) : \mathcal{U} \to \mathbb{R}^n$ and a frame $(e_{(1)}, \ldots, e_{(m)})$ for E over some open subset $\mathcal{U} \subset M$, define the components of R by ${R^i}_{jk\ell}$ so that $(R(X, Y)v)^i = {R^i}_{jk\ell} X^j Y^k v^\ell$. Show that

$${R^i}_{jk\ell} = \partial_j \Gamma^i_{k\ell} - \partial_k \Gamma^i_{j\ell} + \Gamma^i_{jm}\Gamma^m_{k\ell} - \Gamma^i_{km}\Gamma^m_{j\ell}.$$

[2] This choice of terminology foreshadows Theorem 5.21, which relates the curvature tensor to the curvature 2-form for fiber bundles described in the previous section.

It may be surprising at first sight that $R(X,Y)v$ doesn't depend on any derivatives of v: indeed, it seems to tell us less about v than about the connection itself. Our main goal in this section is to establish a precise relationship between the new curvature tensor R and the curvature 2-form Ω_K defined in the previous section. In particular, this next result implies that "covariant mixed partials" commute if and only if the connection is flat.

Theorem 5.21. *For any vector bundle $E \to M$ with connection ∇, the curvature tensor R satisfies*

$$R(X,Y)v = -K\left([X_h, Y_h](v)\right)$$

for any vector fields $X, Y \in \mathrm{Vec}(M)$ and $v \in E$.

Note that K on the right hand side of this formula is not quite the same projection as in the previous section: as is standard for linear connections, K is now the connection map $K : TE \to E$ obtained from the vertical projection via the identification $V_vE = E_p$ for $v \in E_p$. In light of this, it is natural to give a slightly new (but equivalent) definition of the curvature 2-form when the connection is linear. We define an antisymmetric bilinear bundle map $\Omega_K : TM \oplus TM \to \mathrm{End}(E)$ by the formula

$$\Omega_K(X,Y)v = -K([X_h, Y_h](v)), \tag{5.4}$$

for any $p \in M$, $X, Y \in T_pM$ and $v \in E_p$, where on the right hand side we choose arbitrary extensions of X and Y to vector fields near p. It's straightforward to check that this expression is C^∞-linear in both X and Y; what's less obvious is that it is also linear with respect to v. This is true because the connection map $K : TE \to E$ satisfies $K \circ Tm_\lambda = m_\lambda \circ K$ (see Definition 3.9), where $m_\lambda : E \to E$ is the map $v \mapsto \lambda v$ for any scalar $\lambda \in \mathbb{F}$. Indeed, since m_λ is a diffeomorphism on E whenever $\lambda \neq 0$, we have $(m_\lambda)_*[\xi, \eta] \equiv [(m_\lambda)_*\xi, (m_\lambda)_*\eta]$ for any $\xi, \eta \in \mathrm{Vec}(E)$, thus

$$\begin{aligned}
\Omega_K(X,Y)(\lambda v) &= -K([X_h, Y_h](\lambda v)) = -K([X_h, Y_h] \circ m_\lambda(v)) \\
&= -K(Tm_\lambda \circ [X_h, Y_h](v)) = -m_\lambda \circ K([X_h, Y_h](v)) \\
&= \lambda \cdot \Omega_K(X,Y)v.
\end{aligned}$$

It follows from this and Lemma 3.10 that $v \mapsto \Omega_K(X,Y)v$ is linear.

With our new definition of the curvature 2-form, Theorem 5.21 can be restated succinctly as

$$R(X,Y)v = \Omega_K(X,Y)v.$$

It follows that $\nabla_X\nabla_Y v - \nabla_Y\nabla_X v - \nabla_{[X,Y]}v \equiv 0$ for all vector fields X, Y and sections v if and only if the connection is flat. As an important special

case, if $\alpha(s,t) \in M$ is a smooth map parametrized by two real variables and $v(s,t) \in E_{\alpha(s,t)}$ defines a smooth section of E along α, we have

$$\nabla_s\nabla_t v \equiv \nabla_t\nabla_s v$$

if and only if ∇ is flat; more generally

$$\nabla_s\nabla_t v - \nabla_t\nabla_s v = R(\partial_s\alpha, \partial_t\alpha)v.$$

We shall prove Theorem 5.21 by relating the bracket to an exterior derivative using a generalization of the standard formula

$$d\alpha(X,Y) = L_X\left(\alpha(Y)\right) - L_Y\left(\alpha(X)\right) - \alpha([X,Y])$$

for 1-forms $\alpha \in \Omega^1(M)$. In particular, the definitions of Ω_K, K and R can all be expressed in terms of *bundle-valued differential forms*. For any vector bundle $\pi : E \to M$, define

$$\Omega^k(M,E)$$

to be the vector space of smooth real multilinear bundle maps

$$\omega : \underbrace{TM \oplus \ldots \oplus TM}_{k} \to E$$

which are antisymmetric in the k variables. By this definition, $\Omega^k(M)$ is simply $\Omega^k(M, M \times \mathbb{R})$, i.e. the space of k-forms taking values in the trivial real line bundle. Similarly, what we referred to in §3.3.3 as $\Omega^k(M, \mathfrak{g})$ (the space of $\mathfrak{g}$-valued k-forms for a Lie algebra $\mathfrak{g}$) is actually $\Omega^k(M, M \times \mathfrak{g})$, though we'll preserve the old notation when there's no danger of confusion. From this perspective, we have

$$K \in \Omega^1(E, \pi^*E) \quad \text{and} \quad \Omega_K \in \Omega^2(M, \mathrm{End}(E)).$$

Defining $\Omega^0(M,E) := \Gamma(E)$, the covariant derivative gives a linear map $\nabla : \Omega^0(M,E) \to \Omega^1(M,E) = \Gamma(\mathrm{Hom}_{\mathbb{R}}(TM,E))$, and by analogy with the differential $d : \Omega^0(M) \to \Omega^1(M)$, it's natural to extend this to a *covariant exterior derivative*

$$d_\nabla : \Omega^k(M,E) \to \Omega^{k+1}(M,E),$$

defined as follows. Every $\omega \in \Omega^k(M,E)$ can be expressed in local coordinates $x = (x^1, \ldots, x^n) : \mathcal{U} \to \mathbb{R}^n$ as

$$\omega = \sum_{1 \le i_i < \ldots < i_k \le n} \omega_{i_1 \ldots i_k}\, dx^{i_1} \wedge \ldots \wedge dx^{i_k}$$

for some *component sections* $\omega_{i_1\ldots i_k} \in \Gamma(E|_{\mathcal{U}})$. Then $d_\nabla\omega$ is defined locally as

$$\begin{aligned} d_\nabla\omega &= \sum_{1\le i_1<\ldots<i_k\le n} \nabla\omega_{i_1\ldots i_k} \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k} \\ &= \sum_{1\le i_1<\ldots<i_k\le n} \nabla_j\omega_{i_1\ldots i_k}\, dx^j \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k}. \end{aligned}$$

This is well defined by the same argument as for ordinary differential forms. Note that one can naturally define wedge products $\alpha\wedge\beta$ or $\beta\wedge\alpha \in \Omega^{k+\ell}(M,E)$ for $\alpha\in\Omega^k(M)$ and $\beta\in\Omega^\ell(M,E)$, but it makes no sense if *both* forms are bundle-valued.

Exercise 5.22. Show that $d_\nabla : \Omega^k(M,E) \to \Omega^{k+1}(M,E)$ can equivalently be defined as the unique linear operator which matches ∇ on $\Omega^0(M,E)$ and satisfies the graded Leibniz rule

$$d_\nabla(\alpha\wedge\beta) = d_\nabla\alpha\wedge\beta + (-1)^k\alpha\wedge d\beta$$

for all $\alpha\in\Omega^k(M,E)$ and $\beta\in\Omega^\ell(M)$.

Exercise 5.23. Show that for $\lambda\in\Omega^1(M,E)$ written in local coordinates over $\mathcal{U}\subset M$ as $\lambda=\lambda_i\,dx^i$, the component sections for $d_\nabla\lambda$ over $\mathcal{U}$ are given by

$$(d_\nabla\lambda)_{ij} = \nabla_i\lambda_j - \nabla_j\lambda_i.$$

Use this to prove the coordinate free formula

$$d_\nabla\lambda(X,Y) = \nabla_X(\lambda(Y)) - \nabla_Y(\lambda(X)) - \lambda([X,Y]). \tag{5.5}$$

Hint: for the last step, the main task is to show that the right hand side of (5.5) gives a well defined bundle-valued 2-form; the rest follows easily from the coordinate formula.

Proof of Theorem 5.21. We will show that both $R(X,Y)v$ and $\Omega_K(X,Y)v$ can be expressed in terms of a covariant exterior derivative of the connection map $K: TE\to E$. In this context, we regard K as a bundle-valued 1-form $K\in\Omega^1(E,\pi^*E)$, and use the connection ∇ on $\pi: E\to M$ to induce a natural connection on the pullback bundle $\pi^*E\to E$: this is the unique connection such that for any smooth path $\gamma(t)\in E$ and section $v(t)\in(\pi^*E)_{\gamma(t)} = E_{\pi\circ\gamma(t)}$ along γ, $\nabla_t v$ matches the covariant derivative of v as a section of E along the path $\pi\circ\gamma(t)\in M$.

We claim first that for any $p\in M$, $v\in E_p$ and $X,Y\in T_pM$,

$$d_\nabla K(\mathrm{Hor}_v(X),\mathrm{Hor}_v(Y)) = \Omega_K(X,Y)v.$$

Indeed, extend X and Y to vector fields on M and use the corresponding horizontal lifts $X_h, Y_h\in\mathrm{Vec}(E)$ as extensions of $\mathrm{Hor}_v(X)$ and $\mathrm{Hor}_v(Y)\in$

T_vE respectively. Then using (5.5) and the fact that K vanishes on horizontal vectors,

$$\begin{aligned} d_\nabla K(X_h(v),Y_h(v)) &= \nabla_{X_h(v)}(K(Y_h)) - \nabla_{Y_h(v)}(K(X_h)) - K([X_h,Y_h](v)) \\ &= \Omega_K(X,Y)v. \end{aligned}$$

We now show that $R(X,Y)v$ can also be expressed in this way. Choose a smooth map $\alpha(s,t)\in M$ for $(s,t)\in\mathbb{R}^2$ near $(0,0)$ such that $\partial_s\alpha(0,0)=X$ and $\partial_t\alpha(0,0)=Y$, and extend $v\in E_p$ to a section $v(s,t)\in E_{\alpha(s,t)}$ along α such that $\nabla_s v(0,0)=\nabla_t v(0,0)=0$. Then expressing covariant derivatives via the connection map (e.g. $\nabla_s v = K(\partial_s v)$) and applying (5.5) once more, we find

$$\begin{aligned} R(X,Y)v &= \nabla_s\nabla_t v(0,0) - \nabla_t\nabla_s v(0,0) \\ &= \nabla_s(K(\partial_s v(s,t))) - \nabla_t(K(\partial_t v(s,t)))|_{(s,t)=(0,0)} \\ &= d_\nabla K(\partial_s v,\partial_t v) = d_\nabla K(\mathrm{Hor}_v(X),\mathrm{Hor}_v(Y)), \end{aligned}$$

where in the last step we used the assumption that $v(s,t)$ has vanishing covariant derivatives at $(0,0)$. □

We close the discussion of curvature on general vector bundles by exhibiting two further ways that it can be framed in terms of exterior derivatives. The first of these follows immediately from Equation (5.5): replacing λ with ∇v for any section $v\in\Gamma(E)$, we have

$$d_\nabla^2 v(X,Y) = R(X,Y)v. \tag{5.6}$$

This elegant (though admittedly somewhat mysterious) expression shows that the covariant exterior derivative does *not* satisfy $d_\nabla^2=0$ in general. In fact:

Exercise 5.24. Show that $d_\nabla^2=0$ on $\Omega^k(M,E)$ for all k if and only if the connection is flat.

Finally, if $\pi: E\to M$ has structure group G, we can also express curvature in terms of the local connection 1-form $A_\alpha\in\Omega^1(\mathcal{U}_\alpha,\mathfrak{g})$ associated to a G-compatible trivialization $\Phi_\alpha: E|_{\mathcal{U}_\alpha}\to\mathcal{U}_\alpha\times\mathbb{F}^m$. Recall that this is defined so that

$$(\nabla_X v)_\alpha = dv_\alpha(X) + A_\alpha(X)v_\alpha(p)$$

for $X\in T_pM$ and $v\in\Gamma(E)$, where $v_\alpha:\mathcal{U}_\alpha\to\mathbb{F}^m$ expresses $v|_{\mathcal{U}_\alpha}$ with respect to the trivialization. We define a *local curvature 2-form* $F_\alpha\in\Omega^2(\mathcal{U}_\alpha,\mathfrak{g})$ by

$$F_\alpha(X,Y) = dA_\alpha(X,Y) + [A_\alpha(X),A_\alpha(Y)].$$

Exercise 5.25. Show that for any $p \in \mathcal{U}_\alpha$, $v \in E_p$ and $X, Y \in T_pM$,

$$(R(X,Y)v)_\alpha = F_\alpha(X,Y)v_\alpha.$$

Exercise 5.26. If $\Phi_\beta : E|_{\mathcal{U}_\beta} \to \mathcal{U}_\beta \times \mathbb{F}^m$ is a second trivialization related to Φ_α by the transition map $g = g_{\beta\alpha} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to G$, show that

$$F_\beta(X,Y) = gF_\alpha(X,Y)g^{-1}.$$

In general the $\mathfrak{g}$-valued curvature 2-form F_α is only defined locally and depends on the choice of trivialization, though Exercise 5.26 brings to light a certain important case in which there is no dependence: if G is *abelian*, then $F_\beta \equiv F_\alpha$ for any two trivializations Φ_α and Φ_β, wherever they overlap. It follows that in this case one can define a global $\mathfrak{g}$-valued curvature 2-form

$$F \in \Omega^2(M, \mathfrak{g})$$

such that $F(X,Y)|_{\mathcal{U}_\alpha} = F_\alpha(X,Y)$ for any choice of trivialization Φ_α on $\mathcal{U}_\alpha$.

Exercise 5.27. Show that if G is abelian, there is a natural G-action on each of the fibers of E, and therefore also a $\mathfrak{g}$-action. In this case, $F(X,Y)$ is simply $\Omega_K(X,Y)$ reexpressed in terms of this action.

Here is an example that will be especially important in the next chapter: if (M,g) is an oriented Riemannian 2-manifold, then $TM \to M$ has structure group $\mathrm{SO}(2)$, which is abelian, and thus there is a globally defined curvature 2-form $F \in \Omega^2(M, \mathfrak{so}(2))$. Observe now that $\mathfrak{so}(2)$ is the 1-dimensional vector space of real antisymmetric 2-by-2 matrices, thus all of them are multiples of

$$J_0 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and there is a real-valued 2-form $\omega \in \Omega^2(M)$ such that $F(X,Y) = \omega(X,Y)J_0$. We can simplify things still further by recalling that the metric g defines a natural volume form $dA \in \Omega^2(M)$ (not necessarily an exact form, despite the notation), such that

$$dA(X,Y) = 1$$

whenever (X,Y) is a positively oriented orthonormal basis. Since $\dim \Lambda^2 T^*M = 1$, there is then a unique smooth function

$$K : M \to \mathbb{R}$$

such that for any $p \in M$ and $X, Y \in T_pM$, $F(X,Y) = -K(p)\, dA(X,Y)J_0$. We see from this that all information about curvature on a 2-manifold can be encoded in this one smooth function: e.g. the curvature tensor can be reconstructed by

$$R(X,Y)Z = -K(p)\, dA(X,Y)J_0Z.$$

We call K the *Gaussian curvature* of (M,g). We will have more to say in the next chapter on the meaning of this function, which plays a starring role in the Gauss-Bonnet theorem.

References

[Spi99] M. Spivak, *A comprehensive introduction to differential geometry*, 3rd ed., Vol. 1, Publish or Perish Inc., Houston, TX, 1999.