Chapter 6

Curvature in Riemannian Geometry

Contents

For most of this chapter, we focus on a Riemannian manifold (M, g) and its tangent bundle $TM \to M$ equipped with the Levi-Civita connection. The curvature tensor defined in the previous chapter is now a type $(1, 3)$ tensor field $R \in \Gamma(T_3^1 M)$, defined most easily as a commutator of covariant derivatives,

$$R(X, Y)Z = (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})Z.$$

In this context R is called the *Riemann tensor*, and it carries all information about the curvature of the Levi-Civita connection: in particular it follows by combining Theorems 5.15 and 5.21 that the Levi-Civita connection is flat if and only if $R \equiv 0$. As we saw in Exercise 5.20, the components $R^i{}_{jk\ell}$ with respect to a coordinate chart can be written in terms of first derivatives of the Christoffel symbols Γ^i_{jk}; it follows that they depend on *second* derivatives of the metric g_{ij}.

6.1 Locally flat manifolds

Our first application of curvature is a characterization of Riemannian manifolds that are locally isometric to Euclidean space $(\mathbb{R}^n, g_0)$; here g_0 is the Riemannian metric defined by applying the standard Euclidean inner product $\langle\ ,\ \rangle_{\mathbb{R}^n}$ to the tangent spaces $T_p\mathbb{R}^n = \mathbb{R}^n$. Observe that the Levi-Civita connection on $(\mathbb{R}^n, g_0)$ is clearly flat: the metric has components $g_{ij} \equiv \delta_{ij}$ in the natural global coordinates, so their derivatives vanish and so therefore does the Riemann tensor. This motivates the following terminology:

Definition 6.1. We call a Riemannian manifold (M, g) *locally flat* if for every $p \in M$, there is a neighborhood $p \in \mathcal{U} \subset M$ and an embedding $\varphi : \mathcal{U} \to \mathbb{R}^n$ such that $\varphi^* g_0 = g$.

The embedding φ in this definition is called a *local isometry* from (M, g) to $(\mathbb{R}^n, g_0)$ near p. It is equivalent to say that every point $p \in M$ is contained in a coordinate chart in which the metric has constant components δ_{ij}. From this last remark, a necessary condition is clearly that the Riemann tensor of (M, g) must vanish identically. The remarkable fact is that this condition is also sufficient—and given the machinery that we've built up by this point, it's surprisingly easy to prove.

Theorem 6.2. *A Riemannian manifold (M, g) is locally flat if and only if its Riemann tensor vanishes identically.*

Proof. Suppose $R \equiv 0$, so the Levi-Civita connection ∇ on $TM \to M$ is flat. Then for every $p \in M$ and $X \in T_pM$, we can extend X to a covariantly constant vector field on a neighborhood of p. In particular, choose any orthonormal basis $(X_1, \ldots, X_n)$ of T_pM and extend these accordingly to covariantly constant vector fields. This defines a framing of TM on some neighborhood $p \in \mathcal{U} \subset M$, and since parallel transport preserves the metric, the basis remains orthonormal at every point in $\mathcal{U}$, so the components of g with respect this frame are $g_{ij} = \delta_{ij}$.

It remains only to show that the vector fields $X_1, \ldots, X_n$ can be considered coordinate vector fields $\partial_1, \ldots, \partial_n$ with respect to some coordinates $(x^1, \ldots, x^n)$ defined near p. This is possible because it turns out that $[X_i, X_j] \equiv 0$ for all i and j, which follows from the symmetry of the connection:

$$[X_i, X_j] = \nabla_{X_i} X_j - \nabla_{X_j} X_i = 0,$$

using the fact that each X_i is covariantly constant. □

This is the fanciest "integrability theorem" we've yet encountered, and in proving it we essentially used every other integrability result that we've seen previously (notably Corollary 5.3 and Theorem 5.15). In the end, it all comes down to one basic fact about vector fields: the flows of X and Y commute if and only if $[X, Y] \equiv 0$.

According to Theorem 6.2, curvature provides an obstruction to finding local coordinates in which the metric looks like the Euclidean metric. One can ask: even if the curvature is nonzero at a point p, what is the "simplest" form that the metric might take in coordinates near p? There are always, for instance, coordinates in which $g_{ij} = \delta_{ij}$ at p itself, though not necessarily in a neighborhood. Actually one can do slightly better. Choose a basis $X_1, \ldots, X_n$ of T_pM and use the exponential map to define an embedding

$$f(t^1, \ldots, t^n) = \exp_p(t^1 X_1 + \ldots + t^n X_n) \tag{6.1}$$

for $(t^1, \ldots, t^n)$ in some neighborhood $0 \in \mathcal{U} \subset \mathbb{R}^n$. The inverse of this map defines a coordinate chart $f^{-1} = x = (x^1, \ldots, x^n) : f(\mathcal{U}) \to \mathbb{R}^n$.

Proposition 6.3. *If ∇ is any symmetric connection (not necessarily compatible with a metric), then the inverse of (6.1) defines coordinates that identify p with $0 \in \mathbb{R}^n$ such that all Christoffel symbols Γ^i_{jk} vanish at 0.*

Moreover if ∇ is the Levi-Civita connection with respect to a metric g and $X_1, \ldots, X_n \in T_pM$ is an orthonormal *basis, then these coordinates have the property that*

$$g_{ij}(0) = \delta_{ij} \quad \text{and} \quad \partial_k g_{ij}(0) = 0.$$

Proof. If $\partial_1, \ldots, \partial_n$ are the coordinate vector fields, the Christoffel symbols can be computed by $\Gamma^i_{jk} = (\nabla_j \partial_k)^i$, so it suffices to show that $\nabla \partial_k = 0$ at p for all k. To see this, we observe that by construction, all *radial* paths in these coordinates are geodesics, i.e. paths with coordinate expressions of the form $(x^1(t), \ldots, x^n(t)) = (Y^1 t, \ldots, Y^n t)$ for constants $Y^i \in \mathbb{R}$. Writing down the geodesic equation at $t = 0$, it follows that any vector field $Y = Y^i \partial_i$ with constant components $Y^i \in \mathbb{R}$ satisfies $\nabla_{Y(p)} Y = 0$. In particular, we then have

$$0 = \nabla_{\partial_i + \partial_j}(\partial_i + \partial_j)\big|_p = \nabla_i \partial_i|_p + \nabla_j \partial_j|_p + \nabla_i \partial_j|_p + \nabla_j \partial_i|_p = 2\nabla_i \partial_j|_p,$$

where in the last step we've used the symmetry of the connection. This proves that $\Gamma^i_{jk}(0) = 0$.

The second statement follows immediately since $\partial_i g_{jk} = g_{\ell k} \Gamma^\ell_{ij} + g_{\ell j} \Gamma^\ell_{ik}$. $\square$

The coordinates constructed above are called *Riemannian normal coordinates*: they place the metric in a standard form up to first order at any point. One cannot do any better than this in general since the *second* derivatives of g_{ij} depend on the curvature, which is an invariant.

6.2 Hypersurfaces and Gaussian curvature

In this section we consider a Riemannian manifold (Σ, j^*g) which is embedded via a map $j : \Sigma \hookrightarrow M$ into a Riemannian manifold (M, g) of one dimension higher, i.e. $\Sigma \subset M$ is a hypersurface. We will be most interested in the case where $(M, g) = (\mathbb{R}^3, g_0)$ and $\dim \Sigma = 2$, though some of what we have to say will make sense more generally than this. Denote by $\langle\ ,\ \rangle$ the bundle metric on TM (and therefore also on $T\Sigma$) defined by g. Its restriction to Σ is sometimes referred to as the *first fundamental form* on Σ. (A *second* fundamental form will be defined shortly.)

We begin by defining the Gaussian curvature of a surface Σ embedded in Euclidean $\mathbb{R}^3$. The definition is quite simple and intuitive, but it will not at all be obvious at first that it is an invariant of the metric. Indeed, if $\Sigma \subset \mathbb{R}^3$ is oriented, there is a natural choice of unit normal vector $\nu(p) \in \mathbb{R}^3$ for each $p \in \Sigma$, which defines a map

$$\nu : \Sigma \to S^2.$$

This is called the *Gauss map* on Σ. Observe that since $T_{\nu(p)}S^2$ is the orthogonal complement of $\nu(p)$ in $\mathbb{R}^3$, there is a natural identification of $T_p\Sigma$ with $T_{\nu(p)}S^2$ as subspaces of $\mathbb{R}^3$. Thus we can regard

$$d\nu(p) : T_p\Sigma \to T_{\nu(p)}S^2$$

as a linear endomorphism on a particular 2-dimensional subspace of $\mathbb{R}^3$. As such, it has a well defined *determinant*

$$K_G(p) := \det d\nu(p) \in \mathbb{R}, \tag{6.2}$$

defined by choosing any basis of $T_p\Sigma$ and expressing $d\nu(p)$ as a 2-by-2 matrix; the determinant computed in this way does not depend on the basis. In fact, it also doesn't change if we switch the sign of $\nu(p)$, thus we can drop the assumption that Σ is oriented and merely define ν locally in order to facillitate the definition of K_G. Thinking more geometrically, one can choose an arbitrary area form on $T_p\Sigma$ and define $K_G(p)$ as the ratio of the signed areas of $d\nu(p)(P)$ and P for any parallelopiped $P \subset T_p\Sigma$. The sign of $K_G(p)$ therefore depends on whether $d\nu(p)$ preserves or reverses orientation. The function $K_G : \Sigma \to \mathbb{R}$ is called the *Gaussian curvature*, and despite appearances to the contrary, we will find that it does not depend on the embedding $\Sigma \hookrightarrow \mathbb{R}^3$ but rather on the Riemannian metric induced on Σ; indeed, we will show that it is the same function that was defined at the end of §5.3 in terms of the curvature 2-form.

Example 6.4. For the unit sphere $S^2 \subset \mathbb{R}^3$, the Gauss map is simply the identity so $K_G \equiv 1$.

Example 6.5. Consider the cylinder $Z = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$. The Gauss map on Z is independent of z, thus $d\nu(p)$ only has rank 1 at every $p \in Z$, implying $K(p) = 0$. This reflects the fact Z is locally flat: unlike a sphere, a small piece of a cylinder can easily be unfolded into a piece of a flat plane without changing lengths or angles on the surface. The same is true of the cone

$$C = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = z^2,\ z > 0\}.$$

Example 6.6. The hyperboloid $H = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 - z^2 = 1\}$ has everywhere negative curvature. This is true of any surface that exhibits a "saddle" shape, for which the Gauss map is orientation reversing.

To understand the relation of K_G to our previous definitions of curvature, let us generalize and assume (M, g) is an arbitrary Riemannian manifold with an embedded hypersurface $j : \Sigma \hookrightarrow M$ and induced metric j^*g. Denote by ∇ the Levi-Civita connection on (M, g), while $\widetilde{\nabla}$ denotes the Levi-Civita connection on (Σ, j^*g). Assume that $\Sigma \subset M$ admits a normal vector field $\nu \in \Gamma(TM|_\Sigma)$, such that $|\nu(p)| = 1$ and $\nu(p)$ is orthogonal to $T_p\Sigma$ for all $p \in \Sigma$. In general this is uniquely determined up to a sign and may be well defined only locally, but this will pose no problem in the end—since our aim is to define a notion of curvature which depends only on local properties of the embedding $\Sigma \hookrightarrow M$, we may as well shrink Σ if necessary and assume that ν is globally defined.

Lemma 6.7. *For any $p \in \Sigma$ and $X \in T_p\Sigma$, $\nabla_X\nu \in T_p\Sigma$.*

Proof. Differentiating the equation $\langle \nu, \nu \rangle \equiv 1$ yields

$$0 = L_X\langle \nu, \nu \rangle = 2\langle \nabla_X\nu, \nu \rangle,$$

thus $\nabla_X\nu$ belongs to the orthogonal complement of $\nu(p)$, which is precisely $T_p\Sigma$. □

In light of this, the transformation $X \mapsto \nabla_X\nu$ defines a linear map

$$\nabla\nu_p : T_p\Sigma \to T_p\Sigma,$$

called the *Weingarten map.* Observe that for the case $M = \mathbb{R}^3$ with the trivial metric, $\nabla\nu_p$ is precisely the derivative of the Gauss map. It is often expressed in the form of a bilinear map on $T_p\Sigma$ as follows:

Definition 6.8. Given the embedding $\Sigma \hookrightarrow M$ and normal vector field ν, the *second fundamental form* is the tensor field $\mathrm{II} \in \Gamma(T_2^0\Sigma)$ defined by

$$\mathrm{II}(X, Y) = -\langle X, \nabla\nu_p(Y) \rangle.$$

Differentiating the expression $\langle Y, \nu \rangle \equiv 0$ for $Y \in \mathrm{Vec}(\Sigma)$ gives $\langle \nabla_Y X, \nu \rangle + \langle X, \nabla_Y \nu \rangle = 0$, thus the second fundamental form actually computes the normal part of the covariant derivative,

$$\mathrm{II}(X, Y) = \langle \nabla_Y X, \nu(p) \rangle. \tag{6.3}$$

It's important to note that the use of the word "form" does not mean $\mathrm{II}(X, Y)$ is antisymmetric, but merely that it is a bilinear form. In fact:

Proposition 6.9. *The second fundamental form is symmetric:* $\mathrm{II}(X, Y) = \mathrm{II}(Y, X)$.

Proof. This follows from (6.3) and the symmetry of the Levi-Civita connection:

$$\mathrm{II}(Y, X) - \mathrm{II}(X, Y) = \langle \nabla_X Y - \nabla_Y X, \nu(p) \rangle = \langle [X, Y](p), \nu(p) \rangle = 0.$$

$\square$

Corollary 6.10. *The Weingarten map $\nabla \nu_p : T_p\Sigma \to T_p\Sigma$ is a self-adjoint operator with respect to the inner product $\langle\ ,\ \rangle$; in particular, the corresponding matrix with respect to any choice of orthonormal basis on $T_p\Sigma$ is symmetric.*

Recalling some notation from §4.3.4, denote by $\pi_\Sigma : TM|_\Sigma \to T\Sigma$ the fiberwise linear projection along the orthogonal complement of $T\Sigma$, so that the connections ∇ on (M, g) and $\widetilde{\nabla}$ on (Σ, j^*g) are related by

$$\widetilde{\nabla} = \pi_\Sigma \circ \nabla.$$

Denote by R and $\widetilde{R}$ the Riemann curvature tensors on (M, g) and (Σ, j^*g) respectively.

Proposition 6.11. *For $p \in \Sigma$ and $X, Y, Z \in T_p\Sigma$,*

$$\widetilde{R}(X, Y)Z = \pi_\Sigma \left(R(X, Y)Z \right) + \mathrm{II}(X, Z) \nabla \nu_p(Y) - \mathrm{II}(Y, Z) \nabla \nu_p(X).$$

Proof. The connections ∇ on $TM|_\Sigma$ and $\widetilde{\nabla}$ on $T\Sigma$ induce a connection on the bundle $\mathrm{Hom}(TM|_\Sigma, T\Sigma) \to \Sigma$, which we will denote also by ∇: then for $Y \in \mathrm{Vec}(M)$ and $X \in T\Sigma$,

$$\widetilde{\nabla}_X(\pi_\Sigma(Y)) = (\nabla_X \pi_\Sigma)(Y) + \pi_\Sigma(\nabla_X Y).$$

This yields the formula

$$\begin{aligned}(\nabla_X \pi_\Sigma)(Y) &= \widetilde{\nabla}_X(\pi_\Sigma(Y)) - \pi_\Sigma(\nabla_X Y) = \pi_\Sigma \left(\nabla_X (Y - \langle Y, \nu \rangle \nu) \right) - \pi_\Sigma(\nabla_X Y) \\ &= -\pi_\Sigma \left(\nabla_X (\langle Y, \nu \rangle \nu) \right) = -\pi_\Sigma \left((L_X \langle Y, \nu \rangle) \nu + \langle Y, \nu \rangle \nabla_X \nu \right) \\ &= -\langle Y, \nu \rangle \nabla_X \nu.\end{aligned}$$

Now, choosing $X, Y, Z \in \mathrm{Vec}(\Sigma)$, we find

$$\begin{aligned}
\widetilde{R}(X,Y)Z &= \widetilde{\nabla}_X \widetilde{\nabla}_Y Z - \widetilde{\nabla}_Y \widetilde{\nabla}_X Z - \widetilde{\nabla}_{[X,Y]} Z \\
&= \widetilde{\nabla}_X \left(\pi_\Sigma(\nabla_Y Z)\right) - \widetilde{\nabla}_Y \left(\pi_\Sigma(\nabla_X Z)\right) - \pi_\Sigma\left(\nabla_{[X,Y]} Z\right) \\
&= (\nabla_X \pi_\Sigma)(\nabla_Y Z) + \pi_\Sigma(\nabla_X \nabla_Y Z) - (\nabla_Y \pi_\Sigma)(\nabla_X Z) \\
&\qquad - \pi_\Sigma(\nabla_Y \nabla_X Z) - \pi_\Sigma\left(\nabla_{[X,Y]} Z\right) \\
&= \pi_\Sigma(R(X,Y)Z) - \langle \nabla_Y Z, \nu\rangle \nabla_X \nu + \langle \nabla_X Z, \nu\rangle \nabla_Y \nu \\
&= \pi_\Sigma(R(X,Y)Z) - \mathrm{II}(Y,Z)\nabla_X \nu + \mathrm{II}(X,Z)\nabla_Y \nu.
\end{aligned}$$

□

We now specialize to the case $(M, g) = (\mathbb{R}^3, g_0)$ and $\dim \Sigma = 2$. Then $R(X,Y)Z$ vanishes and ∇ is the trivial connection, so by Prop. 6.11, the curvature $\widetilde{R}$ of Σ at $p \in \Sigma$ depends only on the Weingarten map $\nabla\nu_p : T_p\Sigma \to T_p\Sigma$. This in turn is simply the derivative of the Gauss map $\nu : \Sigma \to S^2$ at p, thus the Gaussian curvature is

$$K_G(p) = \det(\nabla \nu_p).$$

We can express this in a slightly more revealing form using the fact that $\nabla\nu_p$ is a self-adjoint operator. Applying the spectral theorem for self-adjoint operators, $T_p\Sigma$ admits an orthonormal basis (X_1, X_2) of eigenvectors $\nabla\nu_p(X_j) = \lambda_j X_j$, and the determinant is simply the product of the eigenvalues, thus

$$K_G(p) = \lambda_1 \lambda_2.$$

The eigenvalues λ_1 and λ_2 of $\nabla\nu_p$ are called the *principal curvatures* of Σ at p. Observe that if the sign of ν is switched, then the second fundamental form and the Weingarten map both change sign, but the Gaussian curvature remains the same. This is why K_G can be defined without assuming Σ is oriented. We will continue assuming Σ has an orientation since our considerations are fundamentally local in any case.

The metric and orientation define on Σ a natural area form $dA \in \Omega^2(\Sigma)$ such that

$$dA(X_1, X_2) = 1$$

for any positively oriented orthonormal basis (X_1, X_2) of $T_p\Sigma$.[1] (Note that dA is not necessarily an exact form: this notation for an area form is chosen out of tradition rather than logic.) Now assume that the orthonormal basis (X_1, X_2) above consists of eigenvectors of $\nabla\nu_p$. Define

$$J_0 : T_p\Sigma \to T_p\Sigma$$

to be the unique linear map such that $J_0(X_1) = X_2$ and $J_0(X_2) = -X_1$.

[1] In fancier terms, the bundle $T\Sigma \to \Sigma$ has structure group $\mathrm{SO}(2)$ due to the orientation and bundle metric, and this is contained in $\mathrm{SL}(2)$, so the extra structure of a volume form comes for free.

Exercise 6.12. Show that $J_0 \in \mathrm{End}(T_p\Sigma)$ is a complex structure on $T_p\Sigma$, i.e. $J_0^2 = -\mathrm{Id}$, and with respect to any positively oriented orthonormal basis on $T_p\Sigma$, the matrix representing J_0 is

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

This exercise also shows that J_0 is antisymmetric with respect to the inner product $\langle\ ,\ \rangle$, i.e. $\langle X, J_0 Y\rangle = -\langle J_0 X, Y\rangle$ for all $X, Y \in T_p\Sigma$.

Using Prop. 6.11 with $X = X_1$ and $Y = X_2$, we find

$$\begin{aligned}
\widetilde{R}(X_1, X_2)Z &= \mathrm{II}(X_1, Z)\nabla\nu_p(X_2) - \mathrm{II}(X_2, Z)\nabla\nu_p(X_1) \\
&= \lambda_1\lambda_2\left(\langle X_1, Z\rangle X_2 - \langle X_2, Z\rangle X_1\right) \\
&= K_G(p)\left(\langle -J_0X_1, J_0Z\rangle X_2 - \langle -J_0X_2, J_0Z\rangle X_1\right) \\
&= -K_G(p)\left(\langle X_2, J_0Z\rangle X_2 + \langle X_1, J_0Z\rangle X_1\right) \\
&= -K_G(p)J_0Z,
\end{aligned}$$

or by the definition of the area form

$$\widetilde{R}(X_1, X_2)Z = -K_G(p)\, dA(X_1, X_2)J_0Z. \tag{6.4}$$

Now we can replace X_1 and X_2 by any linear combinations of these two vectors and see from the bilinearity of both sides that (6.4) still holds. The result is precisely the formula advertised at the end of Chapter 5 for the Gaussian curvature:

Theorem 6.13. *Suppose Σ is an oriented surface embedded in Euclidean $\mathbb{R}^3$, dA is the natural area form on Σ, $K_G : \Sigma \to \mathbb{R}$ is its Gaussian curvature, $R(X,Y)Z$ is its Riemann curvature tensor and $J : T\Sigma \to T\Sigma$ is the unique fiberwise linear map such that for any vector $X \in T_p\Sigma$ with $|X| = 1$, (X, JX) is a positively oriented orthonormal basis. Then*

$$R(X,Y)Z = -K_G\, dA(X,Y)JZ.$$

For arbitrary surfaces Σ, not embedded in $\mathbb{R}^3$, Theorem 6.13 can be taken as a definition of the Gaussian curvature $K_G : \Sigma \to \mathbb{R}$, and it shows in fact that all information about curvature on a surface can be expressed in terms of this one real-valued function. Note that once again the result doesn't actually depend on an orientation: locally, if the orientation of Σ is flipped, this changes the sign of both J and dA, leaving the final result unchanged.

For surfaces in $\mathbb{R}^3$, Theorem 6.13 implies the following famous result of Gauss, which has come to be known by the Latin term for "remarkable theorem":

Theorema Egregium. *For a surface Σ embedded in Euclidean $\mathbb{R}^3$, the Gaussian curvature $K_G : \Sigma \to \mathbb{R}$ defined in (6.2) is an invariant of the induced Riemannian metric on Σ. To be precise, if $\Sigma_1, \Sigma_2 \subset \mathbb{R}^3$ are two surfaces embedded in $\mathbb{R}^3$ with induced metrics g_1, g_2 and Gaussian curvatures K_G^1, K_G^2 respectively, and there is a diffeomorphism $\varphi : \Sigma_1 \to \Sigma_2$ such that $\varphi^* g_2 \equiv g_1$, then*

$$K_G^1 \equiv K_G^2 \circ \varphi.$$

Example 6.14. As a simple application, this shows that there are no isometries between any open subsets of the sphere $S^2 \subset \mathbb{R}^3$ and the hyperboloid of Example 6.6.

To derive an explicit formula for K_G, we first observe the following relation between the metric $\langle\ ,\ \rangle$, the natural area form $dA \in \Omega^2(\Sigma)$ and the fiberwise linear map $J \in \Gamma(\mathrm{End}(T\Sigma))$:

$$dA(X, Y) = \langle JX, Y\rangle.$$

This is true by definition for the special case where X is a unit vector and $Y = JX$, implying that $dA(X, JX)$ and $\langle JX, X\rangle$ define identical quadratic forms; then one can easily use bilinearity and symmetry to show that the two bilinear forms are the same. Theorem 6.13 then implies

$$\begin{aligned}\langle R(X, Y)Y, X\rangle &= -\langle K_G\, dA(X, Y)JY, X\rangle \\ &= -K_G\, dA(X, Y)\langle JY, X\rangle = K_G \cdot |dA(X, Y)|^2,\end{aligned}$$

thus

$$K_G(p) = \frac{\langle R(X, Y)Y, X\rangle}{|dA(X, Y)|^2} \tag{6.5}$$

for any pair of linearly independent vectors $X, Y \in T_p\Sigma$. We can rewrite this as follows in terms of an oriented coordinate chart (x^1, x^2) defined near p. If the components of the metric are denoted by g_{ij} and we define the symmetric matrix-valued function

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix},$$

we recall from Chapter 2, Equation (2.4) that dA takes the form

$$dA = \sqrt{\det g}\; dx^1 \wedge dx^2.$$

Then applying (6.5) to the coordinate vectors $X = \partial_1$ and $Y = \partial_2$, we have

$$\langle R(\partial_1, \partial_2)\partial_2, \partial_1\rangle = g_{ij} R^i{}_{12k} \delta^k_2 \delta^j_1 = g_{i1} R^i{}_{122} = R_{1122},$$

where we've used the index raising/lowering conventions described in Appendix A to lower the first index of $R^i{}_{jk\ell}$. This yields the formula

$$K_G = \frac{R_{1122}}{\det g}.$$

Exercise 6.15. Show that the Poincaré half-plane $(\mathbb{H}, h)$ from Example 4.22 has constant Gaussian curvature $K_G \equiv -1$. (Presumably you wrote down the Christoffel symbols when you did Exercise 4.23; you can use these with the formula of Exercise 5.20 to compute the Riemann tensor.)

6.3 The Gauss-Bonnet formula

6.3.1 Polygons and triangulation

We begin this discussion with the following reminiscence from our youth, when life was simpler and, above all, geometry was easier.

Proposition 6.16. *For any triangle in $\mathbb{R}^2$, the angles at the vertices add up to π.*

A simple example on S^2 shows that on curved surfaces this is no longer true: it's easy for instance to find a "triangle" on S^2 whose edges are geodesics but with angles that add up to $3\pi/2$ (see Figure 5.1 in Chapter 5). We will show in fact that the sum of the angles of any triangle with geodesic edges on a Riemannian 2-manifold differs from π by an amount depending on the amount of curvature enclosed. This result can then be generalized to a formula for the integral of the Gaussian curvature over any compact surface, and the answer turns out to depend on a topological invariant which has nothing to do with Riemannian metrics.

We begin by defining some terminology that is needed to formulate a generalization of Prop. 6.16. A *piecewise smooth* curve in a smooth manifold M is a continuous map $\gamma : [a, b] \to M$ for which there are finitely many points $a = t_0 < t_1 < \ldots < t_{N-1} < t_N = b$ such that the restrictions

$$\gamma|_{[t_{j-1},t_j]} : [t_{j-1}, t_j] \to M$$

are smooth immersions for each $j = 1, \ldots, N$. The curve is called a piecewise smooth *simple closed* curve if $\gamma(b) = \gamma(a)$ and there is no other self-intersection $\gamma(t) = \gamma(t')$ for $t \neq t'$.

Definition 6.17. A *smooth polygon* in $\mathbb{R}^2$ is the closure $P \subset \mathbb{R}^2$ of a region bounded by the image of a piecewise smooth simple closed curve $\gamma : [0, 1] \to \mathbb{R}^2$. For each subinterval $[t_{j-1}, t_j]$ on which γ is smooth, the image $\gamma([t_{j-1}, t_j])$ is called an *edge* of P. We denote their union by ∂P.

Observe that if P has a smooth boundary then $\partial P \cong S^1$ inherits from the orientation of $\mathbb{R}^2$ a natural orientation as the boundary of P. This notion of orientation generalizes naturally to the piecewise smooth case so that each edge of ∂P inherits a natural orientation.

Now suppose (Σ, g) is a 2-dimensional oriented Riemannian manifold with Levi-Civita connection ∇ and Gaussian curvature $K_G : \Sigma \to \mathbb{R}$. Denote by $dA \in \Omega^2(\Sigma)$ the natural area form defined by the metric and orientation.

Definition 6.18. A *smooth polygon* $P \subset \Sigma$ is the image of a smooth polygon $P_0 \subset \mathbb{R}^2$ under a smooth map that embeds an open neighborhood of P_0 into Σ. The *edges* of P are defined as the images of the edges of P_0 under this map.

The orientation of Σ restricts to any smooth polygon P and induces a natural orientation on its edges, whose union we again denote by ∂P. The metric also restricts to each edge $\ell \subset \partial P$ and defines a natural "volume form" $ds \in \Omega^1(\ell)$; again the notation is chosen for traditional reasons, and should not imply that ds is the differential of a specific function. Although ∂P is not generally a smooth manifold, it's easy to see that Stokes' theorem still holds:

$$\int_P d\lambda = \int_{\partial P} \lambda$$

for any $\lambda \in \Omega^1(\Sigma)$, where the integral over ∂P is defined by summing the integrals over the edges. One can prove this by an approximation argument, perturbing ∂P to a smooth loop that bounds a region P_ϵ on which $\int_{P_\epsilon} d\lambda$ is almost the same.

For any oriented 1-dimensional submanifold $\ell \subset \Sigma$ one can define the *geodesic curvature*

$$\kappa_\ell : \ell \to \mathbb{R}$$

as follows. Choose any parametrization $\gamma(t) \in \ell$ such that $|\dot{\gamma}(t)| \equiv 1$ and $\dot{\gamma}(t)$ points in the positive direction along ℓ. Differentiating $\langle \dot{\gamma}, \dot{\gamma} \rangle \equiv 1$ gives $\langle \nabla_t \dot{\gamma}, \dot{\gamma} \rangle$, thus $\nabla_t \dot{\gamma}(t)$ is always orthogonal to $T_{\gamma(t)}\ell$. The orientation of Σ defines a unique unit normal vector $\nu(t) \in T_{\gamma(t)}\Sigma$ along ℓ such that $(\dot{\gamma}(t), \nu(t))$ gives a positively oriented orthonormal basis of $T_{\gamma(t)}\Sigma$ for all t. Then $\kappa_\ell(\gamma(t))$ is defined as the unique real number such that

$$\nabla_t \dot{\gamma}(t) = \kappa_\ell(\gamma(t))\nu(t).$$

The next observation follows immediately from the definition and the geodesic equation.

Proposition 6.19. *A* 1*-dimensional submanifold* $\ell \subset \Sigma$ *can be parametrized by a geodesic if and only if its geodesic curvature* $\kappa_\ell : \ell \to \mathbb{R}$ *vanishes identically.*

We are now ready to state the first version of the Gauss-Bonnet theorem.

Theorem 6.20. *Assume $P \subset \Sigma$ is a smooth polygon with $N \geq 1$ smooth edges $\ell_1 \cup \ldots \cup \ell_N = \partial P$, and the angles at the vertices are $\alpha_1, \ldots, \alpha_N \in (0, 2\pi)$. Then*

$$\sum_{j=1}^{N} \alpha_j = (N-2)\pi + \int_P K_G \, dA + \sum_{j=1}^{N} \int_{\ell_j} \kappa_{\ell_j} \, ds.$$

Corollary 6.21. *If $P \subset \Sigma$ is a smooth polygon with N edges which are all geodesics, then the angles $\alpha_1, \ldots, \alpha_N$ at the vertices satisfy*

$$\sum_{j=1}^{N} \alpha_j = (N-2)\pi + \int_P K_G \, dA.$$

We postpone the proof of Theorem 6.20 to the end of this section, and first discuss a related (and arguably more important) result that arises from it. The idea is to compute the integral of K_G over the entirety of a compact surface by decomposing it into smooth polygons.

Definition 6.22. Let Σ be a 2-dimensional manifold, possibly with boundary. A *polygonal triangulation* of Σ is a collection of isolated points $\Sigma_0 \subset \Sigma$ (called *vertices*) and compact 1-dimensional submanifolds with boundary (called *edges*, their union denoted by $\Sigma_1 \subset \Sigma$) such that

1. Both boundary points of each edge are vertices.
2. Every vertex is a boundary point for at least one edge.
3. Each edge is disjoint from the vertices except at its boundary.
4. All edges are disjoint from one another except possibly at their boundaries.
5. Each connected component of $\Sigma \setminus \Sigma_1$ (called a *face*) is the interior of a smooth polygon $P \subset \Sigma$ whose edges are edges in Σ_1.

It is a fact that every 2-manifold Σ admits a polygonal triangulation,[2] and that if Σ is compact, the triangulation can be assumed to be *finite*, meaning it has finitely many faces, edges and vertices. This can be proved by choosing a large number of vertices close enough together so that edges connecting nearby vertices can be constructed as geodesics; see [Spi99b, Problem 4.17].

Definition 6.23. Given a finite polygonal triangulation of Σ with v vertices, e edges and f faces, the *Euler characteristic* of Σ is the integer

$$\chi(\Sigma) = v - e + f.$$

[2]The notion of a triangulation can be generalized naturally to higher dimensions, and every smooth n-manifold admits a triangulation, though this is not easy to prove, and is also not true for *topological* manifolds in general.

The Euler characteristic turns out to be a topological invariant of Σ, though our definition makes this far from obvious—*a priori* it appears to depend rather crucially on a choice of triangulation. It will follow from Theorem 6.25 below that this is not the case, that in fact $\chi(\Sigma)$ depends at most on the differentiable structure of Σ. Proving that it only depends on the topology of Σ requires methods from algebraic topology: the standard approach is to define $\chi(\Sigma)$ in terms of singular homology and use either cellular or simplicial homology to prove that the quantity above matches this definition for any triangulation. Details may be found in [Hat02] or [Bre93].

Exercise 6.24. Taking it on faith for the moment that the Euler characteristic doesn't depend on a choice of triangulation, show that $\chi(S^2) = 2$, $\chi(D) = 1$ and $\chi(T^2) = 0$. (Here $D \subset \mathbb{R}^2$ denotes the closed unit disk, and T^2 is the 2-torus $S^1 \times S^1$.)

We shall now compute the integral of K_G over a compact surface using a finite polygonal triangulation with v vertices, e edges and f faces. Assume $e = e_0 + e_\partial$ where e_∂ is the number of edges contained in $\partial\Sigma$, and similarly $v = v_0 + v_\partial$. Observe that $e_\partial = v_\partial$. By Theorem 6.20, $\int_\Sigma K_G\, dA$ contains a term of the form

$$-\sum_j \int_{\ell_j} \kappa_{\ell_j}\, ds + \sum_j \alpha_j - (N-2)\pi$$

for each face, assuming the face in question has N edges. Adding these up for all faces, we make the following observations:

1. Every edge $\ell \subset \Sigma \setminus \partial\Sigma$ is an edge for two distinct faces and thus appears twice with opposite orientations, so the geodesic curvature terms for these edges cancel in the sum.

2. The geodesic curvature terms for all edges $\ell \subset \partial\Sigma$ add up to
$$-\int_{\partial\Sigma} \kappa_{\partial\Sigma}\, ds.$$

3. The sum of all angles α_j at an interior vertex (for every face adjacent to that vertex) is 2π, and for boundary vertices the sum is π. Thus altogether these terms contribute $2\pi v_0 + \pi v_\partial = 2\pi v - \pi v_\partial$.

4. Every interior edge is counted twice and boundary edges are counted once, so the $-(N-2)\pi$ terms add up to $-\pi(2e_0 + e_\partial - 2f) = 2\pi(f - e) + \pi e_\partial$.

Summing all these contributions, we have

$$-\int_{\partial\Sigma} \kappa_{\partial\Sigma}\, ds + 2\pi v + 2\pi(f - e) - \pi v_\partial + \pi e_\partial = -\int_{\partial\Sigma} \kappa_{\partial\Sigma}\, ds + 2\pi\chi(\Sigma).$$

This proves:

Theorem 6.25 (Gauss-Bonnet)**.** *For any compact 2-dimensional Riemannian manifold with boundary (Σ, g),*

$$\int_\Sigma K_G\, dA + \int_{\partial\Sigma} \kappa_{\partial\Sigma}\, ds = 2\pi\chi(\Sigma).$$

Several wonderful things follow immediately from this formula. Observe that the left hand side has nothing to do with the triangulation, while the right hand side makes no reference to the metric or curvature.

Corollary 6.26. *The Euler characteristic $\chi(\Sigma)$ does not depend on the choice of triangulation, and for any two diffeomorphic surfaces Σ_1 and Σ_2, $\chi(\Sigma_1) = \chi(\Sigma_2)$.*

Corollary 6.27. *For a fixed compact surface Σ, the sum $\int_\Sigma K_G\, dA + \int_{\partial\Sigma} \kappa_{\partial\Sigma}\, ds$ is an integer multiple of 2π, and is the same for any choice of Riemannian metric.*

In particular, the latter statement imposes serious topological restrictions on the kinds of metrics that are allowed on any given surface: e.g. it is impossible to find a metric with everywhere positive Gaussian curvature on a surface with negative Euler characteristic. To get a handle on this, it helps to have some concrete examples in mind; these are provided by the following exercises.

Exercise 6.28. Suppose Σ is a compact oriented surface with boundary and $\ell_1, \ell_2 \subset \partial\Sigma$ are two distinct connected components of $\partial\Sigma$. We can *glue* these two components to produce a new surface Σ' as follows: since ℓ_1 and ℓ_2 are both circles, there is an orientation reversing diffeomorphism $\varphi : \ell_1 \to \ell_2$, which we use to define

$$\Sigma' = \Sigma / \sim$$

where the equivalence identifies $p \in \ell_1$ with $\varphi(p) \in \ell_2$, thus identifying ℓ_1 and ℓ_2 to a single circle, now in the interior of Σ'. Show that $\chi(\Sigma') = \chi(\Sigma)$.

Note that Σ need not be a connected surface to start with: this trick can be used to glue together two separate surfaces along components of their boundaries!

Exercise 6.29. Let Σ be the closed unit disk in $\mathbb{R}^2$ with two smaller disjoint open disks removed: the resulting surface is called a *pair of pants*. Show that $\chi(\Sigma) = -1$.

Similarly, a *handle* is a surface Σ diffeomorphic to the torus T^2 with one open disk removed. Show that $\chi(\Sigma) = -1$.

Exercise 6.30. Suppose Σ is a compact surface with boundary. The operation of *gluing a handle* to Σ is defined as follows: choose a smoothly embedded closed disk in the interior of Σ, remove its interior, and glue the resulting surface along its new boundary component to a handle (see Exercise 6.29). Show that this operation decreases the Euler characteristic of Σ by 2.

Exercise 6.31. A *closed oriented surface of genus g* is any compact surface Σ without boundary that is diffeomorphic to a surface obtained from S^2 by gluing g handles. Special cases include the sphere itself ($g = 0$) and the torus ($g = 1$). Show that $\chi(\Sigma) = 2 - 2g$.

For Σ a compact surface with boundary, we say it has *genus g* if it is diffeomorphic to a closed surface of genus g with finitely many small open disks cut out. Show that if such a surface has m boundary components, then $\chi(\Sigma) = 2 - 2g - m$.

In case you didn't already believe this, we now have a simple proof of the fact that two closed oriented surfaces with differing genus are not diffeomorphic: if they were, then their Euler characteristics would have to match. The converse is also true, but harder to prove; see [Hir94].

The Gauss-Bonnet theorem enables us to make some sweeping statements regarding what kinds of metrics may exist on various compact surfaces. In general, we say that a surface Σ with a Riemannian metric has *positive* (or *zero* or *negative*) curvature if its Gaussian curvature is positive (or zero or negative) at every point.

Theorem 6.32. *Let Σ be a closed oriented surface of genus g. Then Σ admits a Riemannian metric with positive curvature if and only if $\Sigma \cong S^2$, zero curvature if and only if $\Sigma \cong T^2$, and negative curvature if and only if $g \geq 2$.*

Proof. We shall not provide the entire proof, but by this point the result should at any rate seem believable, and in one direction the claim is clear: the stated conditions on the genus are necessary due to the Gauss-Bonnet theorem and the formula $\chi(\Sigma) = 2 - 2g$. It's easy to see that the sphere admits a metric with positive curvature: this is true for the induced metric coming from the standard embedding of S^2 in $\mathbb{R}^3$. Things are similarly simple for the torus, though the usual embedding of T^2 into $\mathbb{R}^3$ (as a doughnut) is the wrong picture to look at. Instead take $\mathbb{R}^2$ with its standard flat metric and define T^2 as $\mathbb{R}^2/\mathbb{Z}^2$: the translation invariance of the Euclidean metric implies that it gives a well defined metric on the quotient, and it is indeed locally flat.

The only part that is less obvious is that every surface of genus $g \geq 2$ admits a metric of negative curvature—in fact, by a famous result in the theory of surfaces, one can always find a metric that has *constant* curvature -1. One approach is to take the Poincaré half-plane $(\mathbb{H}, h)$ as a model (see Exercise 6.15) and show that every such surface can be constructed by drawing a smooth polygon in $(\mathbb{H}, h)$ and identifying certain edges appropriately. We refer to [Spi99b, Chapter 6, Addendum 1] for details. $\square$

Remark 6.33. For a surface Σ of genus $g \geq 2$, the standard way of embedding Σ into $\mathbb{R}^3$ as a surface with g handles is misleading in some respects: as a hypersurface in $\mathbb{R}^3$, its Gaussian curvature is sometimes positive and sometimes negative. The Gauss-Bonnet theorem guarantees at least that the part with negative curvature is the majority. Unfortunately (from the perspective of people who like to visualize things), there is no isometric embedding of any closed surface with everywhere negative curvature into $\mathbb{R}^3$.

We turn finally to the proof of Theorem 6.20, which makes essential use of the local connection 1-form associated to a local trivialization. By way of preparation, suppose $P \subset \mathbb{R}^2$ is a smooth polygon and g is a Riemannian metric (not necessarily the standard Euclidean metric) defined on an open neighborhood of P. We can construct an orthonormal frame for $T\mathbb{R}^2|_P$ by starting from the standard basis of $\mathbb{R}^2$ and using the Gram-Schmidt procedure to make it orthonormal at each point of P. Denote by $\Phi : T\mathbb{R}^2|_P \to P \times \mathbb{R}^2$ the resulting trivialization. Now choose a piecewise smooth positively oriented parametrization $\gamma : [0, T] \to \mathbb{R}^2$ of ∂P with $\gamma(0) = \gamma(T)$, such that $g(\dot\gamma(t), \dot\gamma(t)) \equiv 1$ except at finitely many parameter values

$$0 = t_0 < t_1 < \ldots < t_{N-1} < t_N = T$$

where $\dot\gamma(t)$ may fail to exist. One can then find a piecewise continuous real-valued T-periodic function $\theta : \mathbb{R} \to \mathbb{R}$ such that

$$\Phi(\dot\gamma(t)) = \left(\gamma(t), \begin{pmatrix} \cos\theta(t) \\ \sin\theta(t) \end{pmatrix}\right).$$

We may assume $\theta(t)$ is smooth on $[0, T]$ except for jump discontinuities

$$\Delta\theta_j := \lim_{t \to t_j^+} \theta(t) - \lim_{t \to t_j^-} \theta(t) \in (-\pi, \pi)$$

for each $j = 0, \ldots, N$.

Lemma 6.34. $\int_0^T \dot{\theta}(t)\, dt + \sum_{j=1}^N \Delta\theta_j = 2\pi.$

Proof. We leave it as an exercise for the reader to show that this formula is correct if Φ is defined by the standard framing $(\mathbf{e}_1, \mathbf{e}_2)$ on $\mathbb{R}^2$: in that case it simply measures the total change in the angular coordinate of $\dot{\gamma}(t)$ is the boundary of P is traversed, including the jump discontinuities. In general, the sum in question is clearly some integer multiple of 2π, and since it can only change continuously under continuous deformations of Φ, it will in fact remain unchanged under such deformations. The result follows by choosing a continuous deformation of g to the standard Euclidean metric by the interpolation

$$g_\tau = (1-\tau)\langle\ ,\ \rangle_{\mathbb{R}^2} + \tau g, \qquad \tau \in [0,1].$$

Then the Gram-Schmidt procedure provides a continuous family of orthonormal frames and hence a continuous family of trivializations. $\square$

If $P \subset \Sigma$ is a smooth polygon, the result above provides a trivialization

$$\Phi_\alpha : T\Sigma|_P \to P \times \mathbb{R}^2$$

which preserves the orientations of the fibers and carries the bundle metric to the standard Euclidean metric on $\mathbb{R}^2$; we can then define a piecewise smooth parametrization $\gamma : [0,T] \to \partial P$ and a function $\theta(t)$ in the same manner so that Lemma 6.34 is satisfied. Moreover if the vertices of P at $t = t_j$ have angles $\alpha_j \in (0, 2\pi)$, we have

$$\alpha_j + \Delta\theta_j = \pi.$$

Denote the smooth edges of ∂P by

$$\ell_j = \gamma([t_{j-1}, t_j]) \subset \partial P,$$

for $j = 1, \ldots, N$.

Writing vector fields $X \in \mathrm{Vec}(P)$ via the trivialization as $X_\alpha : P \to \mathbb{R}^2$, the covariant derivative can be written in terms of an $\mathfrak{so}(2)$-valued 1-form $A_\alpha \in \Omega^1(P, \mathfrak{so}(2))$ as

$$(\nabla_X Y)_\alpha = dY_\alpha(X) + A_\alpha(X) Y_\alpha(p)$$

for $p \in P$, $X \in T_p\Sigma$. The curvature 2-form $\Omega_K \in \Omega^2(\Sigma, \mathrm{End}(T\Sigma))$ can then be expressed via the trivialization as

$$(\Omega_K(X,Y)Z)_\alpha = F_\alpha(X,Y) Z_\alpha,$$

where $F_\alpha \in \Omega^2(P, \mathfrak{so}(2))$ is defined by

$$F_\alpha(X,Y) = dA_\alpha(X,Y) + [A_\alpha(X), A_\alpha(Y)].$$

This is the general formula for F_α, but in the present case the second term vanishes since $\mathrm{SO}(2)$ is abelian, implying $F_\alpha = dA_\alpha$. As was observed at the end of Chapter 5, another consequence of $\mathrm{SO}(2)$ being abelian is that F_α does not actually depend on the trivialization. This reflects the fact that one can define a natural action of $\mathrm{SO}(2)$ (and therefore also of $\mathfrak{so}(2)$) on the fibers of $T\Sigma$: simply choose any orthonormal basis to identify a tangent space $T_p\Sigma$ with $\mathbb{R}^2$ and use the natural action of $\mathrm{SO}(2)$ on $\mathbb{R}^2$. It's easy to show that the resulting action on $T_p\Sigma$ doesn't depend on the basis—this is true specifically because elements of $\mathrm{SO}(2)$ commute. From this perspective, $F_\alpha(X,Y)$ is the same thing as $\Omega_K(X,Y)$, simply using an element of $\mathfrak{so}(2)$ to define an endomorphism of $T_p\Sigma$.

Recall now that $K_G : P \to \mathbb{R}$ can be defined as the unique real-valued function such that

$$F_\alpha(X,Y) = -K_G(p)\; dA(X,Y) J_0$$

for $p \in P$, $X, Y \in T_p\Sigma$, where $J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Define a real-valued 1-form $\lambda_\alpha \in \Omega^1(P)$ such that

$$A_\alpha(X) = \lambda_\alpha(X) J_0,$$

then we have $d\lambda_\alpha = -K_G\; dA$. It should now be fairly obvious what approach we intend to take: since $K_G\; dA$ is presented as the exterior derivative of a 1-form, we can integrate it using Stokes' theorem.

Lemma 6.35. *The geodesic curvature of an edge $\ell_j \subset \partial P$ at $\gamma(t) \in \ell_j$ is given by*

$$\kappa_{\ell_j}(\gamma(t)) = \dot\theta(t) + \lambda_\alpha(\dot\gamma(t)).$$

Proof. Let $v(t) = (\dot\gamma(t))_\alpha \in \mathbb{R}^2$, so by definition

$$v(t) = \begin{pmatrix} \cos\theta(t) \\ \sin\theta(t) \end{pmatrix} \quad \text{and} \quad \dot v(t) = \begin{pmatrix} -\sin\theta(t) \\ \cos\theta(t) \end{pmatrix} \dot\theta(t) = \dot\theta(t) J_0 v(t).$$

Thus

$$(\nabla\dot\gamma(t))_\alpha = \dot v(t) + A_\alpha(\dot\gamma(t)) v(t) = \Big(\dot\theta(t) + \lambda_\alpha(\dot\gamma(t))\Big) J_0 v(t).$$

The claim follows since $J_0 v(t) \in \mathbb{R}^2$ is the unique vector such that $(v(t), J_0 v(t))$ is a positively oriented orthonormal basis. ☐

Proof of Theorem 6.20. Define the piecewise smooth parametrization $\gamma(t)$ and the function $\theta(t)$ as described above. Then combining Lemmas 6.34 and 6.35 with Stokes' theorem,

$$\begin{aligned}\int_P K_G\, dA &= -\int_P d\lambda_\alpha = -\int_{\partial P} \lambda_\alpha = -\sum_{j=1}^N \int_{t_{j-1}}^{t_j} \lambda_\alpha(\dot\gamma(t))\, dt \\ &= -\sum_{j=1}^N \int_{t_{j-1}}^{t_j} \left[\kappa_{\ell_j}(\gamma(t)) - \dot\theta(t)\right]\, dt \\ &= -\sum_{j=1}^N \int_{\ell_j} \kappa_{\ell_j}\, ds + 2\pi - \sum_{j=1}^N \Delta\theta_j \\ &= -\sum_{j=1}^N \int_{\ell_j} \kappa_{\ell_j}\, ds + 2\pi - \sum_{j=1}^N (\pi - \alpha_j) \\ &= -\sum_{j=1}^N \int_{\ell_j} \kappa_{\ell_j}\, ds - (N-2)\pi + \sum_{j=1}^N \alpha_j\end{aligned}$$

□

6.3.2 Counting zeros and the Euler number

Although the focus of this chapter is primarily on Riemannian manifolds, the proof of Theorem 6.20 invites a certain amount of generalization beyond this context. For instance, if we consider not the tangent bundle but a more general vector bundle $E \to \Sigma$ with structure group $\mathrm{SO}(2)$, then we still have a well defined $\mathfrak{so}(2)$-valued curvature 2-form $F \in \Omega^2(M, \mathfrak{so}(2))$, which is locally the exterior derivative of any connection 1-form. What happens if we integrate this 2-form over Σ? As we will see, the integral does indeed compute an invariant, called the *Euler number* of the bundle $E \to \Sigma$, which happens to equal the Euler characteristic when $E = T\Sigma$.

The Euler number is an integer that can be associated to any *oriented* vector bundle E of rank n over a closed n-manifold M. In this discussion we shall deal exclusively with the 2-dimensional case, but much of what we will say can be generalized. The integer arises most naturally as the answer to the following question:

> *Given an oriented vector bundle E of rank 2 over a closed surface Σ, how many zeros does a "generic" smooth section of E have?*

It may seem surprising at first that this question has a well defined answer; we must of course be careful to specify precisely what we mean by "generic" and "how many". A simple example shows in any case that the answer is not arbitrary: in particular, not every such bundle admits a smooth section with no zeros. If indeed $E \to \Sigma$ has such a section $v \in \Gamma(E)$, then using the orientation we can find another section $w \in \Gamma(E)$ such that (v, w) defines an oriented basis at every point, which implies that E is trivial. Clearly then, for nontrivial oriented bundles the answer is not zero; we will see in fact that if zeros are counted with appropriate *multiplicities*, then the answer is independent of the choice of section, and defines an invariant that one can use to distinguish non-isomorphic bundles.

To facilitate the following definition, define $\gamma : \mathbb{R} \to \mathbb{R}^2$ to be the obvious parametrization of the unit circle:

$$\gamma(t) = (\cos t, \sin t).$$

Definition 6.36. Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a smooth map with $f(0) = 0$ and $f(p) \neq 0$ for all p in some neighborhood of 0. Then the *index* of f at 0 is defined as the integer

$$\operatorname{ind}(f; 0) = \frac{1}{2\pi}(\theta(2\pi) - \theta(0))$$

where $\theta(t)$ is any smooth function such that $f(\epsilon\gamma(t)) = r(t)\gamma(\theta(t))$ with $r(t) > 0$ for $\epsilon > 0$ arbitrarily small.

Put another way, $\operatorname{ind}(f; 0)$ is the *winding number* of the path $f \circ \gamma$ about 0; it is positive if the path winds counterclockwise and negative if it winds clockwise. Take a moment to convince yourself that this definition doesn't depend on the choice of the function $\theta(t)$ or the (small!) number $\epsilon > 0$.

Exercise 6.37. Using the natural identification of $\mathbb{R}^2$ with $\mathbb{C}$, show that for any integer $k \geq 0$, the function $f(z) = z^k$ has $\operatorname{ind}(f; 0) = k$, while $g(z) = \bar{z}^k$ has $\operatorname{ind}(g; 0) = -k$.

For the remainder of this section, Σ will be a closed oriented 2-manifold and $E \to \Sigma$ will be an oriented vector bundle of rank 2. The notion of the index of a zero can be extended naturally to smooth sections $v \in \Gamma(E)$.

Definition 6.38. Let $v \in \Gamma(E)$, suppose $v(p) = 0$ and that p is an *isolated* zero of v, i.e. it is contained in a neighborhood $p \in \mathcal{U} \subset \Sigma$ such that $v(q) \neq 0$ for all $q \in \mathcal{U} \setminus \{p\}$. Then choosing any oriented coordinate chart and oriented local trivialization near p so that v is identified with a smooth map v_α from a neighborhood of $0 \in \mathbb{R}^2$ to $\mathbb{R}^2$, define the index of v at p by

$$\operatorname{ind}(v; p) = \operatorname{ind}(v_\alpha; 0).$$

There is of course something to prove here: how do we know that $\operatorname{ind}(v; p)$ doesn't depend on the choice of coordinates and trivialization? If

v_β is the expression for v in a different oriented trivialization, then there exists a transition map $g_{\beta\alpha}$ taking a neighborhood of p to $\mathrm{GL}(2,\mathbb{R})$ such that $\det g_{\beta\alpha}(q) > 0$ and

$$v_\beta(q) = g_{\beta\alpha}(q) v_\alpha(q).$$

We need to show that for sufficiently small loops $\gamma : S^1 \to \Sigma$ around p, $v_\beta \circ \gamma$ and $v_\alpha \circ \gamma$ have the same winding number about 0. The key is that $g_{\beta\alpha} \circ \gamma$ can be deformed continuously to the identity: simply shrink γ to the constant loop at p and then move $g_{\beta\alpha}(p)$ by a continuous path in $\mathrm{GL}(2,\mathbb{R})$ to $\mathbb{1}$, which is possible since $\det g_{\beta\alpha}(p) > 0$. The result is a continuous deformation of $v_\beta \circ \gamma$ to $v_\alpha \circ \gamma$ through loops that never pass through 0, implying that both have the same winding number.

Definition 6.39. The *Euler number* of $E \to \Sigma$ is defined as

$$e(E) = \sum_{p \in v^{-1}(0)} \operatorname{ind}(v; p)$$

where v is a section of E with finitely many zeros.

At this stage it is far from obvious that $e(E)$ doesn't depend on the choice of section, but this will follow from Theorem 6.40 below, which computes $e(E)$ as an integral of the curvature 2-form for a metric connection on E. Before getting into that, it would be a shame not to mention, at least as an aside, that the existence of the Euler number can also be proved by a quite beautiful argument that has nothing to do with curvature. The idea is roughly as follows: say that a section $v \in \Gamma(E)$ is *generic* if it has finitely many zeros, all of which have either index 1 or -1. Recalling the discussion of transversality in §3.3.4, this is precisely the case in which the submanifold $v(\Sigma) \subset E$ intersects the zero-section transversely, and the sign of $\operatorname{ind}(v; p)$ for each $p \in v^{-1}(0)$ depends on whether the linearization

$$dv(p) : T_p\Sigma \to E_p$$

preserves or reverses orientation. One can then regard the zero set $v^{-1}(0)$ as a compact oriented 0-dimensional submanifold of Σ, with the orientation of each point defined by the sign of $\operatorname{ind}(v; p)$. Now if $w \in \Gamma(E)$ is another generic section, we can find a smooth *homotopy* between them, i.e. a map

$$H : [0, 1] \times \Sigma \to E$$

such that $H(t, \cdot) \in \Gamma(E)$ for each t, with $H(0, \cdot) = w$ and $H(1, \cdot) = v$. By a nontrivial bit of transversality theory, one can always make a small perturbation of H so as to assume without loss of generality that its image in E meets the zero-section transversely, in which case $H^{-1}(0) \subset [0,1] \times \Sigma$ is a smooth 1-dimensional submanifold with boundary. Then

$$\partial \left(H^{-1}(0)\right) = \left(\{1\} \times v^{-1}(0)\right) \cup \left(\{0\} \times (-w^{-1}(0))\right),$$

where the minus sign on the right hand side indicates reversal of orientation. The 1-manifold $H^{-1}(0)$ will generally have multiple connected components, which come in three flavors:

1. Circles in the interior of $[0,1] \times \Sigma$
2. Arcs with one boundary point in $\{1\} \times v^{-1}(0)$, and the other a point in $\{0\} \times w^{-1}(0)$ with the same orientation
3. Arcs with both boundary points in either $\{1\} \times v^{-1}(0)$ or $\{0\} \times w^{-1}(0)$, having opposite orientations

The result is that the points in the disjoint union of $v^{-1}(0)$ with $w^{-1}(0)$ come in pairs: matching pairs of zeros of v and w, or cancelling pairs of zeros of v alone or w alone. Thus the count of positive points in $v^{-1}(0)$ minus negative points in $v^{-1}(0)$ is the same as the corresponding count for w. This argument can be extended to arbitrary sections with isolated zeros by a simple perturbation: if p is a zero with $\mathrm{ind}(v;p) \neq \pm 1$, then one must perturb v near p to a section that has only zeros of index ± 1, and one can check that the signed count of these is always equal to the original index. Details of these arguments may be found in [Mil97].

Notice that we've so far assumed very little structure on $E \to \Sigma$: the definition of the Euler number required a choice of section, but the bundle itself has intrinsically only a smooth structure and an orientation. One can always add to this by choosing a bundle metric $\langle\ ,\ \rangle$: then $E \to \Sigma$ becomes an oriented Euclidean vector bundle, with structure group $\mathrm{SO}(2)$. Further, one can choose a connection ∇ compatible with the bundle metric: this choice is again far from unique, but we'll find that the ambiguity matters surprisingly little. We now recall an important fact that was used in our proof of the Gauss-Bonnet theorem: since $\mathrm{SO}(2)$ is abelian, the connection has a well defined $\mathfrak{so}(2)$-valued curvature 2-form

$$F \in \Omega^2(\Sigma, \mathfrak{so}(2)),$$

which is the exterior derivative of any connection 1-form $A_\alpha \in \Omega^1(\mathcal{U}_\alpha, \mathfrak{so}(2))$ associated to an $\mathrm{SO}(2)$-compatible local trivialization $\Phi_\alpha : E|_{\mathcal{U}_\alpha} \to \mathcal{U}_\alpha \times \mathbb{R}^2$. If two such trivializations are related by the transition map

$$g_{\beta\alpha} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to \mathrm{SO}(2),$$

then by Exercise 3.20, the corresponding connection 1-forms satisfy

$$A_\alpha(X) = g_{\beta\alpha}^{-1} A_\beta(X) g_{\beta\alpha} + g_{\beta\alpha}^{-1} dg_{\beta\alpha}(X).$$

In the present case this can be simplified considerably: writing

$$g_{\beta\alpha}(p) = \begin{pmatrix} \cos\theta(t) & -\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) \end{pmatrix}$$

for some function $\theta : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to \mathbb{R}$, we find

$$A_\alpha = J_0 \; d\theta + A_\beta \tag{6.6}$$

where $J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. It's obvious from this expression why the 2-form $F = dA_\alpha$ doesn't depend on the trivialization.

Theorem 6.40. *Define $\omega \in \Omega^2(\Sigma)$ by $F(X,Y) = \omega(X,Y)J_0$, where $F \in \Omega^2(\Sigma, \mathfrak{so}(2))$ is the curvature 2-form for any metric connection on the oriented Euclidean bundle $E \to \Sigma$. Then*

$$-\int_\Sigma \omega = 2\pi e(E).$$

Proof. Choose a section $v \in \Gamma(E)$ with isolated zeros, and for each $p \in v^{-1}(0)$ choose a small disk-shaped open neighborhood $p \subset \mathcal{D}_p \subset \Sigma$, which is the preimage of a small open ball in $\mathbb{R}^2$ under a coordinate chart defined near p. Denote

$$\Sigma_\epsilon = \Sigma \setminus \bigcup_{p \in v^{-1}(0)} \mathcal{D}_p.$$

Since v is nowhere zero in Σ_ϵ, we can rescale it without loss of generality and assume that $\langle v, v\rangle \equiv 1$ on an open neighborhood Σ'_ϵ of Σ_ϵ; this does not change the computation of the indices $\mathrm{ind}(v; p)$ for $p \in v^{-1}(0)$. We can then find a unique section $w \in \Gamma(E|_{\Sigma_\epsilon})$ such that (v, w) defines an oriented orthonormal frame over Σ'_ϵ, and hence an SO(2)-compatible trivialization

$$\Phi_\alpha : E|_{\Sigma'_\epsilon} \to \Sigma'_\epsilon \times \mathbb{R}^2$$

in which $v_\alpha : \Sigma_\epsilon \to \mathbb{R}^2$ is the constant unit vector $\mathbf{e}_1 \in \mathbb{R}^2$. Define the 1-form $\lambda_\alpha \in \Omega^1(\Sigma'_\epsilon)$ such that $A_\alpha(X) = \lambda_\alpha(X)J_0$, thus $\omega = d\lambda_\alpha$ and by Stokes' theorem,

$$-\int_{\Sigma_\epsilon} \omega = -\int_{\partial\Sigma_\epsilon} \lambda_\alpha = \sum_{p \in v^{-1}(0)} \int_{\partial\overline{\mathcal{D}}_p} \lambda_\alpha. \tag{6.7}$$

Focusing now on a particular zero $p \in v^{-1}(0)$, we pick another SO(2)-compatible trivialization on some open neighborhood $\mathcal{D}'_p$ of $\overline{\mathcal{D}}_p$:

$$\Phi_\beta : E|_{\mathcal{D}'_p} \to \mathcal{D}'_p \times \mathbb{R}^2.$$

Denote $\lambda_\beta \in \Omega^1(\mathcal{D}')$ such that $A_\beta(X) = \lambda_\beta(X) J_0$, so again $d\lambda_\beta = \omega$. Now if the transition function from Φ_α to Φ_β has the form

$$g_{\beta\alpha} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

then we have $v_\beta = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ on $\mathcal{D}'$, and by (6.6), $\lambda_\beta = \lambda_\alpha + d\theta$. Thus applying Stokes' theorem again, the integral over $\partial\overline{\mathcal{D}}_p$ in (6.7) becomes

$$\int_{\partial\overline{\mathcal{D}}_p} \lambda_\alpha = \int_{\partial\overline{\mathcal{D}}_p} \lambda_\beta + \int_{\partial\overline{\mathcal{D}}_p} d\theta = \int_{\overline{\mathcal{D}}_p} \omega + 2\pi \operatorname{ind}(v; p).$$

Combining this expression for all $p \in v^{-1}(0)$ with the integral of ω over Σ_ϵ yields the stated formula. □

Corollary 6.41. *The Euler number $e(E)$ does not depend on the choice of section $v \in \Gamma(E)$.*

Corollary 6.42. *If $\omega \in \Omega^2(\Sigma)$ is defined in terms of the curvature 2-form $F \in \Omega^2(\Sigma, \mathfrak{so}(2))$ by $F(X,Y) = \omega(X,Y)J_0$, then the integral*

$$\int_\Sigma \omega$$

is always an integer multiple of 2π and has no dependence on the choice of bundle metric $\langle\ ,\ \rangle$ or metric connection ∇.

Combining Theorem 6.40 with Theorem 6.25, we obtain a formula for the Euler number of tangent bundles which justifies its name:

Corollary 6.43. *For any closed oriented surface Σ, $e(T\Sigma) = \chi(\Sigma)$.*

Exercise 6.44. *Why settle for one proof when you can have two?!*

1. Show by explicitly constructing vector fields that $e(TS^2) = 2$ and $e(TT^2) = 0$. (It suffices to draw pictures.)

2. Show that a pair of pants (see Exercise 6.29) admits a smooth vector field which is tangent to the boundary and nonzero there, and has exactly one interior zero, of index -1.

3. Conclude that the same thing is true for a handle.

4. Use the above results and a gluing argument as in Exercises 6.30 and 6.31 to show that a closed oriented surface Σ of genus g has $e(T\Sigma) = 2 - 2g$.

As the simplest possible application of the above results, one obtains a theorem that is often described by the phrase "you can't comb the hair on a sphere."

Theorem 6.45. *There is no smooth vector field without zeros on S^2.*

A simple consequence is that TS^2 cannot be trivialized globally, and in fact the formula $e(T\Sigma) = 2 - 2g$ implies:

Corollary 6.46. *A closed oriented surface has trivial tangent bundle if and only if it is a torus.*

What we've seen in this section is a snippet of a rather large subject known as *Chern-Weil theory*, which is part of the theory of *characteristic classes*. The latter are invariants of bundles (of which the Euler number is an example), usually defined in homological terms, which can be used to distinguish non-isomorphic bundles of various types. These invariants can be constructed in many ways: the approach taken in Chern-Weil theory is to make the (mostly arbitrary) choice of a connection on the bundle and integrate some form that can be constructed from the curvature, showing in the end that the result doesn't depend on the connection. Among other things, this leads to some higher dimensional generalizations of the Gauss-Bonnet formula, which are covered in detail in [Spi99c]. For a comprehensive survey of characteristic classes from a more algebraic topological perspective, we refer to [MS74].

6.4 Geodesics, arc length and sectional curvature

We conclude the chapter by revisiting the relationship between geodesics and arc length. Assume (M, g) is an n-dimensional Riemannian manifold and ∇ is the Levi-Civita connection on $TM \to M$. The question of the hour is:

> *If $\gamma : [a, b] \to M$ is a geodesic with $\gamma(a) = p$ and $\gamma(b) = q$, is there any other geodesic from p to q near γ? Is γ in fact the* shortest *path between p and q?*

For example, consider the geodesics in $S^2 \subset \mathbb{R}^3$ from the north pole to the south pole. These are far from unique: there is a whole 1-parameter family of them, all equally long. We'll find that this is only allowed because S^2 has positive curvature; it *cannot* happen on a surface with negative or zero curvature.

6.4.1 The shortest path between nearby points

Let us first address a related issue that has nothing to do with curvature: *locally*, a "short" geodesic is always the shortest path between two nearby points. This was mentioned in Chapter 4, Prop. 4.18, and the proof really should have appeared there; in a future version of these notes, it probably will.

Proposition 6.47. *For every point $p \in M$, there is a neighborhood $p \in \mathcal{U}_p \subset M$ such that each $q \in \mathcal{U}_p$ can be expressed uniquely as $\exp_p(X)$ for some $X \in T_pM$, and the geodesic segment*

$$\gamma : [0,1] \to M : t \mapsto \exp_p(tX)$$

parametrizes the shortest path in M from p to q.

The existence of the neighborhood $\mathcal{U}_p$ is the easy part: it follows from the inverse function theorem, as we can easily differentiate the exponential map $\exp_p : T_pM \to M$ at $0 \in T_pM$ and find

$$d\exp_p(0) = \mathrm{Id} : T_pM \to T_pM.$$

Since this map is invertible, there is an open neighborhood $0 \in \mathcal{O}_p \subset T_pM$ such that $\exp_p$ restricts to a diffeomorphism from $\mathcal{O}_p$ to an open neighborhood $\mathcal{U}_p$ of p in M. Without loss of generality, we may assume $\mathcal{O}_p$ has the form

$$\mathcal{O}_p = \{X \in T_pM \mid |X| < \epsilon\}$$

for some $\epsilon > 0$.

Recall now that the Levi-Civita connection was defined by a pair of natural conditions—symmetry and compatibility with the metric—neither of which have anything directly to do with arc length. So what it is it about this particular connection that ensures that the "straight" paths are also the "shortest"? The key turns out to be the following lemma.

Lemma 6.48. *For any $r \in (0, \epsilon)$, define the hypersurface*

$$\Sigma_r = \{X \in T_pM \mid |X| = r\} \subset \mathcal{O}_p.$$

Then every geodesic through p is orthogonal to all of the hypersurfaces $\exp_p(\Sigma_r) \subset M$.

Proof. For any smooth path $X(t) \in T_pM$ with $|X(t)| \equiv 1$, defined for $t \in \mathbb{R}$ near 0, let $\alpha(s,t) \in M$ denote the smooth map

$$\alpha(s,t) = \exp_p(sX(t))$$

for t near 0 and $s \in (-\epsilon, \epsilon)$. It suffices to show that

$$\langle \partial_s \alpha(s,0), \partial_t \alpha(s,0) \rangle = 0$$

for every $s \in (0, \epsilon)$. Using the symmetry and compatibility of the connection, we have

$$\begin{aligned}\partial_s \langle \partial_s \alpha(s,0), \partial_t \alpha(s,0) \rangle &= \langle \nabla_s \partial_s \alpha(s,0), \partial_t \alpha(s,0) \rangle + \langle \partial_s \alpha(s,0), \nabla_s \partial_t \alpha(s,0) \rangle \\ &= \langle \partial_s \alpha(s,0), \nabla_t \partial_s \alpha(s,0) \rangle,\end{aligned} \tag{6.8}$$

where we used the geodesic equation to eliminate $\nabla_s \partial_s \alpha(s,0)$. Likewise, for any fixed t the path $s \mapsto \alpha(s,t)$ is a geodesic with fixed speed $|\partial_s \alpha(s,t)| = |X(t)| = 1$, so

$$0 = \partial_t \langle \partial_s \alpha(s,t), \partial_s \alpha(s,t) \rangle|_{t=0} = 2 \langle \partial_s \alpha(s,0), \nabla_t \partial_s \alpha(s,0) \rangle,$$

implying that (6.8) vanishes. Moving s to 0, we therefore have

$$\langle \partial_s \alpha(s,0), \partial_t \alpha(s,0) \rangle = \langle \partial_s \alpha(0,0), \partial_t \alpha(0,0) \rangle = 0$$

since $\alpha(0,t) = p$ is independent of t and thus $\partial_t \alpha(0,0) = 0$. $\square$

Proof of Prop. 6.47. Consider a path $\gamma : [0,1] \to \mathcal{U}_p$ of the form

$$\gamma(t) = \exp_p(r(t)X(t))$$

where $r(t)$ is an increasing real-valued function with $r(0) = 0$ and $r(1) \in (0, \epsilon)$, and $X(t) \in T_pM$ is a smooth path with $|X(t)| \equiv 1$. We will show that

$$\int_0^1 |\dot{\gamma}(t)| \, dt \geq r(1),$$

with equality if and only if $X(t)$ is constant, implying that $\gamma(t)$ can be reparametrized as a geodesic. This implies that the geodesics

$$[0,1] \to M : t \mapsto \exp_p[t \cdot r(1)X]$$

parametrize the shortest paths from p to any point on the hypersurface $\exp_p(\Sigma_{r(1)})$.

Let $\alpha(s,t) = \exp_p(sX(t))$ as in the proof of Lemma 6.48; then $\gamma(t) = \alpha(r(t), t)$ and

$$\dot{\gamma}(t) = \dot{r}(t)\partial_s \alpha(r(t), t) + \partial_t \alpha(r(t), t).$$

By the lemma, the two terms in this sum are orthogonal to each other, and since $|\partial_s \alpha(s,t)| = |X(t)| = 1$, we have

$$\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle = [\dot{r}(t)]^2 + |\partial_t \alpha(r(t), t))|^2,$$

implying $|\dot{\gamma}(t)| \geq \dot{r}(t)$, with equality if and only if $\partial_t \alpha \equiv 0$, implying $X(t)$ is constant. Clearly then

$$\int_0^1 |\dot{\gamma}(t)| \, dt \geq \int_0^1 \dot{r}(t) \, dt = r(1).$$

$\square$

6.4.2 Sectional curvature

With that out of the way, we turn to questions of a more global nature: assuming that p and q are not necessarily nearby, when is it guaranteed that a geodesic from p to q is shorter than all other nearby paths? The answer has to do with a quantity called the *sectional curvature* of (M, g), which generalizes the Gaussian curvature to higher dimensional manifolds.

Definition 6.49. For any $p \in M$ and a 2-dimensional subspace $P \subset T_pM$, we define the *sectional curvature* $K_S(P) \in \mathbb{R}$ as follows. Choose a sufficiently small neighborhood $0 \in \mathcal{O}_p \subset T_pM$ so that $\exp_p$ restricts to a diffeomorphism from $\mathcal{O}_p$ to a neighborhood of p in M. Then

$$\Sigma_P := \exp_p(\mathcal{O}_p \cap P) \subset M$$

is a 2-dimensional submanifold containing p, and we set

$$K_S(P) = K_G(p),$$

where $K_G : \Sigma_P \to \mathbb{R}$ is the Gaussian curvature of Σ_P with respect to the Riemannian metric induced by its embedding in (M, g).

In the case $\dim M = 2$, there is only one 2-dimensional subspace of T_pM to choose from, and we recover the Gaussian curvature $K_S(T_pM) = K_G(p)$. Then $K_S(T_pM)$ is easily computed via (6.5), and we'd like to generalize this formula to the higher dimensional case.

Lemma 6.50. *For the 2-dimensional submanifold $\Sigma_P = \exp_p(\mathcal{O}_P \cap P) \subset M$ in Definition 6.49, denote by $\widetilde{R}$ its Riemann tensor with respect to the induced metric. Then for any vectors $X, Y \in P = T_p\Sigma_P$,*

$$\langle \widetilde{R}(X, Y)Y, X\rangle = \langle R(X, Y)Y, X\rangle,$$

where R is the Riemann tensor of (M, g).

Proof. It suffices to prove this under the assumption that X and Y form an orthonormal basis of P, as the general case follows by taking linear combinations of these. We shall now make use of the Riemannian normal coordinates constructed in Prop. 6.3. Namely, extending $(X_1, X_2) := (X, Y)$ to an orthonormal basis $(X_1, \ldots, X_n)$ of T_pM, we define a coordinate chart $x = (x^1, \ldots, x^n)$ on a neighborhood of p via the inverse of the map

$$(x^1, \ldots, x^n) \mapsto \exp_p(x^1 X_1 + \ldots + x^n X_n)$$

for $(x^1, \ldots, x^n)$ near the origin in $\mathbb{R}^n$. We then have $\partial_j|_p = X_j$, and the submanifold

$$\{x^3 = x^4 = \ldots = x^n = 0\}$$

is a neighborhood of p in Σ_P, which inherits the coordinates (x^1, x^2). Since now $\partial_1|_p = X$ and $\partial_2|_p = Y$, the claim can be expressed in coordinates as

$$\widetilde{R}_{1122}(0) = R_{1122}(0).$$

By Prop. 6.3, the metric on M has components $g_{ij} = \delta_{ij}$ at 0, while its first derivatives and thus the corresponding Christoffel symbols vanish there. The formulas from Exercises 4.19 and 5.20 then give

$$\begin{aligned} R_{1122}(0) &= \delta_{1i} R^i{}_{122} = \delta_{1i}\left(\partial_1 \Gamma^i_{22} - \partial_2 \Gamma^i_{12}\right) = \partial_1 \Gamma^1_{22} - \partial_2 \Gamma^1_{12} \\ &= \frac{1}{2}\partial_1\left(\delta^{1\ell}(\partial_2 g_{2\ell} + \partial_2 g_{2\ell} - \partial_\ell g_{22})\right) - \frac{1}{2}\partial_2\left(\delta^{1\ell}(\partial_1 g_{2\ell} + \partial_2 g_{1\ell} - \partial_\ell g_{12})\right) \\ &= \frac{1}{2}\partial_1\left(2\partial_2 g_{21} - \partial_1 g_{22}\right) - \frac{1}{2}\partial_2\partial_2 g_{11}. \end{aligned}$$

By the same argument, $\widetilde{R}_{1122}(0)$ satisfies the same formula in terms of the components $\tilde{g}_{ij}$ of the induced metric for $i, j \in \{1, 2\}$; but these are simply the restrictions of the corresponding components of g_{ij} to the submanifold $\{x^3 = \ldots = x^n = 0\}$, implying that the two expressions are the same. □

Proposition 6.51. *For any $p \in \Sigma$ and pair of orthogonal unit vectors $X, Y \in T_pM$, if $P \subset T_pM$ is the subspace spanned by X and Y, then*

$$K_S(P) = \langle R(X, Y)Y, X\rangle.$$

Proof. By the lemma, this equals $\langle \widetilde{R}(X, Y)Y, X\rangle$ for the induced Riemann tensor $\widetilde{R}$ on the submanifold Σ_P, and (6.5) implies that this is the Gaussian curvature of Σ_P at p. □

Recall that in the case $\dim M = 2$, the Riemann tensor of M is entirely determined by its Gaussian curvature. The generalization of this statement is also true, though we will not prove it: the Riemann tensor of a higher dimensional manifold is determined by the sectional curvature $K_S(P)$ for every plane $P \subset TM$. A proof may be found in [Spi99a].

Definition 6.52. We say that a Riemannian n-manifold (M, g) has *positive* (*zero*, *negative*) *sectional curvature* if $K_S(P)$ is positive (zero, negative) for every plane $P \subset T_pM$ at every point $p \in M$.

6.4.3 The second variation

To better understand the global relationship between geodesics and length, we can apply an infinite dimensional version of the "second derivative test" to the energy and length functionals. Recall the following notation from Chapter 4: for two parameter values $a < b \in \mathbb{R}$ and points $p, q \in M$, we denote by

$$\mathcal{P} = C^\infty([a, b], M; p, q)$$

the space of smooth paths $\gamma : [a, b] \to M$ starting at p and ending at q. We think of this space intuitively as an infinite dimensional smooth manifold, with tangent spaces

$$T_\gamma \mathcal{P} := \{\eta \in \Gamma(\gamma^* TM) \mid \eta(a) = 0 \text{ and } \eta(b) = 0\}.$$

We then have two functionals $\ell, E : \mathcal{P} \to \mathbb{R}$, the length functional

$$\ell(\gamma) = \int_a^b |\dot{\gamma}(t)| \, dt$$

and energy functional

$$E(\gamma) = \frac{1}{2} \int_a^b |\dot{\gamma}(t)|^2 \, dt,$$

where an extra factor of 1/2 has been inserted in front of the energy functional to make some of the expressions below look a bit nicer. For a smooth 1-parameter family of paths $\gamma_s \in \mathcal{P}$ with $\gamma_0 = \gamma$ and $\partial_s \gamma_s|_{s=0} = \eta \in T_\gamma \mathcal{P}$, we computed in Chapter 4 the *first variation* of the energy functional:

$$dE(\gamma)\eta := \left. \frac{d}{ds} E(\gamma_s) \right|_{s=0} = \int_a^b \langle -\nabla_t \dot{\gamma}(t), \eta(t) \rangle \, dt.$$

We can express this more succinctly by defining an inner product on the space of sections $\Gamma(\gamma^* TM)$ for each $\gamma \in \mathcal{P}$: for two such sections ξ and η, let[3]

$$\langle \xi, \eta \rangle_{L^2} = \int_a^b \langle \xi(t), \eta(t) \rangle \, dt.$$

Informally, we can think of $\langle \ , \ \rangle_{L^2}$ as defining a Riemannian metric on $\mathcal{P}$. Now the first variation can be expressed as

$$dE(\gamma)\eta = \langle \nabla E(\gamma), \eta \rangle_{L^2},$$

where

$$\nabla E(\gamma) := -\nabla_t \dot{\gamma} \in \Gamma(\gamma^* TM)$$

is the so-called L^2-*gradient* of the energy functional. In this notation, γ is a geodesic if and only if $\nabla E(\gamma) = 0$.

Informally again, we think of ∇E as a vector field on $\mathcal{P}$ which represents the first derivative of E, and we'd now like to compute the second derivative. For $\eta \in T_\gamma \mathcal{P}$, we choose a 1-parameter family $\gamma_s \in \mathcal{P}$ with $\gamma_0 = \gamma$ and $\partial_s \gamma_s|_{s=0} = \eta$ and define a "covariant derivative" $\nabla_\eta \nabla E \in \Gamma(\gamma^* TM)$ by

$$(\nabla_\eta \nabla E)(t) := \left. \nabla_s (\nabla E(\gamma_s)(t)) \right|_{s=0}.$$

[3]The subscript L^2 refers to the standard notation for the Hilbert space completion of $\Gamma(\gamma^* TM)$ with respect to this inner product.

A quick computation shows that this does indeed only depend on η rather than the 1-parameter family γ_s:

$$\begin{aligned}\nabla_s(\nabla E(\gamma_s))|_{s=0} &= -\nabla_s\nabla_t\partial_t\gamma_s|_{s=0}\\ &= -\nabla_t\nabla_s\partial_t\gamma_s - R(\partial_s\gamma_s,\partial_t\gamma_s)\partial_t\gamma_s|_{s=0}\\ &= -\nabla_t^2\eta - R(\eta,\dot\gamma)\dot\gamma.\end{aligned}$$

With this calculation as motivation, define for any $\gamma \in \mathcal{P}$ a linear operator

$$\nabla^2 E(\gamma) : \Gamma(\gamma^*TM) \to \Gamma(\gamma^*TM) : \eta \mapsto -\nabla_t^2\eta - R(\eta,\dot\gamma)\dot\gamma.$$

We can now state the *second variation formula*:

Proposition 6.53. *Suppose $\gamma \in \mathcal{P}$ is a geodesic and $\gamma_{\sigma,\tau} \in \mathcal{P}$ is a smooth 2-parameter family of paths with $\gamma_{0,0} = \gamma$, with variations $\xi, \eta \in T_\gamma\mathcal{P}$ defined by*

$$\xi = \partial_\sigma\gamma_{\sigma,\tau}|_{\sigma=\tau=0} \qquad \text{and} \qquad \eta = \partial_\tau\gamma_{\sigma,\tau}|_{\sigma=\tau=0}.$$

Then

$$\left.\frac{\partial^2}{\partial\sigma\partial\tau}E(\gamma_{\sigma\tau})\right|_{\sigma=\tau=0} = \langle\nabla^2E(\gamma)\xi,\eta\rangle_{L^2}.$$

Proof. Compute:

$$\begin{aligned}\left.\frac{\partial^2}{\partial\sigma\partial\tau}E(\gamma_{\sigma,\tau})\right|_{\sigma=\tau=0} &= \left.\frac{\partial}{\partial\sigma}\left(\left.\frac{\partial}{\partial\tau}E(\gamma_{\sigma,\tau})\right|_{\tau=0}\right)\right|_{\sigma=0}\\ &= \left.\frac{\partial}{\partial\sigma}\left\langle\nabla E(\gamma_{\sigma,0}),\, \partial_\tau\gamma_{\sigma,\tau}|_{\tau=0}\right\rangle_{L^2}\right|_{\sigma=0}\\ &= \left.\frac{\partial}{\partial\sigma}\int_a^b\left\langle\nabla E(\gamma_{\sigma,0})(t),\, \partial_\tau\gamma_{\sigma,\tau}(t)|_{\tau=0}\right\rangle\, dt\right|_{\sigma=0}\\ &= \int_a^b\left\langle\nabla_\sigma\nabla E(\gamma_{\sigma,0})(t)|_{\sigma=0},\eta(t)\right\rangle\, dt\\ &\qquad + \int_a^b\left\langle\nabla E(\gamma)(t),\, \nabla_\sigma\partial_\tau\gamma_{\sigma,\tau}|_{\sigma=\tau=0}\right\rangle\, dt\\ &= \int_a^b\langle(\nabla_\xi\nabla E)(t),\eta(t)\rangle\, dt = \langle\nabla^2E(\gamma)\xi,\eta\rangle_{L^2}.\end{aligned}$$

Note that we used the assumption that γ is geodesic, so $\nabla E(\gamma) = 0$. □

With this machinery in place, we can prove an important fact about geodesics in manifolds with zero or negative sectional curvature.

Lemma 6.54. *If (M, g) has nonpositive sectional curvature and $\gamma \in \mathcal{P}$ is a geodesic, then the operator $\nabla^2E(\gamma) : T_\gamma\mathcal{P} \to \Gamma(\gamma^*TM)$ is positive definite, i.e. for all $\eta \in T_\gamma\mathcal{P}$,*

$$\langle\nabla^2E(\gamma)\eta,\eta\rangle_{L^2} \geq 0,$$

with equality if and only if $\eta \equiv 0$.

Proof. We compute

$$\begin{aligned}\langle \nabla^2 E(\gamma)\eta, \eta\rangle_{L^2} &= \int_a^b \langle -\nabla_t^2 \eta, \eta\rangle \, dt - \int_a^b \langle R(\eta, \dot\gamma)\dot\gamma, \eta\rangle \, dt \\ &= \int_a^b \langle \nabla_t \eta, \nabla_t \eta\rangle \, dt - \int_a^b \langle R(\eta, \dot\gamma)\dot\gamma, \eta\rangle \, dt \ge \int_a^b |\nabla_t \eta|^2 \ge 0,\end{aligned}$$

where in the last line we've integrated by parts and used the curvature assumption to estimate the second term. Note that since $\eta(a) = 0$ and $\eta(b) = 0$ by assumption, $\nabla_t \eta \equiv 0$ would imply $\eta \equiv 0$, so the inequality above is strict unless $\eta \equiv 0$. □

Theorem 6.55. *Suppose (M, g) has nonpositive sectional curvature and $\gamma : [a, b] \to M$ is a geodesic connecting $\gamma(a) = p$ to $\gamma(b) = q$. Then for any smooth 1-parameter family of paths $\gamma_s : [a, b] \to M$ with $\gamma_s(a) = p$, $\gamma_s(b) = q$ and $\gamma_0 \equiv \gamma$ such that $\partial_s \gamma_s|_{s=0}$ is not identically zero, there is a number $\epsilon > 0$ such that:*

1. *γ is the only geodesic among the paths γ_s for $s \in (-\epsilon, \epsilon)$*
2. *For all paths γ_s with $s \in (-\epsilon, \epsilon)$ and $s \neq 0$,*
$$\ell(\gamma_s) > \ell(\gamma).$$

Proof. We first prove the second statement for the energy functional instead of the length functional, using the second variation formula and Lemma 6.54 to estimate $\partial_s^2 E(\gamma_s)|_{s=0}$ in terms of $\eta := \partial_s \gamma_s|_{s=0}$:

$$\left.\frac{d^2}{ds^2} E(\gamma_s)\right|_{s=0} = \langle \nabla^2 E(\gamma)\eta, \eta\rangle_{L^2} > 0.$$

It follows by a straightforward calculation that the same holds for the second derivative of the length functional; we leave this to the reader as an exercise, referring to §4.3.2 for inspiration.

To see that γ is the only geodesic among the family γ_s for s close to 0, denote $\eta_s = \partial_s \gamma_s \in \Gamma(\gamma_s^* TM)$ and compute

$$\begin{aligned}\left.\frac{d}{ds}\langle \nabla E(\gamma_s), \eta_s\rangle_{L^2}\right| &= \langle \nabla^2 E(\gamma)\eta, \eta\rangle_{L^2} + \langle \nabla E(\gamma), \nabla_s \eta_s|_{s=0}\rangle_{L^2} \\ &= \langle \nabla^2 E(\gamma)\eta, \eta\rangle_{L^2} > 0.\end{aligned}$$

Thus $\langle \nabla E(\gamma_s), \eta_s\rangle \neq 0$ for sufficiently small $|s| \neq 0$, implying that $\nabla E(\gamma_s)$ itself cannot be 0, so γ_s is not a geodesic. □

Exercise 6.56. In the case $\dim M = 2$, use the Gauss-Bonnet formula to provide an alternative proof of the first statement in Theorem 6.55, that there is no other geodesic near γ connecting the same end points. *Hint:* you can reduce this to the case where γ and any hypothetical nearby geodesic do not intersect: then they form a smooth polygon.

References

[Bre93] G. E. Bredon, *Topology and geometry*, Springer-Verlag, New York, 1993.

[Hat02] A. Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002.

[Hir94] M. W. Hirsch, *Differential topology*, Springer-Verlag, New York, 1994.

[Mil97] J. W. Milnor, *Topology from the differentiable viewpoint*, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 1997. Based on notes by David W. Weaver; Revised reprint of the 1965 original.

[MS74] J. W. Milnor and J. D. Stasheff, *Characteristic classes*, Princeton University Press, Princeton, N. J., 1974. Annals of Mathematics Studies, No. 76.

[Spi99a] M. Spivak, *A comprehensive introduction to differential geometry*, 3rd ed., Vol. 2, Publish or Perish Inc., Houston, TX, 1999.

[Spi99b] ______, *A comprehensive introduction to differential geometry*, 3rd ed., Vol. 3, Publish or Perish Inc., Houston, TX, 1999.

[Spi99c] ______, *A comprehensive introduction to differential geometry*, 3rd ed., Vol. 5, Publish or Perish Inc., Houston, TX, 1999.