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# h-Principles for the curvature of (semi-)Riemannian metrics

| | |
|---|---|
| $M$ | smooth manifold |
| $E_+ = \mathrm{Sym}^2_+ \, T^*M$ | fibre bundle over $M$ |

sections in $E_+$ = Riemannian metrics on $M$

| | |
|---|---|
| $\mathrm{sec}_g \colon \; \mathrm{Gr}_2 \, TM \to \mathbb{R}$ | sectional curvature of $g \in \Gamma(E_+)$ |
| $\mathrm{Ric}_g \in \Gamma(\mathrm{Sym}^2 \, T^*M)$ | Ricci curvature |
| $\mathrm{scal}_g \colon M \to \mathbb{R}$ | scalar curvature |

$\mathrm{ric}_g \colon \mathbb{P}TM = \mathrm{Gr}_1 \, TM \to \mathbb{R}$: defined by $\mathrm{ric}_g([v]) := \dfrac{\mathrm{Ric}_g(v,v)}{g(v,v)}$

Some interesting second-order PDRs $\mathscr{R} \subseteq J^2 E_+$:

- $\mathscr{R} = \mathit{sec}{>}0$
- $\mathscr{R} = \mathit{ric}{>}0$
- $\mathscr{R} = \mathit{scal}{>}0$
- analogous relations with "$<0$"
- more generally, for $a, b \in \mathbb{R} \cup \{\pm\infty\}$: $a{<}\mathit{sec}{<}b$ etc.

E.g., $\mathit{sec}{>}0$ is defined to be

$$\left\{ j^2_x g \;\middle|\; g \in \Gamma(E_+),\ x \in M,\ \forall \sigma \in \mathrm{Gr}_2\, T_x M\colon\ \sec_g(\sigma) > 0 \right\}.$$

A solution of $\mathit{sec}{>}0$ is a Riem. metric $g$ on $M$ with $\sec_g > 0$.

All these relations are open and $\mathrm{Diff}(M)$-invariant.

Thus Gromov's h-principle theorems apply if $M$ is open.

Recall:

**The parametric h-principle for diff-inv. PDRs on open manifolds**

Let $E \to M$ be a natural fibre bundle over an open manifold.
Let $\mathscr{R} \subseteq J^r E$ be open and $\mathrm{Diff}(M)$-invariant.

Then $j^r : Sol(\mathscr{R}) \to \Gamma(\mathscr{R})$ is a homotopy equivalence.

In our situation, this becomes (as we'll see in a moment):

**Theorem**

*Let $a, b \in \mathbb{R} \cup \{\pm\infty\}$ satisfy $a < b$.*
*Let $M$ be an open manifold of dimension $\geq 2$.*
*Let $\mathscr{R}$ be one of the PDRs $a < \mathit{sec} < b$, $a < \mathit{ric} < b$, $a < \mathit{scal} < b$ on $M$.*

*Then $Sol(\mathscr{R})$ is contractible (w.r.t. the compact-open $C^r$-top.).*
*In particular it is nonempty and connected.*

To prove this, we have to check that $\Gamma(\mathscr{R})$ is contractible...

$$
\begin{tikzcd}
\mathscr{R} \subseteq J^2 E_+ \arrow[d, "p^2_1|_{\mathscr{R}}"'] \\
J^1 E_+ \arrow[d, "p^1_0"'] \\
E_+ \arrow[d, "p^0"'] \\
M
\end{tikzcd}
$$

fibre bundle with contractible fibres: convex, $\neq \varnothing$

contractible fibres (vector spaces)

contractible fibres: $\mathrm{Sym}^2_+ \, T^*_x M$ convex, $\neq \varnothing$

*Why $(p^2_1)^{-1}(\xi) \cap \mathscr{R}$ is nonempty and convex for each $\xi \in J^1 E_+$:*

Since $\dim M \geq 2$, metrics with $a < \sec_g < b$ exist locally.
$\Rightarrow \forall x \in M \colon \exists \xi_0 \in J^1_x E_+ \colon (p^2_1)^{-1}(\xi_0) \cap \mathscr{R} \neq \varnothing$.
All 1-jets of metrics at $x$ look the same in normal coordinates.
$\mathscr{R}$ is Diff-invariant $\Rightarrow \forall \xi \in J^1 E_+ \colon (p^2_1)^{-1}(\xi) \cap \mathscr{R} \neq \varnothing$.

### Curvature in local coordinates:

$(x^1, \dots, x^n)$: local coordinates; $\partial_i \equiv \frac{\partial}{\partial x^i}$

Riemann tensor: $\mathrm{R}^l_{ijk} = \partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik} + \sum_\mu \Gamma^\mu_{jk} \Gamma^l_{i\mu} - \sum_\mu \Gamma^\mu_{ik} \Gamma^l_{j\mu}$ ,

where $\Gamma^k_{ij} := \frac{1}{2} \sum_\mu g^{k\mu} \big( \partial_i g_{j\mu} + \partial_j g_{i\mu} - \partial_\mu g_{ij} \big)$

sectional curvature: $\sec(\mathrm{span}\{u, v\}) = \frac{\mathrm{R}(u,v,v,u)}{g(u,u)g(v,v) - g(u,v)^2}$

Ricci tensor: $\mathrm{Ric}_{ij} = \sum_k \mathrm{R}^k_{kij}$

scalar curvature: $\mathrm{scal} = \sum_{i,j} g^{ij} \mathrm{Ric}_{ij}$

For fixed 0th and 1st derivatives of the metric components $g_{ij}$,
all curvatures are **affine** functions of the 2nd derivatives of $g$.

Thus, for each $\xi \in J^1 E_+$,
$(p^2_1)^{-1}(\xi) \cap \mathscr{R}$ is a convex subset of the fibre $(p^2_1)^{-1}(\xi)$.

This proves our nonempty-and-convex-fibres claim.

Thus $\mathscr{R} \to M$ has contractible fibres $\Rightarrow$ $\Gamma(\mathscr{R})$ is contractible. $\square$

Let $M$ be a(n open) manifold. Let $A \subseteq M$ be a closed subset s.t. each connected component of $M \backslash A$ has an exit to infinity.

**The relative h-principle for diff-inv. PDRs on open manifolds**

Let $E \to M$ be a natural fibre bundle.
Let $\mathscr{R} \subseteq J^r E$ be open and $\mathrm{Diff}(M)$-invariant.
Let $\varphi_0 \in \Gamma(\mathscr{R})$ be holonomic on a neighbourhood of $A$.

Then there exists a continuous map $\varphi\colon [0,1] \to \Gamma(\mathscr{R})$ such that

- $\varphi(0) = \varphi_0$;
- $\forall t \in [0,1]\colon \varphi(t)|_A = \varphi_0|_A$;
- $\varphi(1)$ is holonomic.

**Corollary**

*Let $\dim M \geq 2$. Let $a, b \in \mathbb{R} \cup \{\pm\infty\}$ satisfy $a < b$.*
*Let $\mathscr{R}$ be one of the PDRs $a{<}\mathit{sec}{<}b$, $a{<}\mathit{ric}{<}b$, $a{<}\mathit{scal}{<}b$ on $M$.*
*Let $g_0$ be a Riemannian metric which solves $\mathscr{R}$ on $A$.*

*Then there is a metric $g$ on $M$ which solves $\mathscr{R}$ everywhere and is equal to $g_0$ on $A$.*

There's also a relative parametric h-principle on open manifolds, but let's not spell it out here.

**Could convex integration yield additional information?**

A priori clear: many of the PDRs $a < sec < b$, $a < ric < b$, $a < scal < b$ are not ample. Otherwise they would have solutions on arbitrary closed manifolds $M$, but there are obstructions:

- The solution spaces of $scal > 0$, $ric > 0$, $sec > 0$ are often empty.
- When they are nonempty, they are usually not connected.
- $Sol(sec < 0) = \varnothing$ if $M$ is closed and its universal cover is not diffeomorphic to $\mathbb{R}^n$ (e.g. because $\pi_1(M)$ is finite).
- Many open manifolds do not admit complete solutions of $scal > 0$, $ric > 0$, $sec > 0$, $sec < 0$.
  (E.g. $T^n \times \mathbb{R}$ does not admit a complete $scal > 0$-metric.)
- This shows also that the $C^0$-dense h-principle fails even on open manifolds for $scal > 0$, $ric > 0$, $sec > 0$, $sec < 0$.

But what about the remaining relations?

It's easy to see directly that none of our curvature PDRs is ample!

For $x \in M$ and $W \in \mathrm{Gr}_{n-1}\, T_x M$, let $J^2_{\perp W} E_+$ denote the set of equivalence classes of sections in $E_+ \to M$ w.r.t. the equivalence relation of having at $x$ the same 1-jet and the same $W$-directional derivatives of the 1-jet.
$p^2_{\perp W} \colon J^2_x E_+ \to J^2_{\perp W} E_+$ denotes the obvious projection.

By definition, one of our curvature PDRs $\mathscr{R}$ is *ample* iff:
$\forall W \in \mathrm{Gr}_{n-1}\, TM\colon\ \forall \xi \in J^2_{\perp W} E_+\colon\ (p^2_{\perp W})^{-1}(\xi) \cap \mathscr{R}$ is ample
(i.e., each of its connected comp.s has convex hull $(p^2_{\perp W})^{-1}(\xi)$).

For each of our PDRs $\mathscr{R}$ in dim. $\geq 2$ with $(a, b) \neq (-\infty, \infty)$, each $(p^2_{\perp W})^{-1}(\xi) \cap \mathscr{R}$ is $\neq \varnothing$ and contained in a half-space.

Thus ampleness fails.

Nevertheless...

Lohkamp's theorems (1992–1995) for $ric<0$ and $scal<0$;
we state only the $ric$ versions, $scal$ is analogous:

Let $M$ be a manifold of dimension $n \geq 3$.

**Theorem (existence $=$ $\pi_0$-surjective h-principle)**

*$M$ admits a complete Riemannian metric $g$ with $\mathrm{ric}_g < 0$.*

*Even better: For each $n \geq 3$, there are numbers $a_n < b_n < 0$ s.t. every $n$-mf. admits a complete Riem. metric $g$ with $a_n \leq \mathrm{ric}_g \leq b_n$.*

*Remark.* For $n \geq 5$, it is not known whether we can take $a_n = b_n$.

**Theorem (relative h-principle)**

*Let $c \in \mathbb{R}$. Let $A$ be a closed subset of $M$.*
*Let $g_0$ be a metric on a nbhd. of $A$ with $\mathrm{ric}_{g_0} < c$.*
*Then there is a metric $g$ on $M$ with $\mathrm{ric}_g < c$ and $g|_A = g_0|_A$.*

*Remark.* The same holds with $\mathrm{ric} \leq c$ instead of $\mathrm{ric} < c$.

**Theorem (parametric h-principle)**

*For every $c \in \mathbb{R}$, the space Sol($ric{<}c$) of metrics $g$ on $M$ with $\mathrm{ric}_g < c$ is contractible.*

**Theorem ($C^0$-dense h-principle)**

*For every $c \in \mathbb{R}$, the set Sol($ric{<}c$) is dense in the space $\mathrm{Metr}(M)$ of Riem. metrics w.r.t. the fine (= Whitney) $C^0$-topology.*

*Remark.* Using the Bochner formula

$$\mathrm{d}\mathrm{d}^*\alpha + \mathrm{d}^*\mathrm{d}\alpha = \nabla_g^*\nabla_g\alpha + \mathrm{Ric}_g(\alpha^\sharp, \_) \text{ for 1-forms } \alpha,$$

and the fact that $\mathrm{d}^*\alpha$ and $\nabla_g\alpha$ depend only on the 1-jet of $g$, one can show that

Sol($ric{\le}0$) and Sol($scal{\le}0$) are $C^1$-closed in $\mathrm{Metr}(M)$.

Hence Sol($ric{<}0$) and Sol($scal{<}0$) are not $C^1$-dense in $\mathrm{Metr}(M)$.

How does Lohkamp prove that every manifold of dimension $\geq 3$ admits a complete metric with $ric < 0$?

For each $n \geq 3$, consider the following statements:

$A(n)$: There exists a Riemannian metric $g$ on $\mathbb{R}^n$ which is equal to eucl outside the open unit ball $B^n$ and satisfies $\mathrm{ric}_g < 0$ on $B^n$.

$B(n)$: Each $n$-manifold $M$ admits a complete $\mathrm{ric} < 0$ metric.

Lohkamp's proof consists of 3 steps (we'll see no details today):

1. $A(3)$ is true.
2. $\forall n \geq 3$: $A(n) \Rightarrow B(n)$.
3. $\forall n \geq 3$: $B(n) \Rightarrow A(n+1)$.

inductive construction $\rightsquigarrow$ hard to understand the metrics for $n \gg 3$.

$C^0$-dense h-principle holds, $C^1$-dense fails... what about $C^{0,\alpha}$?
($C^{0,0} = C^0$; $C^{0,1}$-topology $= C^1$-topology)

For simplicity, let's consider only the relation $\mathit{scal} < c$ from now on.

Unlike $\mathit{ric} < c$, this makes sense also for semi-Riemannian metrics!

Difference to Riemannian (= positive definite) or neg. def. metrics:

- For $p, q$ with $pq \neq 0$, **not every** $(p+q)$-manifold admits a semi-Riem. metric of signature $(p, q)$.
- If a manifold $M$ admits a metric of signature $(p, q)$, the space $\mathrm{Metr}_{p,q}(M)$ of such metrics is usually not connected.

**Example:** Lorentzian (i.e. $q = 1$) metrics on closed 2-manifolds.

- Only the 2-torus and the Klein bottle admit Lor. metrics.
- The set of conn. comp.s of the space of Lor. metrics on $\mathbb{T}^2$ is in canonical bijective correspondence to $\mathbb{Z} \times \mathbb{Z}$.

Analogous to what we've seen before, Gromov's theorems yield:

**Theorem (h-principle on open manifolds)**

*Let $M$ be an open manifold of dimension $p+q \geq 2$.*
*Let $a, b \in \mathbb{R} \cup \{\pm\infty\}$ satisfy $a < b$.*
*Then the inclusion*
*from the space of $g \in \mathrm{Metr}_{p,q}(M)$ with $a < \mathrm{scal}_g < b$*
*to $\mathrm{Metr}_{p,q}(M)$ is a homotopy equivalence.*

**Theorem (relative h-principle on open manifolds)**

*Let $M$ be a(n open) manifold of dimension $p+q \geq 2$.*
*Let $A \subseteq M$ be a closed subset such that*
*each connected component of $M \backslash A$ has an exit to infinity.*
*Let $a, b \in \mathbb{R} \cup \{\pm\infty\}$ satisfy $a < b$.*
*Let $g_0 \in \mathrm{Metr}_{p,q}(M)$ satisfy $a < \mathrm{scal}_{g_0} < b$ on $A$.*
*Then the connected component of $\mathrm{Metr}_{p,q}(M)$ that contains $g_0$*
*contains also a metric $g$ with $g|_A = g_0|_A$*
*which satisfies $a < \mathrm{scal}_g < b$ on $M$.*

I proved:

**Theorem (semi-Riem. relative $C^{0,\alpha}$-dense h-principle for $scal < c$)**

*Let $c \in \mathbb{R}$. Let $A$ be a closed subset of a manifold $M$.*

*Let $g_0 \in \mathrm{Metr}_{p,q}(M)$ satisfy $\mathrm{scal}_{g_0}|_A < c$ [resp. $\mathrm{scal}_{g_0}|_A > c$].*

*Let $0 \leq \alpha < 1$, let $\mathscr{U} \subseteq \mathrm{Metr}_{p,q}(M)$ be a fine $C^{0,\alpha}$-nbhd. of $g_0$.*

*If $p \geq 3$, or $p \geq 1$ and $q \geq 2$, [resp. if $q \geq 3$, or $q \geq 1$ and $p \geq 2$,] then $\mathscr{U}$ contains a metric $g$ with $g|_A = g_0|_A$ and $\mathrm{scal}_g < c$ [resp. $\mathrm{scal}_g > c$].*

Thus, in dimension $p + q \geq 3$,
scalar curvature can be decreased **and** increased
except in the signatures $(p, 0)$, $(0, q)$ and maybe $(1, 2)$, $(2, 1)$.

*Idea of proof.* Let $U$ be an open nbhd. of $A$ with $\mathrm{scal}_{g_0}|_U < c$.

We choose locally finite covers $(\hat{B}_i)_{i\in\mathbb{N}}$ and $(B_i)_{i\in\mathbb{N}}$ of $M\setminus U$ by smooth open balls, with closures contained in $M\setminus A$, such that $\forall i$: $\mathrm{closure}(B_i) \subset \hat{B}_i$.

Then we apply the following lemma iteratively to each $i \in \mathbb{N}$:

**Lemma**

*Let $\varepsilon \in \mathbb{R}_{>0}$, let $c \in \mathbb{R}$. Let $M := \mathbb{R}^{p+q}$, let $g_0 \in \mathrm{Metr}_{p,q}(M)$.*

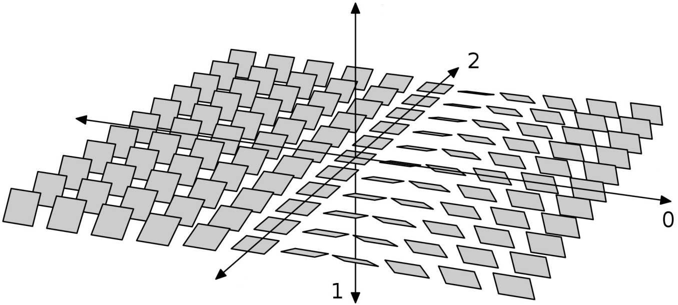
*Let $\hat{B}, B \subseteq M$ be open smooth balls with $\mathrm{closure}(B) \subset \hat{B}$.*

*Let $0 \le \alpha < 1$, let $\mathscr{U} \subseteq \mathrm{Metr}_{p,q}(M)$ be a fine $C^{0,\alpha}$-nbhd. of $g_0$.*

*If $p \ge 3$, or $p \ge 1$ and $q \ge 2$, then there is a metric $g \in \mathscr{U}$ with $g|_{M\setminus\hat{B}} = g_0|_{M\setminus\hat{B}}$ and $\mathrm{scal}_g \le \mathrm{scal}_{g_0} + \varepsilon$ and $\mathrm{scal}_g|_B \le c - 1$.*

This proves the theorem. It remains to prove the lemma.

This involves a picture you might find familiar:




We choose on $M = \mathbb{R}^{p+q}$

- a $g_0$-orthonormal frame $(e_0, \ldots, e_{n-1})$ such that the $\varepsilon_i := g_0(e_i, e_i) \in \{\pm 1\}$ satisfy $\varepsilon_1 = \varepsilon_2$;
- a fct. $\omega_0 \in C^\infty(M, \mathbb{R})$ s.t. $d\omega_0(e_0) > 0$, $\forall i \geq 1$: $d\omega_0(e_i) = 0$;
- a cutoff $\beta \in C^\infty(M, [0,1])$ with $\beta|_B = 1$ and $\beta|_{M \setminus \hat{B}} = 0$.

For $C \in \mathbb{R}$, consider $\omega \in C^\infty(M, \mathbb{R})$ given by $\omega(x) := \omega_0(Cx)$.

For $a \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, consider $f := 1 + a\beta \in C^\infty(M, \mathbb{R}_{>0})$.

For $C \in \mathbb{R}$, consider $\omega \in C^\infty(M, \mathbb{R})$ given by $\omega(x) := \omega_0(Cx)$.
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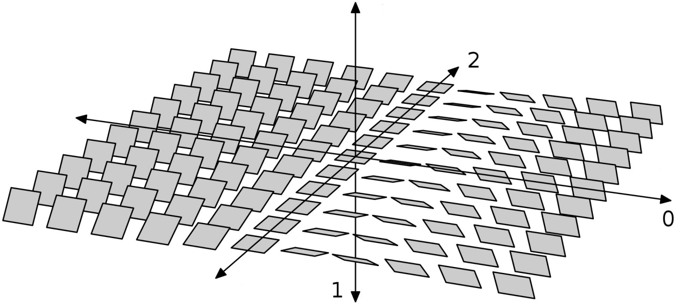


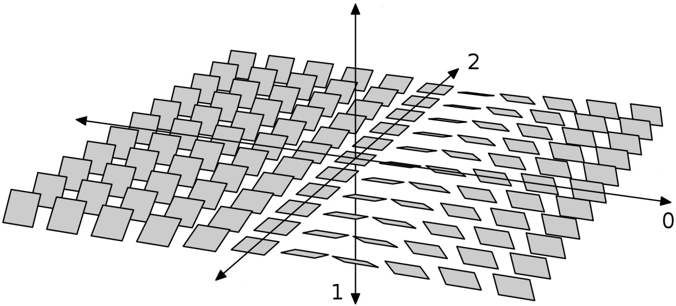

We define another $g_0$-orthonormal frame $(\overline{e}_0, \dots, \overline{e}_{n-1})$ by

$$\begin{aligned} \overline{e}_i &:= e_i && \text{if } i \notin \{1,2\} \\ \overline{e}_1 &:= \cos(\omega)e_1 + \sin(\omega)e_2 && \\ \overline{e}_2 &:= -\sin(\omega)e_1 + \cos(\omega)e_2 && . \end{aligned}$$

Now we modify the frame $(\overline{e}_0, \dots, \overline{e}_{n-1})$ slightly:

$$\hat{e}_i := \overline{e}_i \quad \text{if } i \neq 1\,, \qquad \hat{e}_1 := f\overline{e}_1 \quad .$$

For $C \in \mathbb{R}$, consider $\omega \in C^\infty(M, \mathbb{R})$ given by $\omega(x) := \omega_0(Cx)$.
For $a \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, consider $f := 1 + a\beta \in C^\infty(M, \mathbb{R}_{>0})$.




We define $g$ by declaring $(\hat{e}_0, \dots, \hat{e}_{n-1})$ to be $g$-orthonormal.

If $|a|$ is small, then $g$ is obviously $C^0$-close to $g_0$.
If $|aC|$ is large, then $\mathrm{scal}_g \leq \mathrm{scal}_{g_0} + \varepsilon$ and $\mathrm{scal}_g|_B \leq c - 1$.

By choosing $C > 0$ depending on $a > 0$ such that
$|aC|$ is large but $|aC^\alpha|$ is small,
we can make $g$ even $C^{0,\alpha}$-close to $g_0$ for any $\alpha < 1$. $\square$