LONDON h-PRINCIPLE LEARNING SEMINAR, SPRING 2013

TALK 1 — 17/1/2013 — C. WENDL

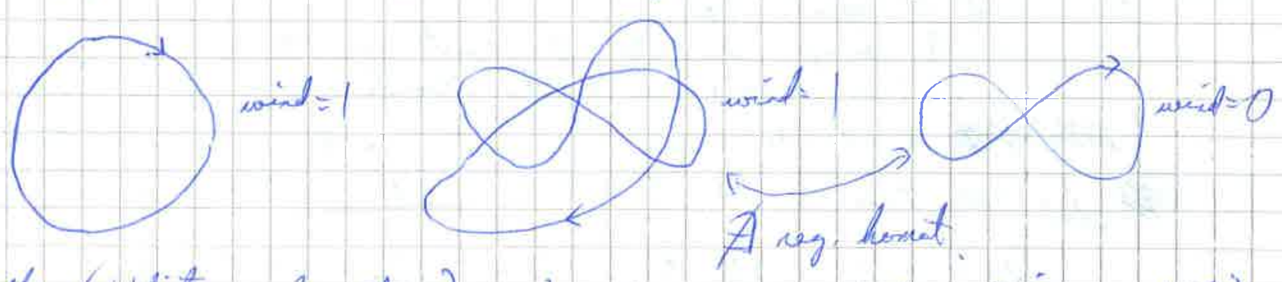
www.homepages.ucl.ac.uk/~ucahcwe/h-principle.html

SAMPLE PROBLEMS

(1) ~~[illegible]~~ $f_i : S^1 \looparrowright \mathbb{R}^2$, $i = 0, 1$

$\exists$? regular homotopy of f_0 to f_1? (i.e. homot. through immersions)

$\dot{f}_i : S^1 \to \mathbb{R}^2 \setminus \{0\}$ $\rightsquigarrow$ can define $\mathrm{wind}(\dot{f}_i) \in \mathbb{Z}$.



thm (Whitney–Graustein)(1937): $f_0 \underset{\text{reg. h.}}{\sim} f_1$ iff $\mathrm{wind}(\dot{f}_0) = \mathrm{wind}(\dot{f}_1)$.

(1′) M, N mflds, $\dim M < \dim N$ or M open.

$f : M \looparrowright N$ immersion $\rightsquigarrow$ $\left\{ \begin{array}{l} f : M \to N \text{ map} \\ \Phi : TM \to f^*TN \\ \text{fiberwise inj. bndl map} \\ (\text{here } \Phi = df) \end{array} \right\}$ =: "formal immersion"

thm (Smale–Hirsch)(1950's): the natural map

$$\{\text{immersions } f : M \looparrowright N\} \longrightarrow \left\{\text{formal immersions } \begin{pmatrix} f : M \to N \\ \Phi : TM \hookrightarrow f^*TN \end{pmatrix}\right\}$$

$$f \longmapsto (f, df)$$

is a homotopy equivalence.

cors: – surj. on π_0 $\Rightarrow$ any formal immersion $\simeq$ an immersion.

– inj. on π_0 $\Rightarrow$ 2 immersions reg. homot. iff homot. as formal immersions.

ex: $\{S^2 \overset{f}{\looparrowright} \mathbb{R}^3\} \sim \{\underbrace{S^2 \times \mathbb{R}}_{\text{"nbhd of } S^2\text{"}} \overset{f}{\looparrowright} \mathbb{R}^3 \text{ orientation preserving}\}$

$T(S^2 \times \mathbb{R}) = (S^2 \times \mathbb{R}) \times \mathbb{R}^3$, $\quad f^*T\mathbb{R}^3 = (S^2 \times \mathbb{R}) \times \mathbb{R}^3$.

$\therefore$ $\{\text{formal immersions}\} = \{\Phi : S^2 \times \mathbb{R} \to GL_+(\mathbb{R}, 3)\} \underset{\text{h.e.}}{\simeq} \{\Phi : S^2 \to GL_+(\mathbb{R}, 3)\}$

$\Rightarrow$ $\pi_0(\{\text{formal immersions}\}) = \pi_2(GL_+(\mathbb{R}, 3)) = \pi_2(SO(3)) = 0$.

$\therefore$ any 2 formal immersions are homotopic

$\Rightarrow$ cor ("SMALE PARADOX"): all $S^2 \looparrowright \mathbb{R}^3$ are regularly homotopic.

(See sphere inversion video — linked from website)

(2) $\dim M = 2n$, M oriented. $\exists$? symplectic form on M? (i.e. 2-form ω s.t. $d\omega = 0$, $\omega^n > 0$)

observation: given ω symp., $\exists$ nonempty, contractible space
$$\{ J \in \Gamma(\mathrm{End}(TM)) \mid J^2 = -\mathrm{Id} \ \& \ \omega(\cdot, J\cdot) \text{ is a Riem. metric} \}$$
"almost complex structures compatible w/ ω"

thm: If M open, then

- $\exists$ symp. ω $\iff$ $\exists$ almost cpx. J (and cohomologous)
- ω_0, ω_1 symp. are homot. through symp. forms $\iff$ $\exists$ homot. a.c.s.'s J_0, J_1 compat. w/ ω_0, ω_1 resp.

rk: Not true if M closed, e.g. must have $[\omega] \in H^2(M)$ s.t. $[\omega]^n \neq 0$. With this condition: ~~still~~ not true for $\dim = 4$ (Taubes), open for $\dim > 4$!

(3) Let $(S^2, g_0) = \partial B \subset (\mathbb{R}^3, g_{Eucl})$.

Any C^2-isometric $(S^2, g_0) \hookrightarrow (\mathbb{R}^3, g_{Eucl})$ is essentially standard, (Gauss map!)

$\Rightarrow \nexists\, C^2$-isometric $(S^2, g_0) \hookrightarrow (B^3_\varepsilon, g_{Eucl})$ for $\varepsilon > 0$ small.

thm (Nash-Kuiper 1954): If $\dim M < n$, then for any Riem. metric g, any C^0 $f : M \to \mathbb{R}^n$ is homot. to a C^1-isometric $f_0 : (M, g) \hookrightarrow (\mathbb{R}^n, g_{Eucl})$.
Under a technical condition, can also assume f_0 C^0-close to f.

cor: $\exists\, C^1$-isom. $(S^2, g_0) \hookrightarrow (B^3_\varepsilon, g_{Eucl})$ $\forall\, \varepsilon > 0$

(4) thm (Lohkamp): For $\dim M \geq 3$, every Riem. metric is C^0-close to one with $\mathrm{Ric} < 0$ or $\mathrm{Scal} < 0$.
(not true for curvature > 0 if M closed!)

GENERAL FRAMEWORK

$V :=$ mfd, $X \to V$ fiber bndl.

$r \in \mathbb{N}$, $x \in V$. An r-jet at x is an equiv. class of sections of X on a nbhd of x, w/ $s \sim s'$ iff their derivatives up to order r (in some local coords) match.

$\rightsquigarrow$ fiber bndls $X^{(r)} = \{r\text{-jets}\} \to V$

$$\underbrace{X^{(r)} \xrightarrow{\text{"forget } r\text{th deriv."}} X^{(r-1)} \longrightarrow \cdots \longrightarrow X^{(1)} \longrightarrow X^{(0)}}_{\text{these are affine bndls}} = X.$$

eg: maps $f: V \to N$ are sections of $X := V \times N \to V$

$x \in V$, $p = (x,y) \in X_x$, then $X^{(1)}_{(x,y)} = \mathrm{Hom}(T_x V, T_y N)$.

If $N = \mathbb{R}$, this means $X^{(1)} = (V \times \mathbb{R}) \times T^*V$.

notation: $\mathrm{Sec}\, X := \{$sections of $X\}$ — we'll usually assume at least C^r.
— for now, C^∞.

$\exists$ natural map $J^r : \mathrm{Sec}\, X \to \mathrm{Sec}\, X^{(r)} : f \mapsto [f]$
"the" r-jet of f

Say $F \in \mathrm{Sec}\, X^{(r)}$ is holonomic if $F = J^r f$ for some $f \in \mathrm{Sec}\, X$

· a partial differential relation (of order r) imposed on sections of $X \to V$ is a subset $\mathcal{R} \subset X^{(r)}$.

- a (genuine) solution of $\mathcal{R}$ is any $f \in \mathrm{Sec}\, X$ s.t. $J^r f$ has image in $\mathcal{R}$.

- a formal solution of $\mathcal{R}$ is any $F \in \mathrm{Sec}\, X^{(r)}$ s.t. image $\subset \mathcal{R}$,
i.e. $F \in \mathrm{Sec}\, \mathcal{R}$.

$\mathrm{Hol}\, X^{(r)} := \{ F \in \mathrm{Sec}\, X^{(r)} \mid F \text{ holonomic} \} \overset{1:1}{\longleftrightarrow} \mathrm{Sec}\, X$

$\{$genuine solutions$\} \overset{1:1}{\longleftrightarrow} \mathrm{Hol}\, \mathcal{R} := \mathrm{Sec}\, \mathcal{R} \cap \mathrm{Hol}\, X^{(r)}$.

defn: We say $\mathcal{R}$ satisfies the h-principle if

all $F \in \mathrm{Sec}\, \mathcal{R}$ are homot. in $\mathrm{Sec}\, \mathcal{R}$ to a genuine sol,
i.e. natural ~~map~~ inclusion $\mathrm{Hol}\, \mathcal{R} \hookrightarrow \mathrm{Sec}\, \mathcal{R}$ is surj. on π_0.

Parametric h-principle: $\mathrm{Hol}\, \mathcal{R} \hookrightarrow \mathrm{Sec}\, \mathcal{R}$ is a (weak) homot. equiv.,
i.e. isomorphism on π_k $\forall k \geq 0$.

$\Leftrightarrow$ $\forall k \geq 0$ & $\varphi_0 : (\mathbb{D}^k, S^{k-1}) \to (\mathrm{Sec}\, \mathcal{R}, \mathrm{Hol}\, \mathcal{R})$,
$\exists$ a homot. $\varphi_t : (\mathbb{D}^k, S^{k-1}) \to (\mathrm{Sec}\, \mathcal{R}, \mathrm{Hol}\, \mathcal{R})$
fixed at S^{k-1} s.t. $\varphi_1(\mathbb{D}^k) \subset \mathrm{Hol}\, \mathcal{R}$.

C^0-dense h-principle: h-principle holds & homot. $F_t \in \mathrm{Sec}\, \mathcal{R}$ can be chosen s.t. underlying sections $f_t \in \mathrm{Sec}\, X$ C^0-close.