h-principle seminar

TALK 2, 31/1/2013 — J. LOTAY

The holonomic approximation thm

notation: V mfld, $X \to V$ fibre bndl

($r \in \mathbb{N}$), $X^{(r)}$ bndl of r-jets

$f \in \mathrm{Sec}(X) \to J^r f \in \mathrm{Sec}(X^{(r)})$. $F \in \mathrm{Sec}(X^{(r)})$ is holonomic if $F = J^r f$.

Q1: Can we approximate $F \in \mathrm{Sec}\, X^{(r)}$ by $J^r f$ for some $f \in \mathrm{Sec}\, X$?

ex1: $u, v : [0,1] \to \mathbb{R}$; can we find, given $\varepsilon > 0$, $f : [0,1] \to \mathbb{R}$ s.t. $|f(x) - u(x)|, |f'(x) - v(x)| < \varepsilon \quad \forall\, x \in [0,1]$?

NO: take $v = 0 \Rightarrow |f(1) - f(0)| < \varepsilon$.

Q2: Given a submfd $A \subseteq V$ (of codim > 0) & a section $F : \mathcal{O}p\, A \to X^{(r)}$, can approximate F by $J^r f$?

$\mathcal{O}p\, A$ = "an open nbhd of A"

ex2: Let $I = [0,1] \times \{0\} \subseteq \mathbb{R}^2$ & $u, v_1, v_2 : \mathcal{O}p\, I \to \mathbb{R}$, can we find f s.t. $\left(f, \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}\right)$ is ε-close to (u, v_1, v_2)?

NO again.

But now $\exists$ freedom to wiggle I!
Given $\varepsilon, \delta > 0$, $\exists\ \varphi : [0,1] \to (-\delta, \delta)$ & $f : \mathcal{O}p\{\text{graph of } \varphi\} \to \mathbb{R}$ s.t. $\left(f, \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}\right)$ is ε-close to (u, v_1, v_2).
(nbhd of graph need not contain I!)
but can fix $\varphi = 0$ at endpts

How?

For $t \in [0,1]$, defn $f_t(x_1, x_2) := u(t,0) + v_1(t,0)(x_1 - t) + v_2(t,0)x_2$

Now $\left(f_t, \frac{\partial f_t}{\partial x_1}, \frac{\partial f_t}{\partial x_2}\right)$ approximates (u, v_1, v_2) near $(t, 0)$.

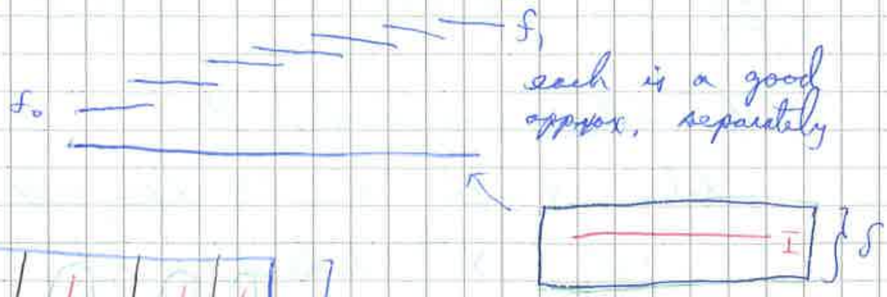
Making δ smaller if needed, can ensure that

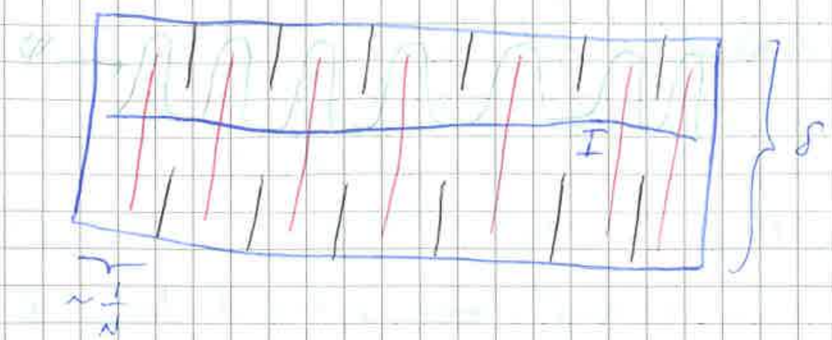
$$\left| f_t(x) - u(x) \right|, \left| \frac{\partial f_t(x)}{\partial x_1} - v_1(x) \right|, \left| \frac{\partial f_t}{\partial x_2}(x) - v_2(x) \right| << \varepsilon$$

if $|x - (t,0)| < \delta$.

Pick $N \in \mathbb{N}$ s.t. $\frac{1}{N} << \delta$ and consider $f_{k/N}$, $k = 0, \dots, N$.

Schematically:

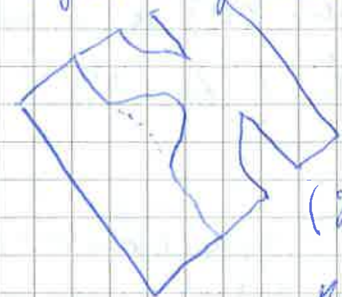




Consider region removing these vertical lines $=: \Omega$.

Defn f on Ω as $f_{k/N}$ on each  (where it's a good approximation);

then adjust as follows:



(graph of f — smooth only because we removed those vertical lines)

Now draw wiggling fn. φ so as to avoid vertical lines.

<u>holonomic approximation thm</u>: Let $A \subset V$ be a polyhedron (a subcomplex of a triangulation of V) of codim > 0, and $F : Op\, A \to X^{(r)}$ a section. Then for any $\varepsilon, \delta > 0$, $\exists$ a diffeotopy $h^\tau : V \to V$, $\tau \in [0,1]$ C^0-close to Id ("δ-small": $dist(h^\tau(v), v) < \delta$ $\forall$ $v \in V$, $\tau \in [0,1]$) & a holonomic $\tilde{F} : Op\, h^1(A) \to X^{(r)}$ s.t. $dist(\tilde{F}(v), F(v)) < \varepsilon$ $\forall$ $v \in Op\, h^1(A)$.

"<u>pf</u>": - Any $F \in Sec\, X^{(r)}$ is holonomic over a point: given $v \in V$, $\exists$ $f : Op\, v \to X$ s.t. $J^r f(v) = F(v)$.

- Any $F \in Sec\, X^{(r)}$ is fibrewise holonomic w.r.t. $Id : V \to V$ (the "stupid fibration"), i.e. holonomic on every fibre.

By induction over the skeleton of A & the fact that $X \to Y$ is trivial over simplices, it is enough to prove a statement for cubes rel. boundary:

$I^k = [0,1]^k \times \{0\} \subseteq \mathbb{R}^n$, $k < n$,

$F : \mathcal{O}p\, I^k \longrightarrow \underset{\left((\mathbb{R}^n \times \mathbb{R}^2)^{(r)} \right)}{\underset{\parallel}{J^r(\mathbb{R}^n, \mathbb{R}^2)}}$, holonomic over $\mathcal{O}p\, \partial I^k$.

Then $\forall\ \varepsilon, \delta > 0$, $\exists$ a δ-small diffeo. $h : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ of form $h(x_1, \dots, x_n) = (x_1, \dots, x_{n-1}, \varphi(x_1, \dots, x_n))$

& $\tilde{F} : \mathcal{O}p\, h(I^k) \longrightarrow J^r(\mathbb{R}^n, \mathbb{R}^2)$

s.t. (a) $h = \mathrm{Id}$ & $\tilde{F} = F$ on $\mathcal{O}p\, \partial I^k$

(b) $\mathrm{dist}(\tilde{F}(x), F(x)) < \varepsilon$ $\forall$ $x \in \mathcal{O}p\, h(I^k)$

(c) $\tilde{F}$ is holonomic.

pf is by induction.

<u>Inductive hypothesis</u>: $l = 0, \dots, k$

$A^{(l)}$: $\exists$ h, $\tilde{F}^l$ as above s.t. (a) & (b) hold, & this:

(d) $\tilde{F}^l$ is fibrewise holonomic w.r.t. $\pi_{k-l} \circ h : I^k \longrightarrow I^{k-l}$

(π_{k-l} = proj. to 1st $k-l$ coords.)

$A^{(k)} \Rightarrow$ (c).

$A^{(0)}$: is true: can take $h = \mathrm{Id}$ & $\tilde{F}^0 = F$

(fibrewise holonomic w.r.t. Id)

For $k=1$, $n=2$, $q=1$, $r=1$: our ex. 2 gives $A^{(0)} \Rightarrow A^{(1)}$.
The same argument works $\forall$ k, n, q, r to show $A^{(0)} \Rightarrow A^{(1)}$ in general.

$A^{(l)} \Rightarrow A^{(l+1)}$ is a bit subtler for $l > 0$, b.c. my $\tilde{F}^l$ is only defined on a "wiggled" cube, $h(I^k)$.

Consider

$h_* : J^r(\mathbb{R}^n, \mathbb{R}^2) \longrightarrow J^r(\mathbb{R}^n, \mathbb{R}^2)$.

Let $G = (h_*)^{-1} \tilde{F}^l$. This is defined on the <u>undeformed</u> cube & is fibrewise hol. w.r.t. $\pi_{k-l} : I^k \longrightarrow I^{k-l}$.

ex 2^+ $\Rightarrow$ hol. approx. $\tilde{G}^{l+1}$ over deformed cube $g(I^k)$, g is δ-small.

If $\tilde{G}^{l+1}$ is suff. C^0-close to G, then $\tilde{F}^{l+1} = h_* \tilde{G}^{l+1}$ is the required section on $\mathcal{O}p\, h \circ g(I^k)$

$\therefore$ $A^{(l)} \Rightarrow A^{(l+1)}$.

$\boxtimes$

The following are proved analogously:

<u>relative holonomic approximation thm:</u> Let $B \subseteq A \subseteq V$, B a subpolyhedron of a polyhedron A, $\operatorname{codim} A > 0$. Let $F : \mathcal{O}p\, A \to X^{(r)}$ s.t. is hol. over $\mathcal{O}p\, B$, then $\forall\, \varepsilon, \delta > 0$, $\exists$ δ-small diffeotopy $h^\tau : V \to V$, $\tau \in [0,1]$ w/ $h^\tau = \mathrm{Id}$ on $\mathcal{O}p\, B$ & holonomic $\tilde{F} : \mathcal{O}p\, h^1(A) \to X^{(r)}$ s.t. $\tilde{F}(v) = F(v)$ $\forall\, v \in \mathcal{O}p\, B$ & $\operatorname{dist}(\tilde{F}(v), F(v)) < \varepsilon$ $\forall\, v \in \mathcal{O}p\, h^1(A)$.

<u>parametric holonomic approximation thm:</u> Let $A \subseteq V$ polyhedron of $\operatorname{codim} > 0$ & $F_z : \mathcal{O}p\, A \to X^{(r)}$ sections $\forall\, z \in I^m_{(m \in \mathbb{N})}$ w/ $F_z : \mathcal{O}p\, A \to X^{(r)}$ holonomic $\forall\, z \in \mathcal{O}p\, \partial I^m$, then $\forall\, \varepsilon, \delta > 0$, $\exists$ δ-small diffeotopies $h_z^\tau : V \to V$, $\tau \in [0,1]$, $z \in I^m$ & holonomic $\tilde{F}_z : \mathcal{O}p\, h_z^1(A) \to X^{(r)}$, $z \in I^m$, s.t. $h_z^\tau = \mathrm{Id}$ & $\tilde{F}_z = F_z$ $\forall\, z \in \mathcal{O}p\, \partial I^m$ & $\operatorname{dist}(\tilde{F}_z(v), F_z(v)) < \varepsilon$ $\forall\, v \in \mathcal{O}p\, h_z^1(A)$, $z \in I^m$.