h-principle seminar

TALK 3 - 7/2/2013 — J. NORDSTRÖM

h-principle for open Diff-invariant relations

defn: A fibre bundl $X \to V$ is natural if $\exists$ choice of lift of action of $\mathrm{Diff}(V)$ to fibre-preserving diffeos of X

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X \arrow[d] \\ V \arrow[r, "f"] & V \end{tikzcd}$$

ex: $X = TM$, $\Lambda^r M$, trivial fibr. $V \times W$

If $X \to V$ natural then so is $X^{(r)} \to V$.

$\rightsquigarrow$ notion of $\mathrm{Diff}(V)$-invariant PDR's $\mathcal{R} \subset X^{(r)}$.

ex: Recall $J^1(V,W) := (V \times W)^{(1)} = \{(x,y,f) \mid x \in V,\ y \in W,\ f : T_xV \to T_yW\}$

The relations

$$\mathcal{R}_{imm}(V,W) = \{(x,y,f) \mid f \text{ injective}\}$$

$$\mathcal{R}_{sub}(V,W) = \{(x,y,f) \mid f \text{ surjective}\}$$

are each $\mathrm{Diff}(V)$-inv. & open.

Local h-principle

prop: Let $X \to V$ natural fibre bundl, $\mathcal{R} \subset X^{(r)}$ open & $\mathrm{Diff}(V)$-inv. Then $\mathcal{R}$ satisfies the local h-princ., i.e. near any polyhedron $A \subset V$ with codim $A > 0$, any $F \in \mathrm{Sec}_{Op\,A}\,\mathcal{R}$ is homotopic to some $G \in \mathrm{Hol}_{Op\,A}\,\mathcal{R}$ (through $\mathrm{Sec}_{Op\,A}\,\mathcal{R}$)

parametric version: $\pi_n \mathrm{Hol}_{Op\,A}\,\mathcal{R} \to \pi_n \mathrm{Sec}_{Op\,A}\,\mathcal{R}$

pf (basic version): For any $\varepsilon, \delta > 0$, holonomic approx. thm $\Rightarrow$

- δ-small diffeotopy $h^\tau : V \to V$, $h^0 = \mathrm{Id}$,
- $\tilde{F} \in \mathrm{Hol}_{Op\,h^1(A)}\,\mathcal{R}$ ε-close to $F|_{Op\,h^1(A)}$.

Let $G = (h^1)^* \tilde{F}$, where $(h^1)^* = (h^1)_*^{-1} : X^{(r)} \to X^{(r)}$

(may assume $h^1(Op\,A) \subseteq Op\,h^1(A)$).

$\mathrm{Diff}(V)$-invariance $\Rightarrow$ G is also a section of $\mathcal{R}$.

ε-closeness + $\mathcal{R}$ open $\Rightarrow$ $\exists\ \tilde{F}^t \in \mathrm{Sec}_{Op\,h^1(A)}\,\mathcal{R}$ homotopy with $\tilde{F}^0 = F|_{Op\,h^1(A)}$, $\tilde{F}^1 = \tilde{F}$.

$\mathrm{Diff}(V)$-invariance $\Rightarrow$ concatenating $\tau \mapsto (h^\tau)^* F$ & $t \mapsto (h^1)^* \tilde{F}^t$, we get a homotopy from F to G.

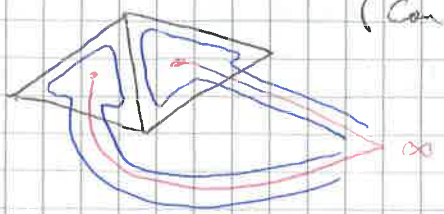
remark (at least when $X \to V$ is trivial): $\exists$ also C^0-dense h-princ.

Can take $bs\, G$ ("base" := section of X underlying $G \in Sec\, X^{(r)}$)
ε-close to $bs\, F$ in $Sec_{Op A}\, X$.

h-principle on open mfds : "open" = "no closed components"

Any open mfd V contains a core : a polyhedron K of codim > 0 s.t. $\exists$ diffeotopy $h^t : V \to V$,
$h^0 = Id$, $h^t|_K = Id$, $h^1(V) \subseteq Op\, h^1(K)$.

(Can take $K = (n-1)$-skeleton w/ holes in $(n-1)$-cells)



thm: Let V be open mfd, $X \to V$ natural, $R \subset X^{(r)}$ open & $Diff(V)$-inv.

Then R satisfies the h-principle:

basic version: $F \in Sec\, R$ is homotopic to some $G \in Hol\, R$

pf: K = core; apply local h-principle $\Rightarrow$ holonomic section over nbhd of K. Let $G = (h^1)^* G_K$ G_K
(h^t as in defn. of core), do construction as before. ☒

- parametric version also still works (adding a layer of notation)

- C^0-dense? NO: expansion from $Op\, K$ to V kills this!

- relative: YES, if each component of $V \setminus B$ (B = subset we want to fix) reaches ∞.

immersions

$R_{imm}(V, W)$ open $Diff(V)$-inv $\rightsquigarrow$ h-principle for immersions $V \hookrightarrow W$ when V is open (Smale-Hirsch).

micro-extension trick

Let $\dim V = n$, $\dim W = q$.
If $n < q$, then immersion $f : V \hookrightarrow W$ $\Longleftrightarrow$
immersion of tubular nbhd of V,
i.e. total space of normal bdl to TV in f^*TW

thm: If $n < q$, then C^0-dense h-principle holds for immersions $V \hookrightarrow W$.

pf: Let $F \in Sec\, R_{imm}(V, W)$ (F is a "formal immersion")

$f = bs\, F : V \to W$. At each $x \in V$, F identifies

$T_x V$ with a subspace of $(f^* TW)_x = T_{f(x)} W$.
Pick arbitrary metric & let

$E_x :=$ ortho-complement. Let $M :=$ total space of $E \to V$.

$T_{(x,e)} M = \pi^* T_x V \oplus \pi^* E_x$, so F lifts to

$\tilde{F} : T_{x,e} M \to T_{f(x)} W$. Then $\tilde{F} \in \mathrm{Sec}\, \mathcal{R}_{imm}(M, W)$.

Local C^0-dense h-princ. near 0-section $V \subset M \Rightarrow$

$G \in \mathrm{Hol}_{Op(V)}\, \mathcal{R}_{imm}(M, W)$ homotopic & ~~C^0-close~~ C^0-close ~~to~~ $\tilde{F}$.

Then $G|_V \in \mathrm{Hol}\, \mathcal{R}_{imm}(V, W)$ is homotopic & C^0-close to F

cor: If $n < q$ (or $n = q$ & V open), then $f : V \to W$ is homotopic to an immersion iff $f^* TW \oplus N_V$ stably isomorphic to some rank $q-n$ bndl E (N_V = stable normal bndl)

pf: $\exists$ formal immersion F w/ bs $F = f \iff f^* TW \cong TV \oplus E$

$\iff f^* TW \underset{stab}{\cong} TV \oplus E$ ($\Leftarrow$ rank q bndls over dim n space)

$\iff f^* TW \oplus N_V \underset{stab}{\cong} E$.

submersions (Phillips submersion thm):

If V is open, then h-princ. holds for submersions $V \to W$.

cor: $\underset{open}{f : V} \to W$ homotopic to a submersion iff

$TV \oplus f^* N_W$ stably isomorphic to rank $n-q$ bndl.

ex: {formal submersions $V \to \mathbb{R}$} $\cong$ {nonvanishing vector fields}

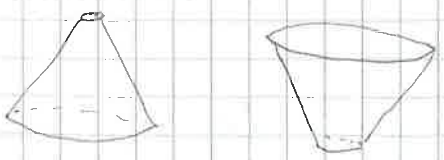
gradient for a genuine submersion

If $A = \{R_1 < r < R_2\} \subset \mathbb{R}^2$, then

{homotopy classes of submersions $A \to \mathbb{R}$} $\cong \pi_1(\mathbb{R}^2 \setminus \{0\}) \cong \mathbb{Z}$

$\rightsquigarrow$ cone inversion: $\pm r : A \to \mathbb{R}$ have gradients with same winding $\# = 1$,

$\therefore$ are homotopic through submersions.



Can invert cone without making it horizontal at any point.

approximation by closed forms

Exterior derivative d on V can be written as a composition

$$\mathrm{Sec}\,\Lambda^p V \xrightarrow{J^1} \mathrm{Sec}\,(\Lambda^p V)^{(1)} \xrightarrow{\tilde{D}} \mathrm{Sec}\,\Lambda^{p+1} V$$

where $\tilde{D}$ is induced by a bundle homomorphism

$$D : \Lambda^p V^{(1)} \longrightarrow \Lambda^{p+1} V \qquad \text{(affine linear map)}$$

Between fibres over $\alpha \in \Lambda^p V$ and $x = bs\,\alpha \in V$,
D is affine, modelled on the natural map

$$\mathrm{Hom}\,(T_x V, \Lambda^p T_x^* V) \longrightarrow \Lambda^{p+1} T_x^* V$$

any $\omega \in \mathrm{Sec}\,\Lambda^{p+1} V$ has a formal primitive $F_\omega \in \mathrm{Sec}\,\Lambda^p V^{(1)}$,
i.e. $\tilde{D} F_\omega = \omega$. ($\omega$ exact $\Leftrightarrow \exists$ holonomic F_ω)

For any $\alpha \in \mathrm{Sec}\,\Lambda^p V$, can choose F_ω s.t. $bs\,F_\omega = \alpha$.

lemma: Let $K \subset V$ polyhedron of codim > 0, $\omega \in \mathrm{Sec}\,\Lambda^p V$,
$\delta, \varepsilon > 0$ (& $\alpha \in \mathrm{Sec}\,\Lambda^{p-1} V$). Then $\exists$:

- δ-small diffeotopy $h^\tau : V \longrightarrow V$, $h^0 = \mathrm{id}$
- $\tilde{\alpha} \in \mathrm{Sec}\,\Lambda^{p-1} V$ s.t. $\tilde{\omega} := d\tilde{\alpha}$ is ε-close to ω on $h^1(K)$ (& $\tilde{\alpha}$ ε-close to α).

pf: Take $F_\omega \in \mathrm{Sec}\,\Lambda^{p-1} V^{(1)}$ formal primitive of ω
(s.t. $bs\,F_\omega = \alpha$), then hol. approx. $\Rightarrow$
$\exists$ h^τ & $\tilde{F} \in \mathrm{Hol}_{\mathcal{O}p\,h^1(K)}\,\Lambda^{p-1} V^{(1)}$ ε-close to $F_\omega|_{\mathcal{O}p\,h^1(K)}$.

Let $\tilde{\alpha} = bs\,\tilde{F}$, extend smoothly from $h^1(K)$ to V

cor: Fix a class $a \in H^p_{dR}(V)$. Then $\exists$ diffeotopy $h^\tau : V \longrightarrow V$,
a closed $\tilde{\omega} \in \mathrm{Sec}\,\Lambda^p V$ s.t. $[\tilde{\omega}] = a$ & $\tilde{\omega}$ ε-close to ω
on $\mathcal{O}p\,h^1(K)$.

pf: Pick $\Omega \in \mathrm{Sec}\,\Lambda^p V$ representing a, apply lemma to $\omega - \Omega$. ☒

$\exists$ parametric versions & so on.

h-principle for symplectic forms

For $R \subset \Lambda^p V$ & $a \in H^p_{dR}(V)$ let $\mathrm{Clo}_a R \subset \mathrm{Sec}\,R$
be subspace of closed p-forms in the class a.

(note: $\mathrm{Clo}_a R$ cannot be described as sections of some PDR — it's just a set of sections.)

prop: Let V = open mfd, $a \in H^p_{dR}(V)$, $\mathcal{R} \subset \Lambda^p V$ open & Diff(V)-invt. Then $\mathrm{Clo}_a \mathcal{R} \hookrightarrow \sec \mathcal{R}$ is a homot. equiv.

cor: h-principle for symp. forms on open mfd in a fixed cohomology class.