h-principle seminar — TALK 7 — 7/3/2013 — C. WENDL

Survey on Stein manifolds

Q: (V, J) open almost cpx mfd: when can J be homotoped to an integrable complex structure?

thm (Gromov–Landweber ~'73): If (V, J) has $\dim_{\mathbb{R}} V = 2n \geq 2$ & $V \underset{h.e.}{\simeq}$ a CW-complex of dim $\leq n$, then J is homotopic to an integrable complex str.

– why dim $\leq n$?

defn: a $\mathbb{C}$-mfd (V, J) is Stein if $\exists$ proper hol. emb. $(V, J) \hookrightarrow (\mathbb{C}^N, i)$ for some N.

rk: If $(V, J) \hookrightarrow (\mathbb{C}^N, i)$ & $f : \mathbb{C}^N \to \mathbb{R} : z \mapsto |z|^2$, then $f|_V$ is exhausting (proper & bdd below) & plurisubharmonic (PSH, A.K.A. "J-convex"): means

$\omega_J := -d(df \circ J)$ satisfies $\omega_J(X, JX) > 0 \;\; \forall \; 0 \neq X \in TV$.
(symplectic!) (i.e. "ω_J tames J")

– f Morse & PSH $\Rightarrow$ all crit. pts. have ind $\leq n = \frac{1}{2}\dim_{\mathbb{R}} V$.
(see below)

– f PSH, $\Sigma := f^{-1}(*) \subseteq V$ reg. level set $\Rightarrow$ Σ is a contact type hypersurface in (V, ω_J): let $\lambda_J = -df \circ J$, $\alpha = \lambda_J|_{T\Sigma}$, then:

- $\xi := \ker \alpha \subseteq T\Sigma$ is a ctct str.; also $\xi = T\Sigma \cap J(T\Sigma) =$ maximal $\mathbb{C}$-subbndl in $T\Sigma$, & $d\alpha|_\xi$ tames $J|_\xi$ ($\Sigma \subseteq (V, J)$ is strictly pseudoconvex)

- V_J defined by $\omega_J(V_J, \cdot) = \lambda_J$ is a Liouville vec. fld
($\mathcal{L}_{V_J}\omega_J = d\underbrace{\iota_{V_J}\omega_J}_{=\lambda_J} + \iota_{V_J}\underbrace{d\omega_J}_{=0} = d\lambda_J = \omega_J$)

& is gradient-like w.r.t. f:

($df(V_J) = \lambda_J(JV_J) = \underbrace{d\lambda_J}_{\omega_J}(V_J, JV_J) > 0$ unless

$V_J = 0 \Leftrightarrow \lambda_J = 0 \Leftrightarrow df = 0$).

EXERCISE: J integrable $\Rightarrow$ $g_J := \omega_J(\cdot, J\cdot)$ is symmetric $\Rightarrow$ is a Riem. metric, & then $V_J = \nabla f$.

thm (Grauert): (V, J) is Stein $\Leftrightarrow$ admits an exhausting PSH fn.

Assume W^{2n} compact, J = integrable $\mathbb{C}$-str.

def: (J,f) is a Stein str. on W if $f:(W,J)\to\mathbb{R}$ is PSH s.t. $\partial W = f^{-1}(*)$ reg. level set.

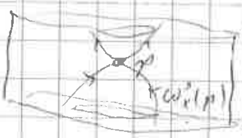
rk: Space of fns $f:(W,J)\to\mathbb{R}$ w/ these properties is convex $\Rightarrow$ contractible. $\therefore$ Can regard f as "auxiliary data".

We call (W,J) a Stein domain.

Stein str. $(J,f) \rightsquigarrow$ Liouville form $\lambda_J = -df\circ J$,
symp. form $\omega_J := d\lambda_J$
Liouville vec. fld $X_J = \nabla f$

defn: (ω, X, f) is a Weinstein str. on W if ω is symplectic, X is a Liouville vec. fld w.r.t. ω ($\mathcal{L}_X\omega=\omega$) & is gradient-like w.r.t. $f:W\to\mathbb{R}$, a (generalized) Morse fn w/ $\partial W = f^{-1}(*)$ reg. level set.

prop: (ω,X,f) Weinstein w/ f Morse, $p\in \mathrm{Crit}(f)$ of index k.
Then $W^s_X(p)$, stable mfd, is an isotropic submfd of (W,ω), (i.e. $\omega|_{\text{submfd}}=0$) & its intersection w/ any reg. level set $f^{-1}(t)=:\Sigma$ is an isotropic submfd of (Σ,ξ_t) (i.e. tangent to ξ_t).



isotropic $\Rightarrow$ $\dim W^s_X(p)\le n$, = if Lagrangian ($\Rightarrow$ Legendrian in (Σ,ξ_t))
$\Rightarrow \mathrm{ind}(p)\le n$.

thm (Weinstein): Every Weinstein domain can be constructed by attaching to $(\mathbb{D}^{2n}, \omega_{std}, X_{std}, f_{std}(z)=|z|^2)$ finitely many symplectic k-handles $\mathbb{D}^k\times\mathbb{D}^{2n-k}$ for $1\le k\le n$ with isotropic cores $\mathbb{D}^k\times\{0\}$, attached along ~~Legendrian~~ isotropic spheres in the ctct bdry.

Big thm 1: W^{2n}, with $(n>2)$!!! admits a Weinstein str. $\Leftrightarrow$ it admits (i) a Morse fn. w/ no crit. pts of ind $>n$, & (ii) a nondeg. 2-form ($\Leftrightarrow$ almost $\mathbb{C}$-str.).

More precisely:
Given Morse $f:W\to\mathbb{R}$ w/ $\mathrm{ind}(p)\le n$ $\forall\, p\in\mathrm{Crit}(f)$ & a nondeg 2-form η, $\exists$ a Weinstein str. (ω,X,f) s.t. $\omega\simeq\eta$ as nondeg 2-forms.

rk: Not true for $n=2$: ex. $S^2\times\mathbb{D}^2$ carries no Weinstein str!

Big thm 1': $n>2$, spse (ω_i,X_i,f_i), $i=0,1$ on W are both subcritical, i.e. all $p\in\mathrm{Crit}(f_i)$ have $\mathrm{ind}(p)<n$: then they are homotopic as Weinstein strs. iff $\omega_0\simeq\omega_1$ as nondeg. 2-forms.

rk: Not true when $\exists\,\mathrm{ind}(p)=n$, but does hold for a special class: "flexible" W. strs.

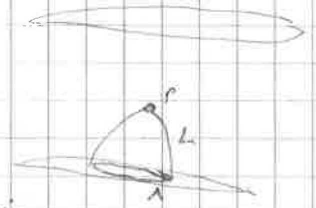
Key thm 2: Natural map $\{\text{Stein strs.}\} \to \{\text{Weinstein strs}\}$
$$(J, f) \longmapsto (\omega_J, X_J, f)$$
is an iso. on π_0.

For any fixed Morse $f_0 : W \to \mathbb{R}$, the map
$$\{\text{Stein strs } (J, f_0)\} \longrightarrow \{\text{Weinstein strs. } (\omega, X, f_0)\}$$
is a homot. equiv.

<u>making a ~~...~~ handlebody Weinstein</u> (inductively)

$W_0 := f^{-1}([t_-, t_+])$, $\partial_\pm W_0 := f^{-1}(t_\pm)$, $\mathrm{Crit}(f|_{W_0}) = \{p\}$, $\mathrm{ind}(p) = k \leq n$.

Given Weinstein str. (ω, X, f) ~~on~~ $\mathcal{O}p(\partial_- W_0)$.
Extend X over W_0 s.t. is ∇-like for f.
$\rightsquigarrow$ stable mfd $W^s_X(p) =: L \subseteq W_0$,
let $\Lambda := L \cap \partial_- W_0 \cong S^{k-1}$.



<u>necessary condition</u>: Λ must be isotopic to a ~~Legendrian~~ isotropic subfd of $(\partial_- W, \xi)$

ctct str. determined by Weinstein str.

<u>h-principle for isotropic embeddings (subcritical)</u>: $k < n$

A <u>formal isotropic embedding</u> of Λ^{k-1} into (Σ^{2n-1}, ξ)
is (φ, Φ^s) w/ $\varphi : \Lambda \hookrightarrow \Sigma$ smooth emb. &
$\{\Phi^s : T\Lambda \to T\Sigma\}_{s \in [0,1]}$ fibrewise inj. bdl maps covering φ s.t.
$\Phi^0 = d\varphi$ & $\Phi^1(T\Lambda) \subseteq \xi$.

<u>thm</u>: $\{$~~I~~ isotropic embs. $\Lambda \hookrightarrow (\Sigma, \xi)\} \longrightarrow \{$formal isot. embs. $(\varphi, \Phi^s)\}$
$$\varphi \longmapsto (\varphi, \Phi^s := d\varphi)$$
is a w.h.e.

<u>rk</u>: Critical case $k = n$: $\exists$ h-princ. for Legendrian immersions $\Lambda^{n-1} \looparrowright (\Sigma^{2n-1}, \xi)$.
Can modify reg. homot. $\rightsquigarrow$ isotopy using <u>Whitney trick</u> if <u>$n > 2$</u>.

<u>making almost complex integrable</u>

$W_0 = f^{-1}([t_-, t_+])$ as above, assume $(J, f) =$ Stein str. on $\mathcal{O}p(\partial_- W)$,
J extended over W_0 as almost $\mathbb{C}$-str. Choose metric on W_0
matching g_J near $\partial_- W_0$

$\rightsquigarrow$ stable mfd $W^s_{\nabla f}(p) =: L \subseteq W_0$, $\Lambda := L \cap \partial_- W_0 \cong S^{k-1}$,

L is <u>totally real</u> near ∂L & $\Lambda \subseteq (\partial_- W, \xi)$ is isotropic.
(i.e. $TL \cap J(TL) = \{0\}$)

<u>idea</u>: If L were totally real everywhere, could extend $\varphi : \mathbb{D}^k \hookrightarrow L \subseteq W_0$
to $\tilde{\varphi} : \mathbb{D}^k \times \mathbb{D}^{2n-k} \hookrightarrow \mathrm{nbhd}(L) \subseteq W_0$ s.t. $\tilde{\varphi}_* i$ C^0-close to J.

Then retract to nbhd(L) $\Rightarrow$ J is an integrable $\mathbb{C}$-str.

h-principle for totally real embeddings

a formal totally real embedding of L^k into (W^{2n}, J)

is (φ, Φ^s), w/ $\varphi : L \hookrightarrow W$ smooth emb. &

$\{\Phi^s : TL \to TW\}$ fiberwise inj. covering φ s.t.

$\Phi^0 = d\varphi$ & $\Phi^1(TL) =$ a tot. real subbndl of $(\varphi^* TW, J)$.

thm: $\{$genuine$\} \to \{$formal$\}$ is a w.h.e.

pf: Open & ample + lots of details. $\boxtimes$