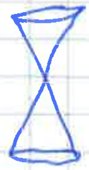
h-principle seminar — TALK 8 — 21/3/2013 — H. GEIGES

ex: $M = \mathbb{R}$, $E = \mathbb{R} \times \mathbb{R}^3 \longrightarrow E^{(1)} = \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3$

relation $\mathcal{R} := \{ (t, y, y') \in E^{(1)} \mid (y_1')^2 + (y_2')^2 - (y_3')^2 < \varepsilon \}$

thickened cone



$\mathcal{R}$ satisfies the h-principle

can produce any tangential condition on average via vectors in the cone:

key property: convex hull of $\mathcal{R} \cap (p_0^1)^{-1}(t,y)$ is the whole fibre $(p_0^1)^{-1}(t,y)$

$p : E \longrightarrow M$

$p_0^1 : E^{(1)} \longrightarrow E$

In general, one needs to consider not the fibres of p_0^1, but those of an intermediate fibration:

$$\begin{tikzcd} E^1 \arrow[r, "p^1_\perp", "\text{fibres } \mathbb{A}^q"'] \arrow[rr, bend right, "\text{affine bundle w/ fibres } \mathbb{A}^{mq}"'] & E^\perp \arrow[r, "p^\perp_0"] & E \end{tikzcd}$$

$E^\perp$: intermediate bundle

$\dim M = m$, $\dim E_x = q$

write a 1-jet as $y_k = y_k^0 + \sum_{i=1}^m a_{ik}(x_i - x_i^0)$, $k = 1, \dots, q$

Say 2 sections σ_1 & σ_2 define same "perp-jet": $j^\perp \sigma_1 = j^\perp \sigma_2 \iff$

$a^1_{kj} = a^2_{kj}$ for $j \neq i$ $\quad \perp = \perp_i$.

ex: Theorem of Smale-Hirsch on immersions $M^m \hookrightarrow N^q$, $m < q$

$E = M \times N \qquad (p_0^1)^{-1}(x,y) = \mathcal{L}(T_xM, T_yN) = \underset{\mathbb{A}^{mq}}{\overset{\shortparallel}{M^{q \times m}}}$ (after choosing coords)

$\mathcal{R} \cap \mathbb{A}^{mq} = \{ (a_{kj}) \in M^{q \times m} \text{ of rank } m \}$

$\mathbb{A}^q = \{ (a_{kj}) \in M^{q \times m} \mid a_{kj} = b_{kj} \text{ fixed for } j \neq i \}$

$$\mathcal{R} \cap \mathbb{A}^q = \begin{cases} \emptyset & \text{if the } b_j := \begin{pmatrix} b_{1j} \\ \vdots \\ b_{qj} \end{pmatrix} \; 1 \le j \le m, \; j \neq i \text{ are lin. dependent} \\ \mathbb{R}^q \setminus \mathrm{Span}\{ b_j \mid j \neq i \} \cong \mathbb{R}^q \setminus \mathbb{R}^{m-1} & \text{otherwise} \end{cases}$$

has codim ≥ 2

Conv$(\mathbb{R}^q \setminus \mathbb{R}^{m-1}) = \mathbb{R}^q$ for $m < q$

For $\mathcal{R} \cap \mathbb{A}^q = \emptyset$, $\nexists$ formal sol., $\therefore$ nothing to do.

ampleness condition on $\mathcal{R}$: Convex hull of each cpnt of $\mathcal{R} \cap \mathbb{A}^q$ is all of $\mathbb{A}^q$.

lemma: For $E = T^*M$ (M any mfd)

$\exists$ commutative diagram
$$\begin{tikzcd} E^{(1)} \arrow[r, "\Delta"] \arrow[d] & \Lambda^2 T^*M \arrow[d] \\ M \arrow[r, "\mathrm{id}"] & M \end{tikzcd}$$

s.t. Δ is a vec. bndl epimorphism, κ an affine fibration &

$$\Delta j^1_{x_0} \alpha = (d\alpha)_{x_0} \quad \text{for any local section } \alpha \text{ of } E \text{ near } x_0$$
(i.e. a 1-form near x_0).

pf: $(x^1, \dots, x^m)$ local coords. near x_0 $\rightsquigarrow$ local 1-forms $dx^1, \dots, dx^m$

polynomial representative of the 1-jet of a local 1-form $\alpha = \alpha_i dx^i$

$$j^1_{x_0} \alpha = (dx^1, \dots, dx^m) \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} + (dx^1, \dots, dx^m) A \begin{pmatrix} x^1 - x^1_0 \\ \vdots \\ x^m - x^m_0 \end{pmatrix}$$

$$a_i = \alpha_i(x_0) \qquad A = (a_{ij}) = \left(\frac{\partial \alpha_i}{\partial x^j}(x_0) \right)$$

write $j^1_{x_0}\alpha = (a_i, a_{ij}) \in \mathbb{R}^{m+m^2}$. This gives the vec. bndl str. on $E^{(1)}$.

$$\Delta j^1_{x_0} \alpha := -(dx^1, \dots, dx^m) A \begin{pmatrix} dx^1 \\ \vdots \\ dx^m \end{pmatrix} = \frac{\partial \alpha_i}{\partial x^j}(x_0)\, dx^j \wedge dx^i = (d\alpha)_{x_0}.$$

$$\Delta^{-1}\left(\sum_{i<j} b_{ij}\, dx^j \wedge dx^i \Big|_{x_0} \right) = \left\{ (a_i, a_{ij}) \mid a_{ij} - a_{ji} = b_{ij} \text{ for } i<j \right\}.$$

$\boxtimes$

prop: Let M = 3-mfd (can actually do this in all dims.)

ω a nowhere vanishing 2-form on M, $a \in H^2_{dR}(M)$.

Then ω is homotopic via nowhere vanishing 2-forms to a closed 2-form in the class a.

pf: $E = T^*M$. Chose closed 2-form $\eta \in a$.

$$R := \{ u \in E^{(1)} \mid \eta + \Delta u \neq 0 \}$$

x^1, x^2, x^3 local coords on M

$$j^1_{x_0} \alpha = (dx^1, dx^2, dx^3) \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + (dx^1, dx^2, dx^3) A \begin{pmatrix} x^1 - x^1_0 \\ x^2 - x^2_0 \\ x^3 - x^3_0 \end{pmatrix}$$

$$\Delta j^1_{x_0}\alpha = \sum_{i<k} \left(a_{ki} - a_{ik} \right) dx^i \wedge dx^k$$

$$\eta_{x_0} = \sum_{i<k} \eta_{ik}\, dx^i \wedge dx^k$$

$$\begin{aligned} \eta_x + \Delta j^1_{x_0}\alpha = {} & \left(\eta_{12} - a_{12} + a_{21} \right) dx^1 \wedge dx^2 \\ & + \left(\eta_{13} - a_{13} + a_{31} \right) dx^1 \wedge dx^3 \\ & + \left(\eta_{23} - a_{23} + a_{32} \right) dx^2 \wedge dx^3 \end{aligned}$$

free variable in $\mathbb{A}^3$ w.r.t. the first coord. direction: a_{11}, a_{21}, a_{31}

choose new affine variables in $\mathbb{A}^3$:

$$\begin{aligned} p_1 &= a_{11} \\ p_2 &= \eta_{12} - a_{12} + a_{21} \\ p_3 &= \eta_{13} - a_{13} + a_{31} \end{aligned}$$

$$\mathcal{R} \cap \mathbb{A}^3 = \begin{cases} \mathbb{A}^3 & \text{if } \eta_{23} - a_{23} + a_{32} \neq 0 \\ \{ p_2^2 + p_3^2 \neq 0 \} & \text{otherwise} \end{cases}$$

$\mathrm{Conv}\left(\{ p_2^2 + p_3^2 \neq 0 \} \right) = \mathbb{A}^3 \Rightarrow$ h-principle.

For the prop., $\left.\begin{array}{l} \eta + (\omega - \eta) = \omega \neq 0 \\ \text{fibres of } \Delta \text{ are contractible} \end{array}\right\} \Rightarrow \omega - \eta$ lifts to a formal sol. $\varphi_0 \in \Gamma\mathcal{R}$.

$\eta + \Delta\varphi_0 \neq 0.$

h-principle $\Rightarrow \varphi_0 \simeq \varphi_1 = j^1\alpha$, $\alpha \in \Gamma_0(E) := \{ \alpha \in \Gamma(E) \mid j^1\alpha \in \Gamma\mathcal{R} \}$

(Using Haefliger's notation: $\Gamma_0 E \xrightarrow{j^1} \Gamma\mathcal{R}$) homotopy $\varphi_t \in \Gamma\mathcal{R}$, $t \in [0,1]$.

$\omega_t := \eta + \Delta\varphi_t \neq 0 \quad \forall\, t \in [0,1] \rightsquigarrow \omega' := \omega_1 = \eta + \Delta\varphi_1 = \eta + d\alpha.$

__Prop__ (G.–Gonzalo ~'92): Let M be a closed orientable 3-mfd. Then M admits a triple of pointwise linearly indep. ctct forms $\alpha_1, \alpha_2, \alpha_3$ with pointwise linearly independent Reeb vector fields R_1, R_2, R_3.

__rk__: (1) Pf assumes existence of ctct forms.

(2) R is defined by $d\alpha(R, \cdot) \equiv 0$ & $\alpha(R) \equiv 1$.
R is "dual" to $d\alpha$: $R \lrcorner\, \alpha \wedge d\alpha = d\alpha$.

$\Rightarrow$ prop. is equiv. to saying $\left\{ \begin{array}{l} \alpha_1, \alpha_2, \alpha_3 \text{ are lin. indep.} \\ d\alpha_1, d\alpha_2, d\alpha_3 \text{ lin. indep.} \end{array} \right.$

pf: Choose ctct form α_1 s.t. ker α_1 is a trivial bundle

$\Rightarrow \exists\ \beta_0, \gamma_0 \in \Omega^1(M)$ s.t. $\alpha_1 \wedge \beta_0 \wedge \gamma_0 \neq 0$.

local extension problem: $\alpha_1 = dw + u\,dv$, $d\alpha_1 = du \wedge dv$

$E = T^*M \oplus T^*M$, $\mathcal{R}_x = \left\{ (j^1_x\beta, j^1_x\gamma) ; (d\alpha_x, \Delta j^1_x\beta, \Delta j^1_x\gamma) \text{ are lin. indep. w/ pos. orientation} \right\}$

$$= \Big\{ \overbrace{(b_{13} - b_{31})}^{x_1} \overbrace{(c_{23} - c_{32})}^{x_2} - \overbrace{(b_{23} - b_{32})}^{x_3} \overbrace{(c_{13} - c_{31})}^{x_4} > 0 \Big\}$$

Free variables in $\mathbb{A}^6$ w.r.t. w-coord. b_{i3}, c_{i3}, $i = 1, 2, 3$

$\mathcal{R}_x \cap \mathbb{A}^6 = \{ x_1 x_2 - x_3 x_4 > 0 \}$. Is ample? $\mathrm{Conv}(\mathcal{R}_x \cap \mathbb{A}^6) = \mathbb{A}^6$?

coord change $\rightsquigarrow = \{ y_1^2 + y_2^2 - y_3^2 - y_4^2 > 0 \}$

(In every y_4-slice, $\mathcal{R}_x$ is exterior of a 1-sheeted hyperboloid.)
$\Rightarrow$ ample!

$\mathcal{R}_x \cap \mathbb{A}^6 \underset{h.e.}{\simeq} S^1$ (in fact $\cong S^1 \times \mathbb{R}^5$)

(In every $S^3 \subset \mathbb{R}^4_{\varphi_1, \varphi_2, \varphi_3, \varphi_4}$, $\mathcal{R}_x$ is interior of a Hopf torus.)

On S^3 $\exists$ such a triple of ctct forms

$\alpha_1 = x_1\,dx_2 - x_2\,dx_1 + x_3\,dx_4 - x_4\,dx_3$

$\left.\begin{array}{l} \alpha_2 = \\ \alpha_3 = \end{array}\right\}$ permute coords

Use branched covering thm of Hilden-Montesinos-Thickstun: one can lift them outside a ball $B^3 \subseteq M$.

$\pi_2(\mathcal{R} \cap \mathbb{A}^6) = 0 \Rightarrow \nexists$ homot. obstruction to extending formal sol. over B^3

$\therefore$ h-principle $\Rightarrow \exists\ \beta, \gamma$ C^0-close to β_0 & γ_0 s.t. $d\alpha_1, d\beta, d\gamma$ are lin. indep. β & γ are not yet ctct forms, but can set $\alpha_2 := \alpha_1 + \varepsilon\beta$, $\alpha_3 := \alpha_1 + \varepsilon\gamma$ for $\varepsilon > 0$ small.

$\blacksquare$