FLEXIBILITY IN HIGHER-DIMENSIONAL CONTACT GEOMETRY BY PATRICK MASSOT AND EMMY MURPHY

1. Fomal legendrian embeddings

We fix a closed manifold L of dimension n and (Y, ξ) a closed contact manifold of dimension $2n+1$.

Definition 1. A formal legendrian embedding is the data (f, F) where $f : L \to Y$ is a smooth embedding and $F = (F_s)_{s\in[0,1]}$ is a homotopy of monomorphisms $F_s : TL \to TY$ such that $F_0 = Tf$ and F_1 is legendrian.
Two legendrian embeddings f_0 and f_1 are called formally isotopic if there is an isotopy (f_t) of smooth embeddings connecting f_0 to f_1 and a homotopy of monomorphisms $(\hat{F}_{s,t})_{s,t\in[0,1]} : TL \to TY$ such that $\hat{F}_{0,t} = Tf_t$, $\hat{F}_{1,t}$ is legendrian, $\hat{F}_{s,0} = Tf_0$ and $\hat{F}_{s,1} = Tf_1$.

Proposition 2. *If $f : L \to (Y, \xi)$ is a legendrian embedding then (f, Tf) is a formal legendrian embedding.*

Question 1. *If $f_0, f_1 : L \to Y$ are two legendrian embeddings that are formally isotopic then are they legendrian isotopic ?*

Answer no : Chekanov in dimension 3 (97) and Ekholm, Etnyre, Sullivan in general (2002).

Theorem 3 (Murphy(2012))**.** *When $n \geqslant 2$ there exists a class of legendrian called loose such that*

1. *For any legendrian embedding $f_0 : L \to Y$ there is a loose legendrian embedding f_1 that coincides with f_0 outside the neighborhood of a point in L that is formally isotopic to f_0 with support in this neighborhood.*
2. *If two legendrian loose embeddings are formally isotopic then they are legendrian isotopic.*

2. ϵ-legendrians and convex integration

We fix a Riemannian metric g on (Y, ξ) and $\epsilon < \frac{\pi}{2}$.

Definition 4. An n-plane $P \subset T_yY$ is called ϵ-legendrian if there exists a legendrian plane $\overline{P} \subset (T_yY, \xi_y)$ such that the angle between P and $\overline{P}$ is less than ϵ for the metric g.

We define the notions of ϵ-legendrian embedding, formal legendrian embedding and formal legendrian isotopy replacing 'legendrian' by ϵ-legendrian in previous section definitions.

Using Gromov convex integration one can prove that the class of ϵ-legendrian satisfy an h-principle :

Theorem 5 (Gromov)**.** *If f_0 and f_1 are two ϵ-legendrians that are formally ϵ-legendrian isotopic then they are ϵ-legendrian isotopic.*

Let's simplify the problem to explain the idea of convex integration :

Proposition 6. *For any smooth $f : [0,1] \to \mathbb{R}^2$ and any $\delta > 0$ there exists a smooth $\overline{f} : [0,1] \to \mathbb{R}^2$ such that $|\overline{f}'(t)| > 50$ and $d(f(t), \overline{f}(t)) < \delta$.*

The key fact here is that the convex hull of $\{v \in \mathbb{R}^2 |\, |v| > 50\}$ is the whole $\mathbb{R}^2$.

Lemma 7. *For any $r > 0$ and $v \in \mathbb{R}^2$ there exists a 1-perodic map $h : \mathbb{R} \to \mathbb{R}^2$ such that $|h(u)| > r$ at every u and $\int_0^1 h = v$.*

Démonstration. FIG □

This lemma also holds when v depends on a parameter p.

Proof of proposition 6. We apply the last lemma with $v = f'(t)$ depending on the parameter $t \in [0,1]$ which gives $h(t,u)$ at time t. Then let

$$f_N(t) = f(0) + \int_0^t h(u, Nu)\ du$$

with N a large positive integer. Then $f_N'(t) = h(t, Nt)$ has norm greater than 50. On the other hand $f_N(t) - f(t) = \int_0^t (h(u,Nu) - f'(u))\ du$ to bound the difference we partition $[0,t]$ in a increasing sequence of $n = \lfloor tN \rfloor$ intervals I_j of time $1/N$ and the interval left I_{n+1}. On each interval I_j with $j \leqslant n$ we have $\int_{I_j} h(u, Nu) = \frac{1}{N}\int_0^1 h(\frac{v+j}{N}, v)$. Then

$$\begin{aligned} f_N(t) - f(t) &= \sum_{j=0}^{n+1} \int_{I_j} \int_0^1 (h(\frac{u+j}{N}, u) - h(v,u))\ du\ dv \\ &\leqslant \sum_{j=0}^{n} \frac{1}{N^2} \sup |\partial_t h| + \frac{2}{N} \sup_{I_{n+1}} |\partial_t h|. \end{aligned}$$

This is arbitrary small when N is large. □

Remark 8. If f was initially embedded then $\overline{f}$ is an embedding to provided that δ is chosen small enough.

Claim 1. Take a smooth embedding $f : C := [0,1]^2 \to \mathbb{R}^5_{std}$ and an ϵ-legendrian plane field along the image of f. Then there exists an ϵ-legendrian immersion $\overline{f} : C \to \mathbb{R}^5_{std}$ that is $\mathcal{C}^0$-close to f.

Démonstration. Consider the jet bundle $J := J^1(C, \mathbb{R}^5)$ over C and set the sub-bundle

$$R := \{(c, y, v_1, v_2) \in J |\ \text{Span}(v_1, v_2) \text{ is } \epsilon\text{-legendrian } \},$$

The ϵ-legendrian plane field over f corresponds to a non holonomic section $\sigma^0 = (c, f, \sigma_1^0, \sigma_2^0)$ of R.
Let's adjust the holonomy of the first derivative considering the bundle $\mathcal{S}^1$ over C whose fiber over $c \in C$ is

$$\mathcal{S}^1_c = \{v_1 \in T_f Y |\ \text{Span}(v_1, \sigma_2^0) \text{ is } \epsilon\text{-legendrian in } T_f Y\}.$$

Using the fact that the convex hull of $\mathcal{S}^1_c$ is $T_f Y$ we can use the loop lemma with $v = \partial_1 f$ and parameter space C. We can then get a one periodic section h^1 of $\mathcal{S}^1$ with $h^1(c, 0) = \sigma^0_1(c)$. Let's set

$$\sigma^1_v(c) = f(0, c_2) + \int_0^{c_1} h^1((u, c_2), N_1 u)\ du$$

where N_1 is a large positive integer and set

$$\sigma^1(c) = (c, \sigma^1_v(c), \partial_1 \sigma^1_v(c), \sigma^0_2(c)).$$

Then $\mathrm{Span}(\partial_1 \sigma^1_v(c), \sigma^0_2(c))$ is ϵ-legendrian in $T_{\sigma^0_v} Y$ but since σ^1_v is $O(1/N_1)$-close to σ^0_v it is also ϵ-legendrian in $T_{\sigma^1_v} Y$ (as being ϵ-legendrian is an open condition).

Now do the second variable considering the bundle $\mathcal{S}^2$ over C whose fiber over $c \in C$ is

$$\mathcal{S}^1_c = \{v_2 \in T_{\sigma^1_v} Y |\ \mathrm{Span}(\partial_1 \sigma^1_0(c), v_2) \text{ is } \epsilon\text{-legendrian in } T_{\sigma^1_v} Y\}.$$

Using the loop lemma again, with $v = \partial_2 \sigma^1_v$ and parameter space C, we can get a one periodic section h^2 of $\mathcal{S}^1$ with $h^2(c, 0) = \sigma^1_2(c)$. Let's set

$$\overline{f} := \sigma^2_v(c) = \sigma^1_v(c_1, 0) + \int_0^{c_2} h^1((c_1, u), N_2 u)\ du$$

where N_2 is a large positive integer and set

$$\sigma^2(c) = j^1 \sigma^2_v(c).$$

Then $\mathrm{Span}(\partial_1 \sigma^1_v(c), \partial_2 \sigma^2_v(c))$ is ϵ-legendrian in $T_{\sigma^1_v} Y$ but since σ^2_v is $O(1/N_2)$-close to σ^1_v and $\partial_1 \sigma^2_v$ is $O(1/N_2)$-close to $\partial_1 \sigma^1_v$ the plane $\mathrm{Span}(\partial_1 \sigma^2_v(c), \partial_2 \sigma^2_v(c))$ is also ϵ-legendrian in $T_{\sigma^2_v} Y$. □

3. Holonomic approximation in Darboux boxes

Goal : Deform the ϵ-legendrian $f : L \to Y$ to a legendrian $\overline{f} : L \to Y$.

Every point $p \in f(L)$ has a neighborhood in Y contactomorphic to a Darboux box $D_{a,b,c} := B_a \times B_b \times B_c \subset \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$ (we denote x, z and x the coordinates in the respective first, second and third factor) such that the contact form in the box is given by $\xi = \ker(dz - \sum_i y_i dx_i)$ and the image of $T_p L$ is given by $\mathrm{Span}(\partial_{x_1}, \dots, \partial_{x_n})$. Shrinking the box ensures that $f(TL)$ stays transverse to $\mathrm{Span}(\partial_z, \partial_{y_i})$ so $f(L)$ is the graph of a function

$$\begin{aligned} \sigma : B_a &\to B_b \times B_c \\ x &\mapsto (z(x), y(x)). \end{aligned}$$

Note that $D_{a,b,c}$ is contactomorphic to $D_{1,1,ac/b}$ by rescaling in each factor, so we can restrict to the use of boxes of the form $D_{1,1,\eta} =: D_\eta$.

In the Darboux box, saying that $f : L \to Y$ is legendrian amounts to saying that σ corresponds to a holonomic section of $J^1(B_a, \mathbb{R})$, so our objective is to deform σ into a holonomic section of this jet bundle with $|\partial_{x_i} z| < \eta$. There are obstructions to do this in general, but we can achieve it over a $\mathcal{C}^0$-small perturbation on the boundary of a n-simplex A that is included in the box. For simplicity take $n = 2$.

Proposition 9 (Eliashberg,Mischachev). *Fix $\eta > 0$, and a closed embedded path γ. Then there is a $\mathcal{C}^0$-small isotopy relative to the boundary that sends γ to a perturbation $\overline{\gamma_i}$ and a section $\overline{\sigma} : B_1 \to D_\eta$ that is holonomic near $\overline{\gamma}$.*

Démonstration. The main idea is to construct a road on a steep mountain (FIG). □

Applying this property to the edges γ_i of the triangle ∂A gives a perturbed triangle $\overline{\partial A}$ and $\overline{f} : L \to Y$ is legendrian over a neighborhood of $\overline{\partial A}$.

If we take a triangulation of L such that each n-simplex is in a Darboux box and apply the latter reasoning we get $\overline{f} : L \to Y$ a legendrian embedding outside finitely many disjoint Darboux boxes D'_η.

4. Wrinkled Legendrian Embeddings

For $\delta \in \mathbb{R}$ define

$$\psi_\delta : [-1,1] \to \mathbb{R}^2$$
$$u \mapsto (u^3 - 3\delta u, \frac{1}{5}u^5 - \frac{2}{3}\delta u^3 + \delta^2 u).$$

(FIG)

Definition 10 (Eliashberg,Mischachev(09)). A smooth map $W : \mathbb{R}^n \to \mathbb{R}^{n+1}$ that is a topological embedding is called a wrinkled embedding if it is a smooth embedding except at finitely many $(n-1)$-spheres S_j such that there exists coordinates $(u,v) \in \mathbb{R} \times \mathbb{R}^{n-1}$ near $S_j = \{u^2 + |v|^2 = 1\}$ such that $W(u,v) = (v, \psi_{1-|v|^2}(u))$. The S_j are called wrinkles.

One can check that the singularities at $\{u \neq 0\} \subset S_j$ are cusps, so the lift of W from the front projection

$$\hat{W} : \mathbb{R}^n \to \mathbb{R}^{2n+1}_{std}$$
$$(u,v) \mapsto (v, \psi_{1-|v|^2}(u), \partial_{v_i}\psi_{1-|v|^2}(u))$$

is a smooth (and legendrian) embedding at $\{u \neq 0\} \subset S_j$. However at the singularities of the equator $S'j := \{u = 0\}$ of S_j the wrinkle embedding does not lift to an immersion.

Proposition 11 (Eliashberg,Mischachev). *Fix $\eta > 0$ and a compact hyper surface in $\mathbb{R}^{n+1}$ transverse to ∂_z such that $(q, T_qS) \in D_\eta$ for q near ∂S. Then there exists a wrinkled hypersurface $\overline{S}$ which coincides with S near ∂S such that $\{(q, T_q\overline{S})\ q \in \overline{S}\} \subset D_\eta$.*

The main idea of the proof is to replace the steep areas by wrinkles.(FIG)

Definition 12 (Murphy(2012)). $f : L \to Y$ is called a wrinkled legendrian if it is a smooth legendrian embedding away from the finitely many Darboux boxes D_η, where its front is a wrinkled hypersurface. The lift of the image of an equator S'_j is called a Legendrian wrinkle.

Theorem 13 (Murphy). *If two legendrian embeddings $f_0, f_1 : L \to Y$ are ϵ-legendrian isotopic then they are connected by an isotopy of wrinkled legendrians.*

5. Loose legendrians

We consider a cut-off version of ψ_δ for $\delta > 0$ (FIG) and we call Λ_0 the lift of its image to $\mathbb{R}^3_{std}$; we also denote Z the zero section of $T^\star\mathbb{R}^{n-1}$ and $B^{2n-2}(R)$ a ball of radius R in $T^\star\mathbb{R}^{n-1}$. Then $\Lambda_0 \times Z$ is legendrian in the contact manifold $\mathbb{R}^3_{std} \times T^\star\mathbb{R}^{n-1}$.

Definition 14. A legendrian embedding $f : \Lambda \to (M, \xi)$ is called loose if for all $R > 0$ there exists an open set $U \subset M$ such that $(U \cap \Lambda, U \cap \Lambda)$ is contactomorphic to $(\Lambda_0 \times Z, \mathbb{R}^3_{std} \times B(R))$.

Proposition 15. *The following assertions are equivalent :*

(1) Λ *is loose ;*
(2) *There exists an open set* $U \subset M$ *verifying* $(U \cap \Lambda, U \cap \Lambda)$ *contactomorphic to* $(\Lambda_0 \times Z, \mathbb{R}^3_{std} \times B(R_0 = 2))$.
(3) *There exists a closed* $(n-1)$*-manifold* V *such that for all open* $W \subset T^\star V$ *with* $Z \subset W$ *there exists an open set* $U \subset M$ *verifying* $(U \cap \Lambda, U \cap \Lambda)$ *contactomorphic to* $(\Lambda_0 \times Z, \mathbb{R}^3_{std} \times W)$.
(4) *For all closed* $(n-1)$*-manifold* V *and open* $W \subset T^\star V$ *with* $Z \subset W$ *there exists an open set* $U \subset M$ *verifying* $(U \cap \Lambda, U \cap \Lambda)$ *contactomorphic to* $(\Lambda_0 \times Z, \mathbb{R}^3_{std} \times W)$.

Démonstration. (4) $\Rightarrow$ (3) is clear. If we suppose (3) there is a contactomorphism of $(\Lambda_0 \times Z, \mathbb{R}^3_{std} \times W)$ that shrinks the wrinkle by a rescaling in x_1 and z directions with same scale. Then we choose a point in V with a small neighborhood in which the wrinkle looks like $(\Lambda_0 \times Z, \mathbb{R}^3_{std} \times B(R_0))$ hence we get (2). If we suppose (2) we can lower the height of the wrinkle to an arbitrary small δ in the x_2 direction on the interval $x_2 \in (R_0 - 1/R_0, R_0 + 1/R_0)$ such that the isotopy is relative to the boundary and $|\partial_{x_2} z| < R_0$. It suffices then to choose a smaller neighbourhood of the wrinkle in the part of height δ to get (1). If we suppose (1) and we fix V^{n-1} closed we introduce a wrinkle at V of type (4) away from the loose chart ; this may change the isotopy class of Λ. By introducing another wrinkle away from the loose chart and the first wrinkle one can get a legendrian Λ' whose formal homotopy class is the same as Λ and is also loose. By theorem 3 the latter are legendrian isotopic. □