

HIGHER STRUCTURE: EXERCISE 1

DINGYU YANG

Exercise 1: Show:

$$\begin{aligned}
 d_H \left(\begin{array}{c} \diagup \quad \diagdown \\ \bullet \\ \diagdown \quad \diagup \\ | \end{array} \right) &:= \begin{array}{c} \diagup \quad \diagdown \\ \bullet \\ \diagdown \quad \diagup \\ | \\ d_H \end{array} \pm \begin{array}{c} \diagup \quad \diagdown \\ \bullet \\ \diagdown \quad \diagup \\ | \\ d_H \end{array} \pm \begin{array}{c} \diagup \quad \diagdown \\ \bullet \\ \diagdown \quad \diagup \\ | \\ d_H \end{array} \pm \begin{array}{c} \diagup \quad \diagdown \\ \bullet \\ \diagdown \quad \diagup \\ | \\ d_H \end{array} \\
 &= \begin{array}{c} \diagup \quad \diagdown \\ \bullet \\ \diagdown \quad \diagup \\ \bullet \\ | \end{array} - \begin{array}{c} \diagup \quad \diagdown \\ \bullet \\ \diagdown \quad \diagup \\ \bullet \\ | \end{array}
 \end{aligned}$$

Exercise 2: Show the ‘higher’ associator is a boundary (of an operation):

$$d_H \left(\begin{array}{c} 1 \quad \dots \quad k \\ \diagdown \quad \diagup \\ \bullet \\ \diagdown \quad \diagup \\ | \end{array} \right) = \sum_{\substack{m+n=k+1 \\ 1 \leq j \leq n \\ n \geq 2 \\ m \geq 2}} \pm \begin{array}{c} 1 \quad \dots \quad m \\ \diagdown \quad \diagup \\ \bullet \\ \diagdown \quad \diagup \\ \dots \quad j \quad \dots \quad n \\ \diagdown \quad \diagup \\ \bullet \\ | \end{array}$$

Exercise 3: Check $\text{Id}_A - i \circ p = d_A \circ h \pm h \circ d_A$. Hint: $\text{LHS} = \text{Id}_{B^n} \oplus \text{Id}_{B^{n+1}}$, and note that under the identification, d_A^n is the following:

$$\begin{array}{c}
 A^n \cong B^n \oplus H^n \oplus B^{n+1} \\
 \downarrow \text{Id}_{B^{n+1}} \\
 0 \oplus 0 \oplus B^{n+1} \oplus H^{n+1} \oplus B^{n+2} \cong A^{n+1}
 \end{array}$$

Exercise 4: (Non-compulsory) The following is a plane projection of the Borromean rings. Express the linking numbers as products in cohomology. Any two rings are separable/unlinked, which means products are trivial and makes the classical (3rd) Massey product defined. Show it is non-trivial (they can’t be separated).

