SEIBERG-WITTEN FLOER HOMOLOGY LECTURE 1

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1. Lecture 1

I will type the lecture notes from the first lecture below, as my stylus was not charged and handwriting was not ideal.

We follow Kronheimer-Mrowka's Monopoles and Three-manifolds [KM] closely.

Let Y be a closed, orientable, Riemannian 3-manifold.

Definition 1.1. A spin^c structure on Y is a unitary rank 2 complex (namely $U(2)$) vector bundle $S \to Y$ (with Hermitian metric on S denoted by h, Riemannian metric on Y denoted by g) with Clifford multiplication $\rho : TY \to \mathrm{Hom}(S,S)$ which is a bundle map with the image $\mathfrak{su}(S) = \{a \mid \mathrm{tr} a = 0, a^* = -a\}$ (where a^* is defined by $h(ax,y) = h(x,a^*y)$), such that, denoting $\tilde{h}(a,b) = \frac{1}{2}\mathrm{tr}(a^*b)$,

$$\rho : (TY, g) \to (\mathfrak{su}(S), \tilde{h})$$

is a bundle isometry.

Exercise 1.2. More concretely, let $\sigma_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$. Then can choose orthonormal basis e_i of T_yY and basis for S_y such that $\rho(e_i) = \sigma_i$.

Exercise 1.3. Look up why for 3-manifold Y, TY is trivial. Then show a spin^c structure for Y always exists.

For any Hermitian line bundle $L \to Y$, any spin^c structure $\mathfrak{s}_0 = (S_0, \rho_0)$, define $\mathfrak{s} = (S, \rho)$ where $S := S_0 \otimes L$ and $\rho := \rho_0 \otimes \mathrm{Id}_L$. We remark (c.f. the main reference [KM]) up to isomorphism (bundle isomorphism intertwining the Clifford multiplications), any two spin^c structures are related this way.

Complex line bundle up to isomorphism via c_1 is $H^2(Y;\mathbb{Z})$, thus space of isomorphism classes of spin^c structures is an affine space over $H^2(Y;\mathbb{Z})$.

Let X be an oriented, Riemannian 4-manifold.

Definition 1.4. A spin^c structure on X is a Hermitian rank 4 (namely $U(4)$) vector bundle $S_X \to X$ with Clifford multiplication $\rho_X : TX \to \mathrm{Hom}(S_X, S_X)$, such that for each $x \in X$, we can find orthonormal basis e_0, e_1, e_2, e_3 such that $\rho(e_0) = \begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix}$, $\rho(e_i) = \begin{pmatrix} 0 & -\sigma_i^* \\ \sigma_i & 0 \end{pmatrix}$ for $i = 1,2,3$ for some orthonormal basis of $(S_X)_x$. Here, I_2 is 2 by 2 identity matrix.

Using the metric g, we can induce $\rho_X : T^*X \to Hom(S_X, S_X)$ with the same notation. We can extend to forms: $\rho(\alpha\wedge\beta) = \frac{1}{2}\big(\rho(\alpha)\rho(\beta) + (-1)^{\deg\alpha\deg\beta}\rho(\beta)\rho(\alpha)\big)$.

Exercise 1.5. $\rho(\mathrm{vol}) = \begin{pmatrix} -I_2 & 0 \\ 0 & I_2 \end{pmatrix}$.

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Definition 1.6. Let S^+ denote the -1 eigenspace of $\rho(\text{vol})$ and S^- the 1 eigenspace of $\rho(\text{vol})$. They are called the positive and negative spin bundle with sections called spinors. So $S_X = S^+ \oplus S^-$.

For $e \in T_xX$, $\rho(e) : S^+_x \to S^-_x$.

Hodge star operator $* : \Lambda^k X \to \Lambda^{\dim X - k} X$ is defined by $\alpha \wedge *\beta = g(\alpha, \beta)\text{vol}$, where the metric on forms is induced from g and denoted by the same notation. Here X is a 4-manifold, and let $\Lambda^\pm$ denote the ± 1 eigenspace of $* : \Lambda^2 X \to \Lambda^2 X$, elements of which are called self-dual and anti-self-dual 2-forms respectively.

Exercise 1.7. $\rho : \Lambda^+ \to \mathfrak{su}(S^+)$ is a bundle isometry. $\rho : \Lambda^+ \to \text{End}(S^-)$ is 0. For any unit vector $e \in T_xX$, $\det\rho(e) : \Lambda^2 S^+_x \to \Lambda^2 S^-_x$ is independent of e.

Definition 1.8. For a spin^c structure (S_X, ρ), $\text{Aut}(S_X, \rho)$ is the group of unitary bundle automorphisms of S_X that commute with ρ. This is precisely $\{u : X \to S^1\}$ called gauge group $\mathcal{G}_X$, which acts on S_X by scalar multiplication. The same is true for 3-manifold Y, $\text{Aut}(S, \rho) = \{u : Y \to S^1\}$.

Definition 1.9. A connection ∇_A on a bundle $S_X \to X$ over Riemannian X is a $\mathbb{C}$-linear map $\nabla_A : \Gamma(S_X) \to \Gamma(T^*X \otimes S_X)$ such that $\nabla_A(fs) = f\nabla_A s + df \otimes s$. We can extend it using Levi-Civita connection ∇_X on X, and recall the curvature $F_A \in \Gamma(\Lambda^2 X \otimes \text{End}(S_X))$ is defined via $\nabla_A(\nabla_A(s)) = F_A(s)$. A connection ∇_A on a unitary bundle S_X with a metric h is called unitary, if

$$d\big(h(s, \tilde{s})\big) = h(\nabla_A s, \tilde{s}) + h(s, \nabla_A \tilde{s}).$$

A connection ∇_A on an oriented Riemannian 4-manifold X with a spin^c structure $\mathfrak{s} = (S_X, \rho)$ is called a spin^c connection, if it is unitary and ρ is parallel, namely

$$(\nabla_A \rho)(v)(s) := \nabla_A\big(\rho(v)(s)\big) - \rho(\nabla_X v)(s) - \rho(v)(\nabla_A s) = 0$$

for all local sections v and s.

Remark 1.10. In particular, parallel transport via ∇_A preserves $S_X = S^+ \oplus S^-$.

Two spin^c connections $\tilde{A}$ and A differ by $a \otimes \text{Id}_{S_X} = a$ for some $a \in \Omega^1(X; i\mathbb{R})$.

Definition 1.11. The Dirac operator $D_A : \Gamma(S_X) \to \Gamma(S_X)$ is defined as the composition $\Gamma(S_X) \xrightarrow{\nabla_A} \Gamma(T^*X \otimes S_X) \xrightarrow{\rho} \Gamma(S_X)$, where the second map is pointwise map $\big(\rho : T^*X \to \text{End}(S_X)\big) = \big(\rho : T^*X \otimes S_X \to S_X\big)$ induced from ρ.

If $\nabla_{\tilde{A}} := \nabla_A + a$, then $\nabla_{\tilde{A}} = D_A + \rho(a)$.

Definition 1.12. As in Remark 1.10, we can decompose $D_A = D^+_A + D^-_A$, where $D^\pm_A : \Gamma(S^\pm) \to \Gamma(S^\mp)$.

A spin^c connection A induces connection on $\Lambda^2 S^+$ and $\Lambda^2 S^-$ which are identified under $\det \rho(e)$ for any unit vector e, denoted by A^t. Note $\nabla_{\tilde{A}^t} = \nabla_{A^t} + 2a$.

Let $u : X \to S^1$ be an element of $\mathcal{G}_X$, it acts on connection via $u(A) = A - u^{-1}du$, due to S^1 is Abelian.

Definition 1.13. In dimension 3, spin^c connection B, Dirac operator D_B is defined the same way.

Let X be oriented Riemannian 4-manifold. $\mathfrak{s}_X = (S_X, \rho)$ a spin^c structure. Consider a pair (A, Φ) where A is a spin^c connection and $\Phi \in \Gamma(S^+)$.

Denote $(\Phi\Phi^*)_0$ the traceless part of $\Phi \otimes \Phi^*$ (omitting the tensor product), which is $\Phi\Phi^* - \frac{1}{2}\mathrm{tr}(\Phi\Phi^*)\mathrm{Id}_{S^+} = \Phi\Phi^* - \frac{1}{2}|\Phi|^2\mathrm{Id}_{S^+}$. It is Hermitian and traceless, thus is i times skew-Hermitian. Thus lies in $i\rho(\Lambda^+) = \rho(\Lambda^+ i)$. So it makes sense to write down the following Seiberg-Witten (SW) equation.

$$\begin{cases} \frac{1}{2}\rho(F^+_{A^t}) - (\Phi\Phi^*)_0 = 0 \\ D^+_A\Phi = 0 \end{cases}$$

Exercise 1.14. Denote the LHS of the above as $\mathcal{F}(A, \Phi)$. Check that

$$D_{(A,\Phi)}\mathcal{F} : (a, \phi) \mapsto (\rho(d^+a) - (\phi\Phi^* + \Phi\phi^*)_0, D^+_A\phi + \rho(a)\Phi).$$

For $u \in \mathcal{G}_X$, we have the action $u : (A, \Phi) \mapsto (u(A), u\Phi)$. $\mathcal{G}_X$ acts freely on

$$\{(A, \Phi) \mid \mathcal{F}(A, \Phi) = 0, \Phi \neq 0\}.$$

If $\Phi = 0$, SW equation reduces to $F^+_{A^t} = 0$, the anti-self-dual equation for A^t.

We return to 3-dimension, let Y be an oriented closed Riemannian 3-manifold.

Definition 1.15. For a spin^c structure $\mathfrak{s} = (S, \rho)$, fix a reference spin^c connection B_0. We define on the configuration space of pairs (B, Ψ), where B is a spin^c connection and $\Psi \in \Gamma(S)$ a spinor, a functional

$$\mathcal{L}(B, \Psi) := -\frac{1}{8}\int_Y (B^t - B^t_0) \wedge (F_{B^t} + F_{B^t_0}) + \frac{1}{2}\int_Y h(D_B\Psi, \Psi)d\mathrm{vol},$$

called Chern-Simons-Dirac functional. Later, we also denote $h(\, , \,)$ by $\langle \, , \, \rangle$.