SEIBERG-WITTEN FLOER HOMOLOGY LECTURE 10

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Please email yangding@math.hu-berlin.de if anything. Lecture notes up to now are available at www.mathematatik.hu-berlin.de/∼yangding/monopole.html. Exercises sprinkled throughout lecture notes have been collected into an exercise sheet at www.mathematik.hu-berlin.de/∼yangding/Exercise_SWF.pdf.

1. Lecture 10

We will discuss gluing and neighborhood of stratum of broken trajectories.

1.0.1. *Compactness of moduli spaces of broken trajectories.* Let $[\mathfrak{a}]$ and $\mathfrak{b}$ be (non-degenerate) zeros of $(\nabla \pounds)^\sigma$.

$M_z([\mathfrak{a}],[\mathfrak{b}])$ is called non-trivial if $[\mathfrak{a}] \neq \mathfrak{b}$ or z is non-trivial.

For any $[\gamma]$ in a non-trivial moduli space, $[\tau_s\gamma]$ will be a different element, where the shift $\tau_s : \gamma \mapsto [\gamma(\cdot + s)]$ descends to a map between the gauge equivalence classes. Let $[\check{\gamma}]$ denote its equivalence class under shift (not to be confused with $\check{\gamma}(t)$ with t variable regarded as a path in $3d$), called an unparametrized trajectory.

$\check{M}_z([\mathfrak{a}],[\mathfrak{b}])$ denotes the moduli space of unparametrized non-trivial trajectories.

Definition 1.1. An unparametrized broken trajectory joining $[\alpha]$ to $[\mathfrak{b}]$ is a tuple $([\check{\gamma}_1],[\check{\gamma}_2],\cdots,[\check{\gamma}_n])$, where

- $n \geq 0$,
- $[\check{\gamma}] \in \check{M}_{z_i}([\mathfrak{a}_{i-1}],[\mathfrak{a}_i])$,
- $[\mathfrak{a}_i]$ for $0 \leq i \leq n$ are zeros of $(\nabla \pounds)^\sigma$, with $\mathfrak{a}_0 = \mathfrak{a}$ and $\mathfrak{a}_n = \mathfrak{b}$, and $z = z_1 * \cdots z_n$ the concatenated homotopy path.

We denote this as $[\check{\gamma}^+]$ (as the mathbb font in the lecture is not native for Greeks in TeX) and call it n-broken (note that we have $n-1$ broken points). If $n = 0$, then $[\check{\gamma}^+] = [\alpha_0]$ by convention.

The moduli space of such broken trajectories is denoted by $\check{M}_z^+([\mathfrak{a}],[\mathfrak{b}])$. $\check{\gamma}_i$ is the representative of the i-th component $[\check{\gamma}_i]$.

We now define the topology for $\check{M}_z^+([\alpha],[\beta])$.

Fix a point $[\check{\gamma}^+] \in \check{M}_z^+([\mathfrak{a}],[\mathfrak{b}])$, we define a neighborhood of it as follows:

Choose $[\gamma_i]$ lifting $[\check{\gamma}_i]$ for all i, and let $U_i \subset B^\tau_{k,loc}(\mathbb{R}\times Y)$ an open neighborhood of $[\gamma_i]$. Let $T \in \mathbb{R}^+$ nonnegative reals.

Consider $[\check{\delta}^+] = ([\delta_1],\ldots,[\delta_m])$, where $m \leq n$ (possibly less components), we need to assign how n components get allocated to those (possibly less) m components via a surjective and order-preserving allocation map $j : \underline{n} := \{1,\cdots,n\} \to \underline{m}$, and if several adjacent components indexed by a subset $i_1, i_k+1,\cdots,i_1+k$ of $\underline{n}$ mapped to same component indexed $j \in \underline{m}$, one should picture that each of those different shifted $[\tau_{s_{i_1+i}}\delta_j]$ is in the neighborhood U_{i_1+i} of $[\gamma_{i_1+i}]$, so $[\delta_j]$ is in the neighborhood of a trajectory 'glued' from $[\gamma_{i_1+i}],\cdots,[\gamma_{i_1+k}]$. Let us package

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the component allocation and shift allocation map by $(j,s) : \underline{n} \to \underline{m} \times \mathbb{R}$, where $1 \le i_1 < i_2 \le n$ implies either $j(i_i) \le j(i_2)$ or '$j(i_1) = j(i_2)$ and $s(i_1) + T \le s(i_2)$ (adjacent components have relative shift of at least T away'.

To summarize the above, define

$$\Omega := \Omega(U_1, \cdots, U_n, T) := \{[\check{\delta}^+] \in \check{M}_z^+([\mathfrak{a}], [\mathfrak{b}]) \mid [\check{\delta}^+] \text{ is } m\text{-broken for some } m \le n \ ,$$

$$\exists \text{ allocation } (j,s) : \underline{n} \to \underline{m} \times \mathbb{R}, \text{ s.t } [\tau_{s(i)}\delta_{j(i)}] \in U_i]\}.$$

Ω is considered as an open neighborhood for $[\check{\gamma}^+]$. Those Ω for different $[\check{\gamma}^+]$, U_i's and T define a basis which gives the topology for $\check{M}_z^+([\mathfrak{a}], [\mathfrak{b}])$.

A misleadingly simple to state, but carrying a lot of heavy-lifting in the proof where the most technical ingredient we have covered before is the following:

Theorem 1.2. *$\check{M}_z^+([\mathfrak{a}], [\mathfrak{b}])$ is compact.*

1.0.2. *Stratified spaces.* We give a primitive version of stratified spaces for our purpose (only counting points in codimension 1 strata).

Definition 1.3. N^d is a d-(dim) stratified space if we have filtered inclusion $\emptyset = N^{-1} \subset N^0 \subset \cdots \subset N^{d-1} \subset N^d$, s.t. $N^e \backslash N^{e-1}$ is either empty or e-dim manifold, and in the later case, it is called e-dim stratum (which can consist of several connected components).

Proposition 1.4. *If $M_z([\mathfrak{a}], [\mathfrak{b}])$ is non-empty and dim d (we can always make it regular so this notion of dim makes sense). Then $\check{M}_z^+([\mathfrak{a}], [\mathfrak{b}])$ is a $(d-1)$-stratified space. If $M_z([\mathfrak{a}], [\mathfrak{b}])$ contains an irreducible (namely, an unbroken $[\gamma]$ with $\gamma = (A, s, \phi)$ with $s > 0$), then $(d-1)$-dim/top stratum of $\check{M}_z^+([\mathfrak{a}], [\mathfrak{b}])$ is $\{$irreducibles$\}$.*

We now want to know more about $(d-2)$-dim stratum (aka codim-1 stratum).

Consider (‡): $\check{M}_{z_1}([\mathfrak{a}_0], \mathfrak{a}_1) \times \cdots \times \check{M}_{z_l}([a_{l-1}], [a_l]) \subset \check{M}_z^+([\mathfrak{a}_0], \mathfrak{a}_l)$.

Let the relative grading of each be $\mathrm{gr}_{z_i}([\mathfrak{a}_{i-1}], [\mathfrak{a}_i]) = d_i - \epsilon_i$, where d_i is its dimension and e_i is 1 if it is boundaryt-obstructed (namely $[\mathfrak{a}_{i-1}]$ is boundary-stable and $[\gamma_i]$ is boundary-unstable) and 0 otherwise. We call $(d_1 - \epsilon_1, \cdots, d_l - \epsilon_l)$ the grading vector, and $(\epsilon_1, \cdots, \epsilon_l)$ the obstruction vector. If we reserve d_i for dimension, then we can read obstruction vector from the grading vector (vice versa).

The $(d-1)$-dim stratum, aka top stratum, is the irreducible part of $\check{M}_z([\mathfrak{a}_0], [\mathfrak{a}_l])$. (Note that $[\mathfrak{a}_l]$ is just a notation as the limit agreeding with a given element under consideration, this by no means implies that the elements are l-broken.)

The $(d-1)$-dim stratum, aka codimension-1 stratum, is the union of

- top stratum of (‡) with grading vector (d_1, d_2) (thus obstruction vector $(0, 0)$),
- top stratum of (‡) with grading vector $(d_1, d_2 - 1, d_3)$ (thus obstruction vector $(0, 1, 0)$), and
- (only if $M_z([\mathfrak{a}_0], [\mathfrak{a}_l])$ contains both reducibles and irreducibles)

$$\check{M}_z([\mathfrak{a}_0], [\mathfrak{a}_l]) \cap \{\text{reducibles}\}.$$

1.0.3. *Moduli space on finite cylinders.* We will capture the behavior of moduli space on infinity cylinder using the finite cylinders where action is most concentrated.

Now consider $Z = I \times Y$ with I compact.

$\mathcal{C}_k^\tau(Z) \subset \tilde{\mathcal{C}}_k^\tau(Z)$ with quotient $B_k^\tau(Z) \subset \tilde{B}_k^\tau(Z)$.

Since we do not have boundary condition imposed on, we do not have a Fredholm problem. But still $M(Z) = \{[\gamma] \in B_k^\tau(Z) \mid \mathcal{F}_q^\tau(\gamma) = 0\} \subset B_k^\tau(Z)$ as a Hilbert submanifold. We also have the tilde version. $M(Z)$ is a Hilbert submanifold with boundary and identified as $\tilde{M}(Z)/i$ where i is the involution changing signs of s variable.

1.0.4. *Spectral boundary condition on $\partial Z = \bar{Y} \sqcup Y$.* Let $R_Y : \tilde{M}(Z) \to B_{k-1/2}^\sigma(Y)$, $R_{\bar{Y}} : \tilde{M}(Z) \to B_{k-1/2}^\sigma(\bar{Y})$ be restricting to the boundary, the regularity $k - 1/2$ is due to Trace theorem, as taking trace costs $1/2$ derivative with $p = 2$ here and is onto. (We came across fractional Sobolev space before and can safely suppress this technicality now).

$[\gamma] \in \tilde{M}(Z)$, $\mathfrak{a} := \gamma|_Y$, $\bar{\mathfrak{a}} := \gamma|_{\bar{Y}}$.

$T_{[\alpha]} B_{k-1/2}^\sigma(Y) \cong K_{k-1/2,\mathfrak{a}}^\sigma$ (transverse to the gauge orbit), so

$$(dR_Y, dR_{\bar{Y}}) : T_{[\gamma]}\tilde{M}(Z) \to K_{k-1/2,\mathfrak{a}}^\sigma(Y) \times K_{k-1/2,\bar{\mathfrak{a}}}^\sigma(\bar{Y}).$$

Using $\mathrm{Hess}_{q,\mathfrak{a}}^\sigma$ operator, we have spectral decomposition $K_{k-1/2,\mathfrak{a}}^\sigma = K_{\mathfrak{a}}^+ \oplus K_{\mathfrak{a}}^-$ (ignoring the $(\cdot)$ signifying the boundary), $K_{\bar{\mathfrak{a}}}^-(\bar{Y}) \cong K_{\mathfrak{a}}^+(Y)$.

Let $\Pi : K_{k-1/2,(\bar{\mathfrak{a}},\mathfrak{a})}^\sigma(\bar{Y} \sqcup Y) \to K_{(\bar{\mathfrak{a}},\mathfrak{a})}^-(\bar{Y} \sqcup Y)$, where the LHS side is defined as $K_{k-1/2,\bar{\mathfrak{a}}}^\sigma(\bar{Y}) \oplus K_{k-1/2,\mathfrak{a}}^\sigma(Y)$ and the RHS is defined as $K_{\bar{\mathfrak{a}}}^+(\bar{Y}) \oplus K_{\mathfrak{a}}^-(Y)$ with the kernel the complement of the range in the above decomposition.

Theorem 1.5. *$\Pi \circ (dR_{\bar{Y}}, dR_Y)$ and $(1 - \Pi) \circ (dR_{\bar{Y}}, R_Y)$ are Fredholm and compact.*

Specifying Lagrangian boundary condition, it becomes Fredholm problem again.

1.1. **Gluing in finite dimension.** L invertible (self-adjoint) SA linear operator $\mathbb{R}^n \to \mathbb{R}^n$. $\dot{\gamma}(t) = -L\gamma(t)$, $M(T) := \{\text{solution } \gamma : [-T, T] \to \mathbb{R}^n\}$.

We have restriction $r : M(T) \to \mathbb{R}^n \times \mathbb{R}^n$, $\gamma \mapsto (\gamma(-T), \gamma(T))$, each factor in the image determines γ. So $\mathrm{im} r = \mathbb{R}^n$.

Want to parametrize $\mathrm{im} r$ so that it converges nicely with respect to $T \to \infty$.

Write $\mathbb{R}^n \times \mathbb{R}^n = (H^+ \oplus H^-) \times (H^+ \oplus H^-)$ spectral decomposition.

$\mathrm{im}(r) = \{(u_+ + e^{2TL}u_-, e^{-2TL}u_+ + u_-) \mid (u_+, u_-) \in H^+ \times H^-\}$ to $H^+ \times H^- = M(\infty)$ decays exponentially as $T \to \infty$.

Here $M(\infty)$ can be thought of as $\{\text{solution } \gamma : R^{geq} \sqcup \mathbb{R}^{\leq} \to \mathbb{R}^n\}$ with restriction $r : M(\infty) \to \mathbb{R}^n \times \mathbb{R}^n$, $r(M(\infty)) = H^+ \times H^-$.

To summarize this above abstractly whose statement can be easily generalized to the SW setting:

For for $T > 0$, there exist parametrizations

$$u(T, \cdot) : \mathbb{R}^n \to M(T) \text{ and } u(\infty, \cdot) : \mathbb{R}^n \to M(\infty)$$

s.t. $\mu_T := r \circ (u(T, \cdot)) : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ converges $\mu_\infty := r \circ (u(\infty, \cdot))$.

Let $W = \bigcup_{T \in (0,\infty]} \mu_T(\mathbb{R}^n)$ has singularity at 0 (all 'slices' $\mu_T(\mathbb{R}^n)$ intersect at 0).

So consider

$$W^0 := \bigcup_{T \in (0,\infty]} \mu_T(\mathbb{R}^n \backslash \{0\}).$$

Unique continuation means $\mu_{T_1}(\mathbb{R}^n \backslash \{0\}) \cap \mu_{T_2}(\mathbb{R}^n \backslash \{0\}) = \emptyset$ if $T_1 \neq T_2$. So W^0 is the injective image of $(0, \infty] \times (\mathbb{R}^n \backslash \{0\})$, and it is a C^0 manifold with boundary $\mu_\infty(\mathbb{R}^\backslash \{0\})$, but not a smooth manifold.

1.2. **Non-linear version of the above in Morse theory.** Let B be a compact Riemannian manifold with Morse $f : B \to \mathbb{R}$. Let K_1, K_2 be closed submanifolds lying in the level sets, $f|_{K_1} = 1$, and $f_{K_2} = -1$ and there exists a unique $a \in f^{-1}([-1,1])$, the flow is linear, consider $\dot\gamma = -\nabla f \circ \gamma = -L\gamma$.

$M_S(K_1, K_2) := \{\gamma : [-S, S] \to B \mid \gamma(-S) \in K_1, \gamma(S) \in K_2, \dot\gamma = -\nabla f \circ \gamma\}$, $M(K_1, K_2) = \bigcup_{S>0} M_S(K_1, K_2)$ has a compactification at infinity by attaching $M_\infty(K_1, K_2) = M(K_1, a) \times M(a, K_2)$, where the first factor is solution $\gamma : \mathbb{R}^{\geq 0} \to B$ with $\gamma(0) \in K_1$ and $\gamma(\infty) = a$ and similarly for the second factor.

If $S_a \pitchfork K_1$ and $U_a \pitchfork K_2$ (transversely intersecting), then the compactification is a C^0 manifold with boundary in the neighborhood of $M_\infty(K_1, K_2)$.

1.2.1. *Abstract statement of gluing theorem.* Let $E_0 \to Y$ and let E denote the pullback of E_0 over $\mathbb{R} \times Y$. Let $Du = \frac{du}{dt} + Lu$ with $L : L^2_k(E_0) \to L^2_{k-1}(E_0)$.

We have $D : L^2_k(Z; E) \to L^2_{k-1}(Z; E)$ where $Z = Z^T$ and Z^∞.

Let $L^2_{k,\delta}(Z^\infty; E) = \{s \mid e^{\delta|t|}s \in L^2_k(Z^\infty; E)\}$ and we have $D : L^2_{k,\delta}(Z^\infty; E) \to L^2_{k-1,\delta}(Z^\infty; E)$.

Suppose $\bar\Pi : L^2_{k-1/2}(\bar Y \sqcup Y; E_0) \to H$ where H is a Hilbert space and $\Pi := \bar\Pi \circ R$ with R the boundary restriction. Write $\mathcal{E}^T := L^2_k(Z^T; E)$ and $\mathcal{F}^\infty = L^2_{k-1}(Z^T; E)$, similarly we have $\mathcal{E}^\infty$, $\mathcal{E}^\infty_\delta$, $\mathcal{F}^\infty$, $\mathcal{F}^\infty_\delta$.

Suppose $(D, \Pi) : \mathcal{E}^\infty \to \mathcal{F}^\infty \oplus H$ invertible (then so is the weighted version for δ close enough to 0).

(†) Let C_0 be a constant dominating the norm of the inverse of (D, Π).

Suppose we have smooth $\alpha : L^2_k([-1,1] \times Y; E) \to L^2_{k-1}([-1,1] \times Y; E)$ of the following form: There exists a continuous $\alpha_0 : C^\infty(E_0) \to L^2(E_0)$ and α is the extension from the induced map $C^\infty(E) \to L^2_{loc}(E)$ defined by $\gamma \mapsto (\alpha_0 \circ \check\gamma(t))\hat{}$, where $\hat{}$ denotes the inverse of $\check{}$. Suppose additionally, $\alpha : L^2_k([-1,1] \times Y; E) \to L^2_{k-1}([-1,1] \times Y; E)$ satisfies $\alpha(0) = 0$ and $d_0\alpha = 0$. This then implies that α is smooth as a map $\mathcal{E}^T \to \mathcal{F}^T$, $\mathcal{E}^\infty \to \mathcal{F}^\infty$, $\mathcal{F}^\infty_\delta \to \mathcal{F}^\infty_\delta$, and for all $\epsilon > 0$, there exists $\eta > 0$ s.t. $\|u\|, \|u'\| \leq \eta$ implies that $\|\alpha(u) - \alpha(u')\| \leq \epsilon \|u - u'\|$.

Suppose η_1 chosen from the above for $\epsilon := \frac{1}{2C_0}$ with C_0 chosen above.

$F^T = D + \alpha : \mathcal{E}^T \to \mathcal{F}^T$ with $M(T) = (F^T)^{-1}(0)$, and $F^\infty = D + \alpha : \mathcal{E}^\infty \to \mathcal{F}^\infty$ with $M(\infty) = (F^\infty)^{-1}(0)$. $M(T) \subset \mathcal{E}^T$ and $M(\infty) \subset \mathcal{E}^\infty$ Hilbert submanifolds.

Then there exists $\eta > 0$ and smooth $u(T, \cdot) : B_\eta(H) \to M(T)$ and $u(\infty, \cdot) : B_\eta(H) \to M(\infty)$ each diffeomorphism onto the image, with $\Pi \circ (u(T, \cdot)) = \mathrm{Id} = \Pi \circ (u(\infty, \cdot))$, and for $T \in [T_0, \infty]$, $\mu_T := r \circ (u(T, \cdot))$ smooth embedding from $B_\eta(H)$ and $[T_0, \infty) \times B_\eta(H) \ni (T, h) \mapsto \mu_T(h)$, $\mu_T \to \mu_\infty$ in C^∞_{loc}, and there exists $\eta' > 0$ (independent of T), s.t. $\mathrm{im}(u(T, \cdot)) \supset \{u \in M(T) \mid \|u\| \leq \eta'\}$.

1.2.2. *Applying to the SW setting.* Let $\mathfrak{a} \in \tilde{\mathcal{C}}^\sigma_k(Y)$ be a non-degenerate (by q) zero of $(\nabla \pounds)^\sigma$. Let $\gamma_\mathfrak{a}$ be the associated translational-invariant solution in $4d$.

For each $T > 0$, think $\gamma_\mathfrak{a}$ lives on $Z^T = [-T, T] \times Y$. Let

$$Z^\infty = (\mathbb{R}^{\leq} \times Y) \sqcup (\mathbb{R}^{\geq} \times Y).$$

We can define $\tilde{\mathcal{C}}^\tau_{k,loc}(Z^\infty)$ etc.

$\tilde M(Z^\infty, [\mathfrak{a}]) \overset{\text{Hilbert submanifold}}{\subset} \tilde B^\tau_{k,loc}(Z^\infty)$, where we have the limit to be $[\mathfrak{a}]$ at the ends of two half cylinders.

We have r restricting to the boundary and we have spectral decomposition.

Let $\mathcal{S}^{\tau}_{k,\mathfrak{a}}(Z^T) = \{(A = A_0 + a, s, \phi) \in \tilde{\mathcal{C}}^{\tau}_k(Z) \mid \langle a|_{\partial Z}, n\rangle = 0, \mathrm{Coul}^{\tau}_{\gamma_{\mathfrak{a}}}(A, s, \phi) = 0\}$ with n be normal to the boundary.

$r : \tilde{\mathcal{C}}^{\tau}_k(Z^T) \to \tilde{\mathcal{C}}^{\sigma}_{k-1/2}(\bar{Y} \sqcup Y) \times L^2_{k-1/2}(i\mathbb{R})$ where the last coordinate is $\langle a|_{\partial Z}, n\rangle$.

$\mathcal{T}^{\sigma}_{k-1/2,\mathfrak{a}} \cong \mathcal{J}^{\sigma}_{k-1/2,\mathfrak{a}}(Y) \oplus \mathcal{K}^{\sigma}_{k-1/2,\mathfrak{a}}(Y)$.

$\hat{\text{H}}$ess ASAFOE hyperbolic ($\mathfrak{a}$ non-degenerate) gives a spectral decomposition $K^+ \oplus K^-$.

Let $H^-_Y = \{0\} \oplus K^- \oplus L^2_{k-1/2}(i\mathbb{R})$ and $H^-_{\bar{Y}} = \{0\} \oplus K^+ \oplus L^2_{k-1/2}(i\mathbb{R})$.

Let $H := H^-_{\bar{Y}} \oplus H^-_Y$ and $\Pi^-_Y : \mathcal{T}^{\sigma} \oplus L^2_{k-1/2}(i\mathbb{R}) \to H^-_Y$ and $\Pi^-_{\bar{Y}}$, and define $\Pi := \Pi^-_{\bar{Y}} \oplus \Pi^-_Y$.

Apply abstract theorem to

$$\begin{cases} \mathcal{F}^{\tau}_q(\gamma) = 0 \\ \mathrm{Coul}^{\tau}_{\mathfrak{a}}(\gamma) = 0 \\ (\Pi \circ i^{-1} \circ r)(\gamma) = h, \end{cases}$$

where i is the identification of $\mathcal{T}^{\tau}$ to a subspace in $\tilde{\mathcal{C}}^{\tau}$, and verify the hypothesis of the abstract theorem.