SEIBERG-WITTEN FLOER HOMOLOGY LECTURE 11

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Please email yangding@math.hu-berlin.de if anything. Lecture notes up to now are available at www.mathematik.hu-berlin.de/∼yangding/monopole.html. Exercises sprinkled throughout lecture notes have been collected into an exercise sheet at www.mathematik.hu-berlin.de/∼yangding/Exercise_SWF.pdf.

1. Lecture 11

Lecture 11 is about gluing and the accompanying exercise session is about orientation of moduli spaces. (Lecture 12 finishes up the construction of SWF, and lectures 13-15 cover some calculations and applications.)

1.0.1. *Concatenation of trajectories on adjacent finite cylinders being a trajectory on the union.* Let I be compact. $I = I_1 \sqcup I_2$ with $I_1 \cap I_2 = \{0\}$.

We have restriction map to sub-cylinders

$$\rho : \tilde{M}(I \times Y) \to \tilde{M}(I_1 \times Y) \times \tilde{M}(I_2 \times Y)$$

and restriction to the common boundary

$$R_i : \tilde{M}(I_1 \times Y) \times \tilde{M}(I_2 \times Y) \to B^{\sigma}_{k-1/2}(\{0\} \times Y).$$

We have $\mathrm{im}\rho \subset \mathrm{Fib}(R_1, R_2) := \{m = (m_1, m_2)] \mid R_1(m) = R_2(m)$.

ρ, being a restriction to a Hilbert submanifold from a smooth map on the configuration space, is smooth.

- $\mathrm{im}\rho = \mathrm{Fib}(R_1, R_2)$, and ρ is a homeomorphism onto this image.
- (R_1, R_2) is transverse to the diagonal in the image, so $\mathrm{Fib}(R_1, R_2)$ is a smooth Hilbert submanifold.
- Then ρ is a diffeomorphism.

The above is true for $\tilde{M}(\mathbb{R} \times Y)$ is fixing $[\mathfrak{a}]$ and $[\mathfrak{b}]$.

If $\tilde{M}([\mathfrak{a}], [\mathfrak{b}])$ is not boundary-obstructed, then (R_1, R_2) is transverse to the diagonal iff $\tilde{M}([\mathfrak{a}], [\mathfrak{b}])$ is regular.

The version for multiple segments also works with R_1 being restriction to the positive boundaries and R_2 being restriction to the negative boundaries.

1.0.2. *Boundary-obstructed case.* For the boundary-obstructed case ($\mathfrak{a}$ and $\mathfrak{b}$ are reducible, and $\mathfrak{a}$ boundary-stable and $\mathfrak{b}$ boundary-unstable.

We have codimension 1 inclusion $\partial B^{\sigma}_{k-1/2}(Y) \subset \tilde{B}^{\sigma}_{k-1/2}(Y)$ with the former defined by $\{s = 0\}$. (Note that we are in 3d case and middle variable is a constant.)

We have a map $\pi^{\partial} : \tilde{B}^{\sigma}_{k-1/2}(Y) \to \partial B^{\sigma}_{k-1/2}(Y)$, $[(B, s, \psi)] \mapsto [(B, 0, \psi)]$.

Let us deal with the general case, which is not much different.

$$\tilde{M}([\mathfrak{a}], [\mathfrak{b}]) \cong \tilde{M}(\mathbb{R} \times Y; [\mathfrak{a}], [\mathfrak{b}]) = \bigsqcup_{i=0}^{n} \tilde{M}(i),$$

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where $\tilde{M}(0)$ and $\tilde{M}(n)$ are half-infinite cylinders with limit point to $\mathfrak{a}$ and $\mathfrak{b}$ respectively.

Let $1 \le i_0 \le n$. Let

$$R'_+, R'_- : \prod_{i=0}^{n} \tilde{M}(i) \to \tilde{B}' := \partial B^\sigma_{k-1/2}(\{t_0\} \times Y) \times \prod_{i \ne 0} \tilde{B}^\sigma_{k-1/2}(\{t_i\} \times Y)$$

be the restriction (R_+, R_-) to the positive and negative boundaries followed by π^∂ at the $\{t_0\} \times Y$-boundary.

ρ, restricting to sub-cylinders, is a homeomorphism from $M([\mathfrak{a}],[\mathfrak{b}])$ to $\mathrm{Fib}(R'_+, R'_-)$.

If regular, then (R'_+, R'_-) is transverse to the diagonal and ρ is a diffeomorphism; vice versa.

In this boundary-obstructed case, $\mathrm{Fib}(R_+, R_-) \subsetneq \mathrm{Fib}(R'_+, R'_-)$.

Let us specialize to a simple example for notational simplicity. Case $n = 1$ and $t_0 = 0$ and $I_0 = \mathbb{R}^{leq0}$ and $I_1 = \mathbb{R}^{\ge 0}$.

Let $E\tilde{M}([\mathfrak{a}],[\mathfrak{b}]) := \mathrm{Fib}(R'_+, R'_-)$ has no translation action.

We have $\tilde{M}([\mathfrak{a}],[\mathfrak{b}]) \hookrightarrow E\tilde{M}([\mathfrak{a}],[\mathfrak{b}]) \xrightarrow{\delta} \mathbb{R}$, where $\delta := s([\gamma_+]|_{\{0\}\times Y}) - s([\gamma_-]|_{\{0\}} \times Y)$.

If $E\tilde{M}$ is regular at m (the fiber product is transverse to the diagonal), then near m, it is a smooth manifold of dimension $d+1$ with $d = \mathrm{gr}_z([\mathfrak{a}],[\mathfrak{b}])$.

1.0.3. *Centered trajectory via localizing.* $\gamma \in \mathcal{C}^\sigma_{k-1/2}(Y)$, $e(\gamma) = \|(\nabla \pounds)^\sigma(\gamma)\|_{L^2(Y)}$ smooth in γ and gauge-invariant.

- Any pair of zeros of $\mathfrak{a}$ and $\mathfrak{b}$ of $(\nabla \pounds)^\sigma$ (if $[\mathfrak{a}] = [\mathfrak{b}]$, z is non-trivial). There exists $\epsilon > 0$, for any component $[\gamma_i]$ of broken $[\check{\gamma}^+] \in \check{M}^+_z([\mathfrak{a}],[\mathfrak{b}])$, there exists t such that $e(\gamma_i(t)) > \epsilon$.
- For such $\epsilon > 0$, let $\beta(t)$ be a cut-off function $\begin{cases} 0, \text{ if } t \le \epsilon. \\ > 0, \text{ if } t > \epsilon. \end{cases}$ Then, if $[\gamma] \in M_z([\mathfrak{a}],[\mathfrak{b}])$, then $(\beta \circ e \circ \gamma)(t) \ge 0$, not identically zero and supported in a compact interval.

Remark 1.1. $[\check{\gamma}] \in \check{M}_z([\mathfrak{a}],[\mathfrak{b}])$, there exists a unique parametrization $[\gamma] \in M_z([\mathfrak{a}],[\mathfrak{b}])$, the center of distribution $c(\gamma) := c(\beta \circ e \circ \gamma) := \frac{\int t(\beta \circ e \circ e)(t)dt}{\int (\beta \circ e \circ \gamma)(t)dt}$ of $(\beta \circ e \circ \gamma)(t)$ is at 0.

Definition 1.2. For finite $I = [t_1, t_2]$ of length > 2, $[\gamma] \in M(I \times Y)$ is centered, if

(i) $e(\gamma(t)) < \frac{\epsilon}{2}$ for all $t \in [t_1, t_1+1] \cup [t_2 - 1, t_2]$.
(ii) $e(\gamma(t)) > \epsilon$ for some $t \in [t_1, t_2]$.
(iii) $c(\gamma)$ is at the center of I, $\frac{1}{2}(t_1 + t_2)$. Let $M^{\mathrm{cen}}(I \times Y) \subset M(I \times Y)$ denote the collection of centered ones.

A local centering interval of $[\gamma] \in M_z([\mathfrak{a}],[\mathfrak{b}])$ is I s.t. $[\gamma|_I]$ is centered.

We can have multiple intervals in this definition that are disjoint and ordered with increasing centers.

We can have this for broken trajectories.

$\{[\gamma] \in M(I \times Y) \mid \text{(i) and (ii)}\}$ is an open subset of $M(I \times Y)$ because conditions are open conditions. Inside it, M^{cen} is a closed smooth submanifold.

1.0.4. *Description of neighborhood of strata in spaces of unparametrized broken trajectories.* Consider $\check{M}_{z_1}([\mathfrak{a}_0],[\mathfrak{a}_1] \times \cdots \times \check{M}_{z_n}([\mathfrak{a}_{n-1}],[\mathfrak{a}_n]) \subset \check{M}^+_z([\mathfrak{a}],[\mathfrak{b}])$, where $[\mathfrak{a}_0] = [\mathfrak{a}]$, $[\mathfrak{a}_\mathfrak{n}] = [\mathfrak{b}]$ and $z = z_1 * \cdots * z_n$.

Let $K_m \subset M([\mathfrak{a}_{m-1}],[\mathfrak{a}_m])$ compact, which induces $\check{K}_m \subset \check{M}$ and denote $\check{\mathbb{K}} = \prod_m \check{K}_m$. Choose ϵ and β as above which gives $M^{\text{cen}}(I \times Y)$.

Proposition 1.3. *We can find L_0 depending on $\check{\mathbb{K}}$ s.t. for any $L \geq L_0$, there exists a neighborhood $\check{W} \supset \check{\mathbb{K}}$ in $\check{M}_z^+([\mathfrak{a}_0],[\mathfrak{a}_n])$ s.t. for all $[\check{\gamma}^+] = ([\check{\gamma}_1],\cdots,[\check{\gamma}_l]) \in \check{W}$ admits a unique complete collection of local centering intervals $\{I_{i,m}\}$ of length $2L$, of n members.*

Here, a complete collection of local centering intervals is for fixed i, $\{I_{i,m}\}_{m=1}^{n_i}$ adjoint, with increasing centers of lengths $2L$ each, each such interval is local centering, and $(\beta \circ e \circ \gamma)(t)$ is supported in $\bigcup_{m=1}^{n_i} I_{i,m}$.

The proposition gives an identification

$$\mu : \{1,\cdots,n\} \to \{(i,m) \mid 1 \leq i \leq l, 1 \leq m \leq n_i\}.$$

Then, we define $S_j = \begin{cases} C_{\mu(j+1)} - C_{\mu(j)} \text{ if } \mu(j+1), \mu(j) \text{ lie in the same component} \\ \infty \text{ otherwise.} \end{cases}$

So we have $\mathbb{S} = (S_1,\cdots,S_{n-1}) : \check{W} \to (0,\infty]^{n-1}$. Local choices of $\mathbb{S}$ agree, fit together into $\mathbb{S}$ defined in $\check{W}$ neighborhood of an entire statum.

Definition 1.4. Let Q be a topological space and $q_0 \in Q$. $\Pi : S \to Q$ continuous. $S_0 \subset \pi^{-1}(q_0)$. π is a topological submersion along S_0, if for all $s_0 \in S_0$, there exist U neighborhood of s_0 in S, Q' neighborhood of q_0 in Q and a homemomorphism $(U \cap S_0) \times Q' \to U$ which identifies the second projection with Π. (S_0 is necessarily is open $\pi^{-1}(q_0)$.)

Denote $\check{M}_i := \check{M}_{z_i}([\mathfrak{a}_{i-1}],[\mathfrak{a}_i])$.

Theorem 1.5. *If none of the factors $\check{M}_i$ in the stratum of the moduli space is boundary-obstructed. Then there exists neighborhood $\check{W}$ of $\prod_{i=1}^n \check{M}_i$ in $\check{M}_z^+([\mathfrak{a}_0],[\mathfrak{a}_n])$ and $\mathbb{S} : \check{W} \to (0,\infty]^{n-1}$ s.t. $\mathbb{S}^{-1}((\infty,\cdots,\infty)) = \prod_i \check{M}_i$ and $\mathbb{S}$ is a topological submersion along $\prod_i \check{M}_i$.*

If for all $i \in O \subset\subset \{1,\cdots,n\}$, $\check{M}_i$ is boundary-obstructed; $i \in O^c$, not boundary-obstructed.

Theorem 1.6. *There exists $\check{W}$ around $\prod_i \check{M}_i$ in $\check{M}^+([\mathfrak{a}_0],[\mathfrak{a}_n])$ s.t.*

(1) $j : \check{W} \subset E\check{W}$ topological embedding with $\mathbb{S} : E\check{W} \to (0,\infty]^{n-1}$ from which $\check{W} \to (0,\infty]^{n-1}$ is restricted via j.
(2) $\mathbb{S} : E\check{W} \to (0,\infty]^{n-1}$ topological submersion alogn fiber at $\infty := (\infty,\cdots,\infty)$.
(3) $j(\check{W}) \subset E\check{W}$ is the zero set of a continuous map $\delta : E\check{W} \to \mathbb{R}^O$ with $\delta|_{\text{fiber at }\infty} = 0$ (so the fiber at ∞ of $E\check{W}$ = the fiber at ∞ of $\check{W} = \prod_i \check{M}_i$).
(4) Let $\check{W}^0$ and $E\check{W}^0$ denote the subset where non of S_i is ∞, then $j|_{\check{W}^0}$ is an embedding between smooth manifolds, and $\delta|_{E\check{W}^0}$ is transverse to zero.
(5) For $i_0 \in O$ and δ_{i_0} of δ associated, then for all $z \in E\check{W}$, we have

$$\begin{cases} \textit{if } i_0 \geq 2 \textit{ and } S_{i_0-1}(z) = \infty, \textit{ then } \delta_{i_0}(z) \geq 0, \\ \textit{if } i_0 \leq n-1 \textit{ and } S_{i_0}(z) = \infty, \textit{ then } \delta_{i_0}(z) \leq 0. \end{cases}$$

To explain the last item, let us look at the simplest boundary-obstructed case:
For $n = 3$, $[\mathfrak{a}_0],\cdots,[\mathfrak{a}_3]$. Let $\check{M}_i := \check{M}_{z_i}([\mathfrak{a}_{i-1}],[\mathfrak{a}_i])$, $i = 1,2,3$, as above.
Suppose $[\mathfrak{a}_1]$, $[\mathfrak{a}_2]$ reducible, and $\check{M}_2$ boundary-obstructed.

Let $\check{M}_1^{\mathrm{irr}} \subset \check{M}_1$, $\check{M}_3^{\mathrm{irr}} \subset \check{M}_3$ be the irreducible parts, the top strata. $\check{M}_1^{\mathrm{irr}}$, $\check{M}_2$, $\check{M}_2^{\mathrm{irr}}$ non-empty.

Then, the theorem says that there exists $E\check{W}$ and topological submersion $\mathbb{S} = (S_1, S_2) : E\check{W} \to (0,\infty] \times (0,\infty]$ and map $\delta : E\check{W} \to \mathbb{R}$ vanishes on the fiber over (∞,∞). The zero set of δ is identified with a neighborhood $\check{W}$ of the subset $\check{M}_1^{\mathrm{irr}} \times \check{M}_2 \times \check{M}_3^{\mathrm{irr}}$ in $\check{M}^+([\mathfrak{a}_0],[\mathfrak{a}_3])$; and $\delta(z) > 0$ if $S_1(z) = \infty$ and $S_2(z)$ finite, and $\delta(z) < 0$ if $S_2(z) = \infty$, $S_1(z)$ finite. Let $E\check{W}^0 := \{\text{finite } \mathbb{S}\}$ and $\check{W}^0$ sits in $E\check{W}^0$ as a smooth submanifold and as the transverse zero of δ.

1.0.5. *Simplistic structure of the proof.* For the unobstructed case:

Fix $L > 0$, let $T_j \in (1,\infty]$, $1 \le j \le n-1$, and let $\mathbb{T} := (T_1, \cdots, T_{n-1})$.

$Z^T = [-T,T] \times Y$ for $T < \infty$, $Z^\infty = Z^+ \sqcup Z^-$ two half-infinite cylinders.

$Z_{\mathbb{T}} = Z^- \sqcup Z^L \sqcup Z^{T_1} \sqcup Z^L \cdots Z^{T_n-1} \sqcup Z^L \sqcup Z^+$. Let $l := \#\{T_j = \infty\}$, and gluing boundaries give l infinite cylinders $\mathbb{R} \times Y$.

Fix $[\mathfrak{a}]$. Let d_k metric on $B_k^\tau(I \times Y)$ defined by

$$d_k([\gamma],[\gamma']) = \inf\{\|u\gamma - \gamma'\|_{L^2_{k,\mathfrak{a}}} \mid u \in \mathcal{G}_{k+1}(I \times Y)\}.$$

$M_\eta(Z^\pm;[\mathfrak{a}]) = \{d_j([\gamma],[\gamma_{\mathfrak{a}}]) \le \eta\}$, $M_\eta(Z^T;[\mathfrak{a}]) = \{\text{ditto}\}$ and

$$M_\eta(Z^\infty;[\mathfrak{a}]) = \{([\gamma^+],[\gamma^-]) \mid d_k([\gamma^-],[\gamma_{\mathfrak{a}}])^2 + d_k([\gamma^+],[\gamma_{\mathfrak{a}}])^2 \le \eta^2\}.$$

$M_{\mathbb{T}} = M(Z^-;[\mathfrak{a}_0]) \times M(Z^L) \times M(Z^{T_1}) \times \cdots \times M(Z^{T_{n-1}}) \times M(Z^L) \times M(Z^+;[\mathfrak{a}_n]$, and contained in there is:

$M^{\mathrm{cen}}_{\mathbb{T},\eta} = M_\eta(Z^-;[\mathfrak{a}_0] \times M^{\mathrm{cen}}(Z^L) \times M_\eta(Z^{T_1};[\mathfrak{a}_1]) \times \cdots \times M_\eta(Z^{T_{n-1}};[\mathfrak{a}_{n-1}]) \times M^{\mathrm{cen}}(Z^L) \times M_\eta(Z^+;[\mathfrak{a}_n])$.

$(R^-_{\mathbb{T}}, R^+_{\mathbb{T}})$ restricts $M_{\mathbb{T}}$ to the positive and negative boundaries.

$\mathrm{Fib}(R^-_{\mathbb{T}}, R^+_{\mathbb{T}}) \subset M_{\mathbb{T}}$ fiber product.

Write $\mathbb{M} = \bigcup_{\mathbb{T}} M_{\mathbb{T}}$, $\mathbb{M}^{\mathrm{cen}}_\eta = \bigcup_{\mathbb{T}} M^{\mathrm{cen}}_{\mathbb{T},\eta}$, and $\mathrm{Fib}(R^-,R^+) = \bigcup_{\mathbb{T}} \mathrm{Fib}(R^-_{\mathbb{T},R^+_{\mathbb{T}}})$.

Concatenation: $\mathbb{c}_{\mathbb{T}} : \mathrm{Fib}(R^-_{\mathbb{T}}, R^+_{\mathbb{T}}) \to \check{M}^+([\mathfrak{a}_0],[\mathfrak{a}_n])$ into l-broken trajectory.

If η small, $m \in \mathrm{Fib}(R^-_{\mathbb{T}}, R^+_{\mathbb{T}}) \cap M^{\mathrm{cen}}_{\mathbb{T},\eta}$, $\mathbb{c}_{\mathbb{T}}(m)$ has a complete collection of n local centering intervals of length $2L$ from those Z^L's.

$$S_i(m) = \begin{cases} 2T_i + 2L & \text{if } T_i < \infty \\ \infty & \text{if } T_i = \infty \end{cases}.$$

Let $\mathbb{c} = \bigcup_{\mathbb{T}} \mathbb{c}_{\mathbb{T}} : \mathrm{Fib}(R^-,R^+) \to \check{M}^+([\mathfrak{a}_0],[\mathfrak{a}_n])$.

Previous lemma on the existence of unique complete collection provides a canonical right inverse for $\mathbb{c}$ on $\check{W}$.

Proposition 1.7. *Let $\check{\mathbb{K}} \subset \prod_i \check{M}_i$ compact. There exists $\eta_0 > 0$ s.t. $\eta < \eta_0$ and for all $L \ge L_1(\eta)$, $im(\mathbb{c}) \supset \check{W} \supset \check{\mathbb{K}}$ and $\mathbb{c}|_{\mathbb{c}^{-1}(\check{W}) \cap \mathbb{M}^{cen}_\eta}$ is injective.*

Obstructed case:

For $i \in O$, take $\tilde{M}([-L,0] \times Y) \times M([0,L] \times Y)$, and R'_+, R'_- restrict to $\partial B^\sigma_{k-1/2}(\{0\} \times Y)$ and repeat.

1.0.6. *Orientation of moduli spaces.* This is explained during the exercise session.