SEIBERG-WITTEN FLOER HOMOLOGY LECTURE 12

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Please email yangding@math.hu-berlin.de if anything. Lecture notes up to now are available at www.mathematatik.hu-berlin.de/∼yangding/monopole.html. Exercises sprinkled throughout lecture notes have been collected into an exercise sheet at www.mathematik.hu-berlin.de/∼yangding/Exercise_SWF.pdf.

1. Lecture 12: Construction for Floer homologies and calculation for S^3

Last time in exercise session,

2 element set (of orientations)

$$\Lambda([\mathfrak{a}], q) := \bigsqcup_{[\mathfrak{a}_0] \text{ reducible}} \Lambda([\mathfrak{a}], q; [\mathfrak{a}_0], 0)/\big(\lambda \sim q(\lambda, \tau^{-1}(0))\big),$$

where $[\mathfrak{a}] \in B_k^\sigma(Y)$ and q a tame perturbation, $\tau : \Lambda([\mathfrak{a}_0], 0; [\mathfrak{a}_0'], 0]) \to \mathbb{Z}/2$ trivialization, and q concatenation.

Remark 1.1. Analogous to orientation set of U_a in f.d. Morse theory.

$\Lambda([\mathfrak{a}_1], q_1)\Lambda([\mathfrak{a}_2], q_2) \to \Lambda([\mathfrak{a}_1], q_1; [\mathfrak{a}_2], q_2), [(\lambda_1, \lambda_2)] \mapsto q(\lambda_1, \rho(\lambda_2))$ where ρ is take the inverse of the path. This way two absolute orientation (relative orientations relative to reducibles, equivalent over all reducibles) induces a relative orientation.

Here, for 2-element sets A and B, $AB := A \times_{\mathbb{Z}/2} B$ where $\mathbb{Z}/2$ acts on both factors non-trivially.

1.1. Inducing orientation of moduli spaces from relative orientation.

Fix q, write $\Lambda([\mathfrak{a}]) := \Lambda([\mathfrak{a}], q)$, and $\Lambda[\mathfrak{a}_1], [\mathfrak{a}_2] := \Lambda([\mathfrak{a}_1], q; [\mathfrak{a}_2], q)$.

How does an element of $\Lambda([\mathfrak{a}_1], [\mathfrak{a}_2])$ determine an orientation of $M([\mathfrak{a}_1], [\mathfrak{a}_2])$?

$[\gamma] \in M([\mathfrak{a}_1], [\mathfrak{a}_2])$, $\zeta = [\check{\gamma}]$ path in $B_k^\sigma(Y)$, $P_\gamma := Q_\gamma = \frac{d}{dt} + L$. Critical points $\mathfrak{a}_i$ nondegenerate so $L(t)$ hyperbolic for $|t| \geq T$ for T large enough. $H^-(L(t))$ varies continuously for $|t| \geq T$, so $\Lambda([\mathfrak{a}_1], [\mathfrak{a}_2]) = \Lambda(\zeta(t_1), \zeta(t_2))$ for any $t_1 \leq -T$ and $t_2 \geq T$.

1.1.1. *∂-unobstructed case.* $\Lambda^{\text{top}} T_{[\gamma]} M([\mathfrak{a}_1], [\mathfrak{a}_2]) \cong \det P_\gamma$.

$\gamma = \gamma_1 \cup \gamma_0 \cup \gamma_+$, where γ_- and γ_+ are for $t \leq t_1$ and $t \geq t_2$ respectively.

The earlier restriction map restricting to the fiber product is an isomorphism, which implies $\det P_\gamma = \det P_{\gamma_-} \otimes \det P_{\gamma_0} \otimes \det P_{\gamma_+}$. $P_{\gamma_\pm}$ close to constant coefficient operator, thus invertible, and orientation is canonical, and P_{γ_0} has orientation set $\Lambda(\zeta(t_1), \zeta(t_2)) = \Lambda([\mathfrak{a}_1], [\mathfrak{a}_2])$.

Date: February 5, 2021.

1.1.2. *∂-obstructed case.* $\ker P_\gamma = M([\mathfrak{a}_1],[\mathfrak{a}_2])$, and $\mathrm{cok} P_\gamma = \mathrm{cok}(\frac{d}{dt}+\lambda(t))$ which is oriented by $(0,0,-1)$ (just as already in definition of $\Lambda([\mathfrak{a}_0],q;[\mathfrak{a}_0'],q')$ done in the exercise session).

$\Lambda([\mathfrak{a}_1],[\mathfrak{a}_2]) = \det P_\gamma = \det T_{[\gamma]}M([\mathfrak{a}_1],[\mathfrak{a}_2]) \otimes \mathrm{cok}^* \cong \Lambda^{\mathrm{top}} T_{[\gamma]}M([\mathfrak{a}_1],[\mathfrak{a}_2])$.

Therefore, regularity of moduli spaces implies orientability.

1.2. **Orientation for the unparametrized moduli spaces.** $\mathbb{R} \hookrightarrow M_z([\mathfrak{a}],[\mathfrak{b}]) \to \check{M}_z([\mathfrak{a}],[\mathfrak{b}])$. $t : [\gamma] \mapsto [\tau_t^* \gamma]$.

For bundle, fiber first convention plus standard orientation for $\mathbb{R}$ implies orientation of $\check{M}$.

1.3. **Orientation under gluing.**

1.3.1. *∂-unobstructed.* $\check{M}_{z_1}([\mathfrak{a}_0],[\mathfrak{a}_1]) \times \check{M}_{z_2}([\mathfrak{a}_1],[\mathfrak{a}_2]) \subset \check{M}_z^+([\mathfrak{a}_0],[\mathfrak{a}_2])$

Let λ_{ij} be the induced relative orientation for $\check{M}([\mathfrak{a}_i],[\mathfrak{a}_j])$.

The LHS has orientation $q(\lambda_{01},\lambda_{12})$. The RHS has an orientation induced from $\check{M}_z([\mathfrak{a}_0],[\mathfrak{a}_2])$ which has $\lambda_{0,2}$ and using outwards normal first, we have an orientation for $\partial \check{M}^+$.

Those two orientations differ by $(-1)^{\dim M_{z_1}([\mathfrak{a}_0],[\mathfrak{a}_1])}$.

1.3.2. *∂-obstructed.* $M_{ii+1} := M_{z_{i+1}}([\mathfrak{a}_i],[\mathfrak{a}_{i+1}])$. Let

$$M := \prod_{i=0}^{2} \check{M}_{ii+1} = \check{M}_{01} \times \check{M}_{12} \times \check{M}_{23}.$$

Let us assume $\check{M}_{12}$ is ∂-obstructed. We have orientation λ_{01}, λ_{12}, λ_{23}, thus an orientation $q(\lambda_{01}\lambda_{12}\lambda_{23})$ on M.

The description of neighborhood of M in $\check{M}^+([\mathfrak{a}_0],[\mathfrak{a}_3]) =: N$. $\exists\, W \overset{\text{open}}{\subset} N$ gives

$$\begin{tikzcd}
M \arrow[r, "\text{fiber}"] \arrow[d] & W \arrow[r, "\text{top. emb. } j"] \arrow[d] & EW \arrow[r, "\delta"] \arrow[d, "\text{top. subm.}"'] & \mathbb{R} \\
\{(\infty,\infty)\} \arrow[r] & (0,\infty]^2 \arrow[r] & (0,\infty]^2 &
\end{tikzcd}$$

This structure is called codim 1 δ-structure along M.

EW locally $M \times (0,\infty]^2$ (fiber first order) and get an orientation from orientation $q(\lambda_{01}\lambda_{12}\lambda_{23})$ and standard one on $(0,\infty]^2$.

$W \hookrightarrow EW \xrightarrow{\delta} \mathbb{R}$ with fiber first convention gives an orientation on W, and thus an orientation on N.

Definition 1.2. M is said to have boundary orientation if the orientation λ_{03} on the top stratum of $N := \check{M}^+([\mathfrak{a}_0],[\mathfrak{a}_3])$ and the induced orientation on W (thus N) differ by $(-1)^{\dim \check{M}([\mathfrak{a}_0],[\mathfrak{a}_3])} = (-1)^{\dim M([\mathfrak{a}_0],[\mathfrak{a}_3])+1}$.

For $M = \prod_{i=0}^2 M_{ii+1}$, the boundary orientation and $q(\lambda_{01}\lambda_{12}\lambda_{23})$ differ by $(-1)^{\dim M_{01}+1}$.

1.4. **Reducible moduli spaces.** The reducible moduli space consists of trajectories in reducible part.

- If $[\mathfrak{a}]$ boundary-unstable and $[\mathfrak{b}]$ boundary-stable, $M^{\mathrm{red}}([\mathfrak{a}],[\mathfrak{b}]) = \partial M([\mathfrak{a}],[\mathfrak{b}])$ which has orientation from outwards normal first convention. Otherwise, $M^{\mathrm{red}} = M$ which inherits an orientation.

- Repeat early using reducible part (which behaves nicely under gluing), this gives orientation on M^{red}.
- Both orientations agree, except when both $[\mathfrak{a}]$ and $[\mathfrak{b}]$ are both boundary-unstable and they differ by $(-1)^{\dim M} = (-1)^{\dim M^{\text{red}}}$.

1.5. **Evaluation at the boundary components.** $\mathcal{U}$, an open cover for topological space B, has covering order $\leq d+1$ if every $d+2$ intersection $U_{i_0} \cap \cdots \cap U_{i_{d+1}} = \emptyset$ for distinct $U_i \in \mathcal{U}$.

A metric space has covering dimension $\leq d$ if every open cover has a refinement with covering order $\leq d+1$.

Čech cohomology $\check{H}^n(B;\mathbb{Z}) := \varinjlim H^n_{\text{Simp}}(K(\mathcal{U});\mathbb{Z})$ where $K(\mathcal{U})$ is the nerve which is a simplicial space to which one can attach simplicial cohomology.

If $B' \subset B, \mathcal{U}' := \mathcal{U}|_B := \{U \cap B\}$. $K(\mathcal{U}|_B) \overset{\text{subcomplex}}{\subset} K(\mathcal{U})$.

We have $\check{H}(B, B';\mathbb{Z}) = \varinjlim H^n_{\text{Simp}}(K(\mathcal{U}), K(\mathcal{U}');\mathbb{Z})$.

If $N^d \supset N^{d-1} \supset \cdots \supset N^0$ stratified, we have covering dimension $\leq d$ (fact).

If each $M^e := N^e \backslash N^{e-1}$ is oriented, we have $\check{H}^d(N^d, N^{d-1};\mathbb{Z}) = H^d_c(M^d;\mathbb{Z})$. Here $M^d = \bigsqcup_\alpha M^d_\alpha$.

Let $I_\alpha : \check{H}^d(N^d, N^{d-1};\mathbb{Z}) \to \mathbb{Z}$, $\mu^d_\alpha \mapsto 1$, where μ^d_α is the generator for M^d_α.

LES for (N^d, N^{d-1}, N^{d-2}) gives coboundary map $\delta_* : H^{d-1}_c(M^{d-1};\mathbb{Z}) \to H^d_c(M^d;\mathbb{Z})$, namely , $\bigoplus_\beta H^{d-1}_c(M^{d-1}_\beta;\mathbb{Z}) \to \bigoplus_\alpha H^d_c(M^d_\alpha;\mathbb{Z})$, defined as $\sum \delta_{\alpha\beta}$ wrt this splitting, where $\delta_{\alpha\beta} = I_\alpha \delta_* \mu^{d-1}_\beta$ is the multiplicity of M^{d-1}_β in the boundary of M^d_α. Boundary multiplicity is $\delta_\beta = \sum_\alpha \delta_{\alpha\beta}$.

N^d d-dimensional stratified space compact with an embedding to a metric B. An open cover $\mathcal{U}$ of B is transverse to N^d if $\mathcal{U}|_{N^e}$ has covering order $\leq e+1$.

Fact: $\{N^{d_k}_k\}$ countable locally finite collection of stratified manifolds. Then every open cover of B has a refinement that is transverse to $\{N^{d_k}_k\}$.

This implies that $\check{H}^n(B;\mathbb{Z}) = \varinjlim H^n_{\text{Sim}}(K(\mathcal{U};\mathbb{Z}))$.

$u \in \check{C}^d(\mathcal{U}|_{N^d};\mathbb{Z}) = C^d_{\text{Sim}}(K(\mathcal{U}|_{N^d});\mathbb{Z})$ is coclosed and vanishes on $C^{d-1}_{\text{Sim}}(K(\mathcal{U}|_{N^{d-1}});\mathbb{Z})$, so $[u \in \check{H}^d(N^d, N^{d-1};\mathbb{Z})]$ can integrate over $\bigsqcup_\alpha M^d_\alpha = M^d = N^d \backslash N^{d-1}$.

$\langle u, [M_\alpha]\rangle := I_\alpha[u|_{N^d}]$

Stokes theorem says for $v \in \check{C}^{d-1}(\mathcal{U};\mathbb{Z})$, $\sum_\beta \delta_{\alpha\beta}\langle v, [M^{d-1}_\beta]\rangle = \langle \delta v.[M^d_\alpha]\rangle$.

For ∂-obstructed case.

Lemma 1.3. *N^d d-dimensional compact stratified and stratum-oriented. If N^d has codim 1 δ-structure along M^{d-1}_β. Then $\delta_\beta = 1$.*

For $d = 1$, N^0 has boundary orientation, then $\#_{sign} N^0 = 0$.

1.6. **Floer homology.** Y compact connected oriented Riemannian 3-manifold (with Riem. metric g) and spin^c structure $\mathfrak{s}$. $\mathcal{P}$ tame perturbation Banach space. $q \in \mathcal{P}^{\text{res}}$ chosen s.t. all zeros of $(\nabla \pounds)^\sigma$ non-degenerate and $M([\mathfrak{a}],[\mathfrak{b}])$ regular. Moreover, if f $c_1(\mathfrak{s})$ not torsion, q small, $\not\exists$ reducible solution/zero (doable, remark after proposition 16.4.3. in [KM]), such q is called admissible.

Define $\check{HM}_*(Y,\mathfrak{s})$, $\widehat{HM}_*(Y,\mathfrak{s})$, $\overline{HM}_*(Y,\mathfrak{s})$. M for monopole.

$\mathfrak{c} \subset B^\sigma_k(Y,\mathfrak{s})$ zeros of $(\nabla \pounds)^\sigma$ split into $\mathfrak{c}^o \sqcup \mathfrak{c}^s \sqcup \mathfrak{c}^u$, irreducible, boundary-stable, boundary-unstable zeros.

$\Lambda = \{x, y\}$ 2-element (orientation) set. $\mathbb{Z}\Lambda = \mathbb{Z}\langle x, y\rangle/(x = -y) \cong \mathbb{Z}$ if preferred element in Λ is chosen.

$C^0 = \bigoplus_{[\mathfrak{a}]\in\mathfrak{c}^o} \mathbb{Z}\Lambda([\mathfrak{a}])$, C^s and C^u similarly.

Define $\check{C} = C^0 \oplus C^s$, $\hat{C} = C^0 \oplus C^u$, $\overline{C} = C^s \oplus C^u$.

A choice of $\Lambda([\mathfrak{a}],[\mathfrak{b}])$ determines an orientation of $\check{M}_z([\mathfrak{a}],[\mathfrak{b}])$, which leads to $\epsilon[\gamma] : \mathbb{Z}]\Lambda([\mathfrak{a}]) \to \mathbb{Z}\Lambda([\mathfrak{b}])$ for all $[\gamma] \in \check{M}_z([\mathfrak{a}],[\mathfrak{b}])$.

For $[\gamma] \in \check{M}^{\text{red}}M_z([\mathfrak{a}],[\mathfrak{b}])$, we have similarly $\bar{\epsilon}[\gamma]$ agrees with $\epsilon[\gamma]$ except $\bar{\epsilon}[\gamma] = -\epsilon[\gamma]$ when both $[\mathfrak{a}],[\mathfrak{b}] \in \mathfrak{c}^u$.

Define $\bar{\partial} : \overline{C} \to \overline{C}$ as $\sum_{[\mathfrak{a}],[\mathfrak{b}]} \sum_{[\gamma]\in\check{M}_z^{\text{red}}([\mathfrak{a}],[\mathfrak{b}]),\ \text{0-dim}} \bar{\epsilon}\gamma]$, here the sum is finite for each $\mathbb{Z}\Lambda([\mathfrak{a}])$. It can be written as $\begin{pmatrix} \bar{\partial}^s_s & \bar{\partial}^u_s \\ \bar{\partial}^s_u & \bar{\partial}^u_u \end{pmatrix}$.

Similar can define ∂^o_o, ∂^o_s, ∂^u_s, ∂^u_o,
e.g. $\partial^o_o = \sum_{[\mathfrak{a}]\in\mathfrak{c}^o \text{ supscript}} \sum_{[\mathfrak{b}]\in\mathfrak{c}^o \text{ subscript}} \sum_{[\gamma]\in\check{M}_z([\mathfrak{a}],[\mathfrak{b}]),\ \text{0-dim}} \epsilon[\gamma]$.

Define $\check{\partial} = \begin{pmatrix} \partial^o_o & -\partial^u_o\bar{\partial}^s_u \\ \partial^o_s & \bar{\partial}^s_s - \partial^u_s\bar{\partial}^s_u \end{pmatrix}$ on $\check{C} = C^o \oplus C^s$, this calculates Morse homology of $B^\sigma_k(Y,\mathfrak{s})$ w.r.t. gradient-like $(\nabla \pounds)^\sigma$.

Define $\hat{\partial} = \begin{pmatrix} \partial^o_o & \partial^u_o \\ -\bar{\partial}^s_u\partial^o_s & -\bar{\partial}^u_u - \bar{\partial}^s_u\partial^u_s \end{pmatrix}$ on $\hat{C} = C^o \oplus C^u$, this calculates Morse homology of $\big(B^\sigma_k(Y,\mathfrak{s}), \partial B^\sigma_k(Y,\mathfrak{s})\big)$ w.r.t. gradient-like $(\nabla \pounds)^\sigma$.

Note that we are able to upgrade mod 2 into signed coefficient after discussion of orientation.

$\bar{\partial}$, $\check{\partial}$, $\hat{\partial}$ squares to 0, by considering the 1-dimensional stratified compactified space of broken trajectory and use $\#_{\text{sign}}N^0(\check{M}^+) = 0$ (same line of reasoning as in Morse theory with vertical boundary and see some proofs earlier in the lecture).

We have L.E.S. relating all three (see verbatim construction and proof in earlier lecture).

Definition 1.4. $HM_*(Y,\mathfrak{s}) = \text{im}(j_*)$ called reduced monopole Floer homology, where $\check{HM}_*(Y,\mathfrak{s}) \xrightarrow{j_*} \widehat{HM}_*(Y,\mathfrak{s})$, $j = \begin{pmatrix} 1 & 0 \\ 0 & -\bar{\partial}^s_u \end{pmatrix}$.

Proposition 1.5. *$HM(Y,\mathfrak{s})$ is of finite rank.*

Proof. If $c_1(\mathfrak{s})$ not torsion, $\mathfrak{c} = \mathfrak{c}^o$ finite due to compactness and $\check{HM} = \widehat{HM} = HM$.

If $c_1(\mathfrak{s})$ torsion, $\bar{\partial}^s_u$, $\bar{\partial}^u_s$, ∂^u_s finitely many nonzero matrix entries, because nonzero only when $\text{gr}_z([\mathfrak{a}_i],[\mathfrak{b}_j]) = d + 2i - 2j = 0$ and $i > 0$ and $j < 0$. Here $\mathfrak{a}_i$ and $\mathfrak{b}_j$ blow down $[\alpha]$ and $[\beta]$. $\mathfrak{a} = (\alpha,\lambda)$ and $\alpha = (B,0)$.

$$i = i(\mathfrak{a}) = \begin{cases} |\text{Spec}D_{q,B} \cap [0,\lambda)|, \lambda > 0 \\ \frac{1}{2} - |\text{Spec}D_{q,B} \cap [\lambda,0]|, \lambda < 0 \end{cases},$$

and $j := i(\mathfrak{b}_j)$ similarly.

$\text{rk}HM_*(Y,\mathfrak{s}) \le \text{rk}C^0 + \text{rk}\bar{\partial}^s_u$.

Suppose $H \subset HM_*(Y,\mathfrak{s})$ generated in $0 \oplus C^s$ such that $j_*|_H$ injective.

Then $j(H) \subset (0, \bar{\partial}^s_u(H))$ finite rank. □

1.7. **Grading.** Define $\mathbb{J}(\mathfrak{s})$ grading for $\check{HM}_*$ etc.

$\mathbb{J}(Y,\mathfrak{s}) := B^\sigma_k(Y, \mathfrak{s} \times \mathcal{P} \times \mathbb{Z})/\sim$, where $([\mathfrak{a}], q_1, m) \sim ([\mathfrak{b}], q_2, n])$ if $\exists$ path $\zeta = [\check{\gamma}]$ joining $[\mathfrak{a}]$ to $[\mathfrak{b}]$, and p 1-parameter perturbation from q_1 to q_2, s.t. $\text{ind}D_{\gamma,p} = n - m$.

$\mathbb{Z}$ acts on $\mathbb{J}(\mathfrak{s})$, $([\mathfrak{a}, q, m]) \mapsto ([\mathfrak{a}, q, m+1])$.

$\text{gr}[\mathfrak{a}] = ([\mathfrak{a}, q, 0])/\sim\, \in \mathbb{J}(\mathfrak{s})$.

$\mathfrak{a}$ reducible, define $\bar{\mathrm{gr}}[\mathfrak{a}] = \begin{cases} \mathrm{gr}[\mathfrak{a}], & \mathfrak{a} \in \mathfrak{c}^s, \\ \mathrm{gr}[\mathfrak{a}] - 1, & \mathfrak{a} \in \mathfrak{c}^u \end{cases}$.

$\check{C}_j = \bigoplus_{[\mathfrak{a}] \in \mathfrak{c}^o \cup \mathfrak{s}^s,\ \mathrm{gr}[\mathfrak{a}]=j} \mathbb{Z}\Lambda([\mathfrak{a}])$, etc.

$\check{\partial}$, $\hat{\partial}$, $\bar{\partial}$ differential of degree -1, $\check{\partial}(\check{C}_j) \subset \check{C}_{j-1}$.

$\mathbb{Z}$ acts transitively on $\mathbb{J}(\mathfrak{s})$ with stabilizer $\mathrm{im}([\sigma] \in H_2(Y;\mathbb{Z}) \mapsto \langle c_1(\mathfrak{s}), [\sigma] \rangle) \subset 2\mathbb{Z}$. Thus free, iff $c_1(\mathfrak{s})$ torsion.

1.8. **Calculation of Monopole Floer homology for S^3 with round metric.** The unperturbed CSD for the unique spin^c structure $\mathfrak{s}$ has 1 critical point, reducible $[B, 0]$ with $F_{B^t} = 0$ (becuase 3d Riemannian Y with scalar curative ≥ 0, the only critical $[B, \psi]$ of unperturbed CSD is reducible).

In $B_k^\sigma(S^3, \mathfrak{s})$, zeros of $(\nabla \pounds)^\sigma$ degenerate (as the spectrum D_B is not simple). Choose small $q \in \mathcal{P}$ still unique critical point, $[B, 0]$ reducible, but $D_{q,B}$ now has simple nonzero eigenvalues. Label increasinglyly λ_i with λ_0 being the first positive eigenvalue. Eigenvalues are 1-1 corresponding to gauge equivalence classes of zeros of $(\nabla \pounds)^\sigma$, $[\mathfrak{a}_i] = ([B, 0], \lambda_i)$.

$[\mathfrak{a}_i], i \geq 0, \in \mathfrak{c}^s$.

$[\mathfrak{a}_i], i < 0, \in \mathfrak{c}^u$.

By assigning $[\mathfrak{a}_0] \mapsto 0$, then $\mathbb{J}(S^3, \mathfrak{s}) \cong \mathbb{Z}$, having free transitive $\mathbb{Z}$-action.

If λ_i, λ_{i-1} have the same sign, then $\mathrm{gr}_z([\mathfrak{a}_i], [\mathfrak{a}_{i-1}]) = 2$ independent of z.

The remaining case: $\mathrm{gr}_z([\mathfrak{a}_0], [\mathfrak{a}_{-1}]) = 1$.

Therefore, $\mathrm{gr}[\mathfrak{a}_i]$ has $\mathbb{Z}$-grading, $= \begin{cases} 2i & i \geq 0 \\ -2i + 1 & i < 0 \end{cases}$.

$\Lambda[\mathfrak{a}_i] = \Lambda([\mathfrak{a}_i], [\mathfrak{a}_i])/\sim$ has preferred element.

So $\check{C}_j = \mathbb{Z}$ for even $j \geq 0$ and 0 otherwise; $\hat{C}_= \mathbb{Z}$ for odd $j < 0$ and 0 otherwise; and $\overline{C}_j = \mathbb{Z}$ for even j (due to the shift in grading as detailed above).

Differentials are 0 trivially, as either domain or the target is 0. $H\check{M}_j \cong \check{C}_j$ etc, homologies equal to the chain groups.

Therefore $HM_*(S^3, \mathfrak{s}) = 0$ as $j_* = 0$.