SEIBERG-WITTEN FLOER HOMOLOGY LECTURE 13

DINGYU YANG

Please email yangding@math.hu-berlin.de if anything. Lecture notes up to now are available at www.mathematik.hu-berlin.de/∼yangding/monopole.html. Exercises sprinkled throughout lecture notes have been collected into an exercise sheet at www.mathematik.hu-berlin.de/∼yangding/Exercise_SWF.pdf.

1. Lecture 13: Non-vanishing of $\widehat{\mathrm{HM}}_*$ and Weinstein conjecture

Coupled Morse theory associates homology group to manifold Q equipped with a family of self-adjoint (s.a.) Fredholm operators of index 0 parametrized by Q. We will introduce it below.

1.1. **Family of self-adjoint operator over Q.** Let H be a separable complex Hilbert space of ∞ dimensionality. $K : H \to H$ compact s.a. with $\ker K = 0$ (injective), and $H_1 := K(H)$ dense in H thus Hilbert, with $\|v\|_1 := \|K^{-1}v\|$.

$K = K^+ \oplus K^-$ splitting into the $\pm$ve parts with respective images $H_1^\pm$ with the closure in H being $H^\pm$. E.g. $H_1 = L_1^2(Y; E) \subset H := L^2(Y; E)$.

Definition 1.1. J compact positive s.a. operator on H with $\ker J = 0$. Let its eigenvalues enumerated as follows: $\mu_i = \lambda_i^{-1}$ with $0 < \lambda_0 \leq \lambda_1 \leq \cdots$. The spectrum of J is called mild if $\exists\, C$ s.t. $\frac{\lambda_{2N}}{\lambda_N} \leq C$.

$B(H : H_1) := \{x : H \to H \mid \text{bounded linear s.t. } x(H_1) \subset H_1, x^*(H_1) \subset H_1$ with finite operator norms in $H_1\}$.

Let $U(H : H_1) := \{x \in B(H : H_1) \mid x^*x = 1\}$.

$S(H : H_1) := \{L : H \to H \mid \text{Fredholm, index } 0, \text{ s.a.}\} \supset \{\text{SAFOE diff operator}\}$ (if H is a function space).

Remark 1.2. $L \in S(H : H_1)$, $\exists$ complete orthonormal (o.n.) system $\{e_i\}_i$ for H with $e_i \in H_1$ and eigenvectors with eigenvalues λ_i.

As $\{\lambda_i\}$ has no accumulation point, we have either (i) bdd from below, (ii) bdd from above, or (iii) unbounded in both directions.

Definition 1.3. $S_*(H : H_1) := \{L \in S(H : H_1) \mid$ satisfying (iii) above, $H_1^\pm(L)$, with H-closure $H^\pm(L)$ eigenspaces of eigenvalues ≥ 0 and < 0 of L, s.t.
$\exists u \in U(H : H_1)$with $u(H^\pm) = H^\pm(L), u(H_1^\pm) = H_1^\pm(L)\}$.

We now define (Q, L): Q a compact Riem. maniold, $P \to Q$ principle bundle with structure group $U(H : H_1)$, with associated vector bundles $\mathcal{H}_1$ and $\mathcal{H}$ over Q.

Definition 1.4. A family of s.a. operators of type $S_*(H : H_1)$ over Q is a principle $U(H : H_1)$ bundle above with a bundle map $L : \mathcal{H}_1 \to \mathcal{H}$ between the associated vector bundles (v.b.), equivalently, a smooth section of $S_*(\mathcal{H} : \mathcal{H}_1) \to Q$ with fiber $S_*(H : H_1)$.

Date: February 12, 2021.

We have the following properties:

- (Kuiper) $U(H)$ contractible.
- If the spectrum of $|K| := K^+ - K^-$ is mild, then $U(H : H_1)$ is contractible (lengthy but ok proof, similar to proving the previous item, see [KM] 33.1.5).
- If K^+ and $-K^-$ have mild spectrum, then $S_*(H : H_1)$ has homotopy type of $U(\infty) = \lim U(n)$.
- By points 2 (we can homotopically uniquely trivialize any $U(H : H_1)$-bundle) and 3, we have that families of s.a. operator of type $S_*(H : H_1)$ over Q are classified by $[Q, U(\infty)]$ homotopy maps between them.

  This means any such family over Q is the pullback of the universal family over $U(\infty)$ via a map (called classifying map) from Q to $U(\infty)$.

Now, we give a description (a model) of the universal family of s.a. operator over $U(\infty)$. Any other over some Q is a pullback via a map above $[Q, U(\infty)]$.

We describe a criterion to tell whether two such maps are homotopic. Let $U^f(H) := \{u \in U(H) \mid u - 1 \text{ has finite rank}\}$.

Lemma 1.5. *Let $u_1, u_2 : Q \to U^f(H)$. Suppose $\exists$ continuous $\theta : Q \to U^f(H)$ s.t. $\theta u_1 = u_2\theta$ and for all $q \in Q$, $\theta|_{\ker(u+1)} : \ker(u_1 + 1) \to \ker(u_2 + 1)$ is isomorphism, then u_1 and u_2 are homotopic.*

$U(\infty) = \lim_N U(N)$, we describe family of L over $U(N)$ and show the classifying map is (homotopic to) inclusion, which then induces the family over $U(\infty)$.

For $z \in U(N)$, let $C^\infty(S^1, z) := \{h : \mathbb{R} \to C^N \mid \text{smooth} h(t + 1) = zh(t)\}$ with L^2 and L^2_1 norms (integration over $[0, 1]$) and the respective completions denoted by $H(z)$ and $H_1(z)$. We denote $H := H(1)$ and $H_1 := H_1(1)$. We have a bundle over $U(N)$ with structure group $U(H : H_1)$. Let $L(z) : H_1(z) \to H(z), h \mapsto -i\frac{d}{dt}h$ of type $S_*(H : H_1)$. $L(z)$ is classified by $\psi : U(N) \to U(\infty)$, which using the above lemma is homotopic to the inclusion.

We have described the object for which we can define a homology theory, and we describe it now:

1.2. **Coupled homology $\overline{H}(Q, L)$.** Let (g, f) be a Morse-Smale pair on Q, which means g is a Riemannian metric and f is Morse function such that the moduli spaces of trajectories of ∇f defined using g are regular. Let ∇ be a connection on a principle bundle, which means 1-form valued in Lie algebra of $U(H : H_1)$ with equivariant properties (it is instructive to recall this exactly, but we will not need this below).

For each $q \in \mathrm{Crit}(f)$, a critical point, assume $L(q) : H_1(q) \to H(q)$ has simple nonzero spectrum.

Assume there is no spectral flow around loops in Q ($\iff$ classifying map $Q \to SU(\infty) \subset U(\infty)$ factoring through $SU(\infty)$), which means we can label eigenvalues of $L(q)$, $q \in \mathrm{Crit}(f)$, $\ldots \lambda_{-1}(q) < \lambda_0(0) < \lambda_1(q) < \cdots$, s.t. any path $q(t)$ joining q_1 to q_2, the spectral flow along $q(t)$ is $s(q_2) - s(q_1)$ where absolute spectral flow can be arranged as $s(q) = |\{i \mid i < 0, \text{and } \lambda_i(1) > 0\}|$.

Let $\phi_i(q)$ be a unit eigenvector for $\lambda_i(q)$.

Define $\overline{C}_n = \overline{C}_n(Q, L) = \bigoplus_{q \in \mathrm{Crit}(f)} \bigoplus_{i, \mathrm{ind} q + 2i = n} \mathbb{Z}\Lambda_q$, where Λ_q is the orientation 2-element set for the unstable manifold at q (from which we can get stable manifold orientation, and thus the moduli space orientation through intersection of two oriented submanifolds).

$$\begin{cases} \frac{d}{dt}\gamma + (\nabla f)_{\gamma(t)} = 0 \\ (\gamma^*\nabla)\phi + (L(\gamma(t))\phi)dt = 0 \end{cases}$$

Define $M(q_0, i; q_1, j)$ the $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$ (acting on the second variable) quotient of (γ, ϕ) of solutions of the above equation s.t.

- $\gamma(t)$ from q_0 to q_1,
- $\phi(t)$ has the leading asymptotic $c_0 e^{-\lambda_i t}\phi_i(q_0)$ as $t \to -\infty$,
- $\phi(t)$ has the leading asymptotic $c_1 e^{-\lambda_j t}\phi_j(q_1)$ as $t \to +\infty$.

We need weighted Sobolev space to make $\phi \mapsto (\gamma^*\nabla)\phi + (L(\gamma(t))\phi)dt$ a Fredholm problem with $\mathrm{ind}_{\mathbb{C}} = i - j + 1$.

We can make the moduli space regular so that $M(q_0, i; q_1, j)$ is a smooth manifold with dimension $\mathrm{ind}(q_0) - \mathrm{ind}(q_1) + 2(i-j)$. We also have $\check{M}$ and $\check{M}^+$ as before.

We define the differential as $\bar{\partial} = \sum_{\check{M},\ \dim 0} \sum_{[\gamma,[\phi]] \in \check{M}} \epsilon([\gamma, [\phi]])$.

Homotopic L's give isomorphic $\overline{H}_*(Q, L)$'s.

$\overline{H}_*(Q, L)$ is a module over $H^*(\mathbb{P}(\mathcal{H}))$

For the tautological line bundle L over $\mathbb{P}(\mathcal{H})$, $u_2 := -c_1(L)$ and we can construct Morse chain level operation $(\tilde{u}_2 \cap \cdot) : \overline{C}_k \to \overline{C}_{k-2}$ which is invertible for all k.

There is also a version with spectral flow, which we suppress, as we do not need it below.

1.3. Calculation of $\overline{HM}_*$ for 3-manifold with torsion spin^c structure.

Let Y be a 3-manifold oriented with spin^c structure $\mathfrak{s}$ s.t. $c_1(\mathfrak{s}) := c_1(S)$ torsion.

Reducible critical points of unperturbed CSD $\mathcal{L}$ in $\mathcal{B}(Y, \mathfrak{s})$
$=\mathbb{T}$ torus of gauge equivalence class of spin^c connection A s.t. $\mathrm{tr}(A)$ is flat.
$\cong H^1(Y; \mathbb{R})/H^1(Y; \mathbb{Z})$.

Let $A_*(Y)$ be the exterior algebra generated by $A_1(Y) := H^1(Y; \mathbb{Z})$. Then $A_*(Y) \cong H_k(\mathbb{T}; \mathbb{Z})$.

We have $\beta_k : A_k(Y) \to A_{k-3}(Y)$,
$\alpha_1\alpha_2\cdots\alpha_k \mapsto \sum_{i_1<i_2<i_3} (-1)^{i_1+i_2+i_3} \langle \alpha_{i_1} \cup \alpha_{i_2} \cup \alpha_{i_3}, [Y]\rangle \alpha_1 \cdots \hat{\alpha}_{i_1} \cdots \hat{\alpha}_{i_2} \cdots \hat{\alpha}_{i_3} \cdots \alpha_k$.

Theorem 1.6. *Y closed connected oriented 3-manifold, $\mathfrak{s}$ has c_1 torsion. Then $\overline{HM} := \overline{HM}(Y, \mathfrak{s}; \mathbb{Q})$ has a filtration $\overline{HM} \supset \cdots \supset \mathcal{F}_s\overline{HM} \supset \mathcal{F}_{s-1}\overline{HM} \supset \cdots$, s.t. the graded pieces $\frac{\mathcal{F}_s\overline{HM}}{\mathcal{F}_{s-1}\overline{HM}} \cong \frac{\ker\beta_s}{im\beta_{s+3}} \otimes \mathbb{Q}[T^{-1}, T]$, the expression to the right of the tensor product is the polynomial ring generated by formal variables T^{-1} and T.*

Proof. We have retraction $p : B(Y, \mathfrak{s}) \to \mathbb{T}$, and f Morse on $\mathbb{T}$. Consider $f \circ p = f_1$ which is an example of a cylinder function. Define $\not{\mathcal{L}} = \mathcal{L} + f_1$.

Reducible critical points of $\not{\mathcal{L}}$is critical points $[\alpha]$ of f in $\mathbb{T} \subset B(Y, \mathfrak{s})$.

The flow of $-\nabla\not{\mathcal{L}}$ preserves $\mathbb{T}$.

In $B^\sigma(Y, \mathfrak{s})$, reducible zeros $[\alpha]$ are gauge orbits of $(\alpha, [\phi])$ where $\alpha \in [\alpha]$ is a critical point of f and $\phi \in L^2(Y; S)$ is an eigenvector of Dirac D_α.

Using perturbation in $\mathcal{P}$ that vanishes at the reducible locus, everything above remains, except D_α becomes $D_{q,\alpha}$ and $[\alpha]$ is now non-degenerate in B^σ and $M_z^{\mathrm{red}}([\alpha], [\beta])$'s are regular.

Claim: $\overline{HM}(Y, \mathfrak{s})$ is homology of $\overline{C}_*(Q, L)$ where $Q = \mathbb{T}$ and $L := \{D_{q,\alpha}\}_{[\alpha]\in T}$.

L corresponds to a classifying map $\mathbb{T} \to SU(2) \hookrightarrow U(\infty)$. The pullback of the 3-d generator of $SU(2) = S^3$ is $\xi_3 : (a_1, a_2, a_3) \mapsto \langle a_1 \cup a_2 \cup a_3.[Y]\rangle \in \Lambda^3 H^1(Y; \mathbb{Z})^*$.

This is seen through a computation involving Chern character ch. We quickly remark $ch(V \oplus W) = ch(V) + ch(W)$, $ch(V \otimes W) = ch(V)ch(W)$, and $ch(V) = \mathrm{tr}(\exp(i\Omega/2\pi))$ where Ω is the curvature of a connection using Chern-Weil theory.

Then $\exists$ a Riemannian metric on Q, Morse f and homotopic $\tilde{L}$ s.t. $\overline{HM}(Q,\tilde{L}) = C_*(Q,f)\otimes\mathbb{Z}[T^{-1},T]$, here the differential $\bar{\partial}x = \partial x + T(\tilde{\xi}\cap x)$ where ∂ is Morse differential, and $\tilde{\xi}$ is Morse chain level cap product with ξ. This squares to 0. $\square$

I gave a lecture in another lecture series about spectral sequence, its intuition generalizing SES inducing LES, and how it works and I will put that part here soon. For now, it is a way to calculate homology using doubly graded complex with differential pointing left and up and homology of it is the next page with induced longer differential, at infinite page if it stabilizes, it is said to converge/abut.

$\exists$ spectral sequence abutting to $\overline{H}_*(Q,L)$ where E^2 and E^3 terms are: $E^2_{s,2j} = E^3_{s,2j} = T^jH_s(Q)$ and differential on E^3 page is

$$d^3_{s,2j}: E^3_{s,2j}\to E^3_{s-3,2j+2}, T^j[x]\mapsto T^{j+1}\xi\cap[x].$$

If the higher differentials for pages 4 or later vanish, then $\overline{H} := \overline{H}_*(Q,L)$ has $\cdots\supset\mathcal{F}_s\overline{H}\supset\mathcal{F}_{s-1}\overline{H}\supset\cdots$ with $\mathcal{F}_s\overline{H}/\mathcal{F}_{s-1}\overline{H}\cong\ker\beta_s/\mathrm{im}\beta_{s+3}\otimes\mathbb{Z}[T^{-1},T]$.

1.4. **Non-vanishing at infinite gradings.** The previous subsection implies that $\overline{HM}_*(Y,\mathfrak{s})$ with $c_1(\mathfrak{s})$ torsion is nonzero and has infinite rank.

Proof. Suffice to show $\frac{\ker\beta_s}{\mathrm{im}\beta_{s+3}}$ has nonzero rank for at least one of s. Let $\zeta = e^{2\pi i/6}$.

$$\begin{aligned}&\sum_s \mathrm{rank}(\frac{\ker\beta_s}{\mathrm{im}\beta_{s+3}})\zeta^s\\ =&\sum_s\mathrm{rank}(A_s)\zeta^s\\ =&\text{evaluation of Poincaré polynomial of } \mathbb{T} \text{ at } \zeta\\ =&(1-\xi)^{b_1(Y)}, \text{ which is always nonzero.}\end{aligned}$$

$\square$

Thus, $\widehat{HM}_*(Y,\mathfrak{s})$ nonzero in infinitely many grading for $\mathfrak{s}$ with c_1 torsion.

Proof. $\widehat{HM}_*(Y,\mathfrak{s})$ graded by $\mathbb{J}(Y,\mathfrak{s})\cong\mathbb{Z}$ if $c_1(\mathfrak{s})$ torsion.

By definition, $\widehat{HM}_*$: non-trivial grading (grading that is not trivial) is bounded from above.

$\check{HM}_*$: non-trivial grading is bounded from below.

$\overline{HM}_*$ has nontrivial gradings that are infinite in both directions, seen above.

LES relating these 3 implies that we have isomorphism between $\overline{HM}$ and $\widehat{HM}$ when grading is sufficiently below. $\square$

1.5. **Application in proof of Weinstein conjecture.** Following Hutchings' exposition, we will give a sketch on how an application of the nonvanishing result in the previous section together with lots of new insights gives a deep result in dynamics in Taubes' proof of Weinstein conjecture. One should argue the following is loosely related to the subject matter in this lecture series, and can omit it without continuity, but partly because of the interests of the department and partly because of lots of interests of HM in recent years have come from outside like dynamics, it is helpful to be aware of reasoning using the concepts and results in the Seiberg-Witten world, even if one is to study SW per se, and we will see lots of notions we have familiarized ourselves with in past lecturees have come into play.

Let Y be a closed 3-manifold, and λ 1-form on Y such that $\lambda \wedge d\lambda$ is nowhere vanishing. $\xi := \ker \lambda$ is called a (cooriented) contact structure.

For such λ, we have $\iota_\bullet d\lambda : TY \to T^*Y, X \mapsto \iota_X d\lambda := d\lambda(X, \cdot)$ and it has 1-dimensional kernel denoted by L, and $\lambda|_L$ is a nowhere zero, so we can define a vector field $R \in \Gamma(L) \subset \Gamma(TY)$ s.t. $\iota_R d\lambda = 0$ and $\lambda(R) = 1$, so R is a nowhere zero vector field, called Reeb vector field. One can study its dynamics e.g. whether R has a closed orbit $\gamma : \mathbb{R}/T\mathbb{R} \to Y$ such that $\dot{\gamma} = R \circ \gamma$ with period $T < \infty$ as the first approximation. This does not do the justice of the following question, but it is an important question that drives lots of development in contact geometry, and earlier work by Hofer on proof for the case of $Y = S^3$ and overtwisted case introduces holomorphic curve into contact geometry and establishes the connection of existence of Reeb orbits and existence of finite energy punctured holomorphic curves, lays the foundation and inspires the creation of a powerful invariant called symplectic field theory (SFT) and its variants.

Weinstein conjecture (3d): For (Y, λ) closed contact 3-manifold. $\exists$ a closed orbit of the Reeb vector field R (associated to λ).

Taubes approached this using Seiberg-Witten theory and used in an essential way the non-vanishing result in the earlier subsection.

How do we get the initial data needed to construct SW from the current setting. First fix a Riemannian metric g on Y such that $|\lambda|_g = 1$ and $d\lambda = 2 *_g \lambda$ (both 2-forms are nonvanishing and one can define $*$ using a multiple of $\lambda \wedge d\lambda$ as volume form and adjust the proportion using g).

$\xi = \ker \lambda = K^{-1}$ (the anticanonical line bundle (for symplectization), at the following one can just regard it as a notation). Denote the associated spin^c structure as $\mathfrak{s}_\xi := (S, c)$ where $S = S_\xi = \underline{C} \oplus K^{-1}$ with the first factor denoting the trivial bundle and the Clifford multiplication acts by $c(\lambda)$ acts as i and $-i$ in each factor respectively.

Any spin^c structure $\mathfrak{s}$ is $E \otimes S = E \oplus K^{-1}E$ for some complex line bundle E.

We perturb SW equation by exact 2-form as follows

$$(\ddagger) \begin{cases} *F_A = r(\langle \rho(\cdot)\psi, \psi \rangle - i\lambda) + i\overline{w} \\ D_A \psi = 0 \end{cases}$$

Here $\overline{w}$ is a harmonic 1-form, such that $[\frac{*\overline{w}}{\pi}] = c_1(K^{-1})$ with $\pi = 3.14...$ the universal constant.

$\exists$ unique Hermitian connection A_0 and spinor $\psi_0 = (1, 0) \in \Gamma(\underline{C} \oplus K^{-1})$ s.t. $D_{A_0}\psi_0 = 0$. A trivial solution to the equation for any r.

We want to let $r \to \infty$.

After scaling, and small perturbation (to achieve regularity etc), IFT shows that we still have:

For all $\delta > 0$ if $r >> 0$ (my notation for being large enough), we have

$\exists$ unique (up to gauge transformation) (A_{triv}, ψ_{triv}) to $(\ddagger)$ for $\mathfrak{s}_\xi$ such that:

$$\begin{cases} 1 - |\psi_{triv}| \le \delta \text{ on } Y \quad (\dagger) \text{ in fact, LES is of order } O(r^{-1/2}) \\ \text{grading of } (A_{triv}, \psi_{triv}) \text{in HM chain group is independent of } r. \end{cases}$$

So for δ small enough, ψ is a nowhere vanishing section.

Fix E, $\mathfrak{s} = E \otimes \mathfrak{s}_\xi$.

Let (A_n, ψ_n) be a sequence of solutions to $(\ddagger)_{r_n}$ with $r_n \to \infty$. Suppose

$$\begin{cases} (1)\ \exists \delta > 0, \text{ with } \sup_Y (1 - |\psi_n|) > \delta \text{ (thus not the trivial solution above)}, \\ (2)\ \exists C < \infty, \text{ with } i\int_Y \lambda \wedge F_{A_n} < C \text{ (energy control for the following convergence)}. \end{cases}$$

Then $\exists$ non-empty orbit set a (a formal sum of closed Reeb orbits with nonzero integer weights) s.t. $[a] = PD(c_1(E))$.

The idea is that $\psi_n = (\alpha_n, \beta) \in \Gamma(E \oplus EK^{-1})$ w.r.t the splitting. We have α_n^{-1} converges to a as a current (in the dual space of smooth compactly supported dR forms) under energy bound (2) and which would be non-empty due to (1) and uniqueness of the perturbed trivial solution, while $|\beta_n| \to 0$. Topological property is one definition of c_1. This is inspired by Taubes' earlier work of getting a holomorphic curve from SW solution (with r parameter) in closed symplectic 4-manifolds.

Fix E s.t. $\mathfrak{s}_E = E \otimes \mathfrak{s}_\xi$ has torsion c_1 (note that $c_1(\mathfrak{s}_E) = c_1(K^{-1}) + 2c_1(E)$ as $S = E \oplus K^{-1}E$).

Theorem 1.7 ([KM] in previous subsection)**.** *$\exists$ solution to $(\ddagger)_r$ for all $r \geq 1$*

Need to find a sequence of solutions satisfying conditions (1) and (2):

1.5.1. *Condition (1).* If $c_1(E) \neq 0$, $\exists c > 0$ s.t. for $r >> 0$ if (A, ψ) is solution to $(\ddagger)_r$, then $\exists$ point $p \in Y$, s.t. $1 - |\psi(p)| \geq 1 - \frac{c}{\sqrt{r}}$.

For $c_1(E) = 0$, since "trivial" solution is unique at one degree (independent of r), but $\widehat{\text{HM}}$ nonzero in infinitely many gradings.

1.5.2. *Condition (2).* In the current perturbed form $CS_{\overline{w}}(A) := -\int_Y (A - A_1) \wedge (F_A + F_{A_1} - 2i * \overline{w})$. Want to show that it is a sum of 2 functionals to control. It is gauge-invariant as c_1 is torsion.

Definition 1.8. $\mathcal{E}(A) := i \int_Y \lambda \wedge F_A$ (appeared in condition (2)).

$CSD(A, \psi) := -\frac{1}{8} \int_Y (A - A_0) \wedge (F_A + F_{A_0} - 2i\mu) + \frac{1}{2} \int_Y \langle D_A \psi, \psi \rangle d\text{vol}$ (with $d\text{vol} = \lambda \wedge d\lambda$).

($\mu = -rd\lambda - iF_{A_0} + 2 * \overline{w}$). Let the reference base point be $A_0 + 2A_1$.

From these, we have $CSD(A, \psi) = \frac{1}{2}(CS(A) - r\mathcal{E}(A)) + \frac{r}{2} \int \langle D_A \psi, \psi \rangle d\text{vol}$.

For SW solution, we have $CSD(A, \psi) = \frac{1}{2}(CS(A) - r\mathcal{E}(A))$. (So indeed we have CS written as a sum of two meaningful quantities).

We first find a piecewise smooth canonical family in r:

$\circledast$: For $r >> 0$, $\exists$ $(A(r), \psi(r))$ to $(\ddagger)_r$.

- $(A(r), \psi(r))$ is piecewise smooth in r.
- $CSD(A(r), \psi(r))$ continuous in r.
- For r. s.t. $(A(r), \psi(r))$ is smooth, $(A(r), \psi(r))$ is nondegenerate and its grading in $\widehat{HM}$ is independent of r.
- $(A(r), \psi(r))$ is not gauge equivalent to (A_{triv}, ψ_{triv}).

Proof. For any grading k, $r >> 0$, all generators defining $\widehat{\text{HM}}_k(Y, \mathfrak{s})$ are irreducible, via a spectral flow argument.

This implies that the differential of Floer chain just counts for pertrubed (-ve) gradient flow of CSD without reducibles.

[KM] in the earlier s says that $\exists$ nonzero clas σ in $\widehat{HM}_*(Y, \mathfrak{s})$ and when $c_1(E) = 0$, we can assume grading of σ is not the same as (A_{triv}, ψ_{triv}). Fix σ.

(Using tame perturbation in $\mathcal{P}$, suppressed in notation,) We can assume $\widehat{\text{HM}}_*$ defined for generic r.

For such r, σ is represented by $\sum_i n_i c_i$ where c_i is a critical point of CSD $\mathcal{L}$(they are distinct), and n_i's are nonzero, and the index set is finite.

Define $h(r) := \min_{\sum_i n_i^r c_i^r \in \sigma} \max_i CSD(c_i^r) =: CSD(c^r_{\text{minmax}})$, where we show the r-dependence as superscripts.

Define $(A(r), \psi(r)) := c^r_{\text{minmax}}$. Need to show $h(r)$ extends to a continuous function for all $r >> 0$, using bifurcation.

We give a baby version of this in Morse theory. (f_r, g_r) fails Morse-Smale condition for finitely many r. Let $r < r'$ be two such that M-S condition does hold.

We have continuation map $\Phi : C_*^M(f_{r'}, g_{r'}) \to C_*^M(f_r, g_r)$, which is defined by counting maps $\gamma : \mathbb{R} \to B$ s.t. $\dot{\gamma}(s) = -\nabla f_{\phi(s)}\gamma(s)$, where $\phi : \mathbb{R} \to [r, r']$ monotone is fixed before hand with $\phi|_{(-\infty,0]} = r'$ and $\phi|_{[1,\infty)} = r$.

By chain rule, $\frac{d}{ds} f_{\phi(s)}(\gamma(s)) = |\nabla f_{\phi(s)}(\gamma(s))|^2 + \frac{d\phi(s)}{ds}\frac{\partial f_r(x)}{\partial r}|_{r=\phi(s),x=\gamma(s)}$. The second factor of the last term is bounded by C as $[r, r'] \times X$ is compact.

If p' and p critical points of $f_{r'}$ and f_r respectively, suppose $\langle \Phi(p'), p\rangle \neq 0$, namely the cofficient of p in $\Phi(p')$ is nontrivial, then integrating, $f_{r'}(p') \geq f_r(p) - C(r' - r)$. Plus the other direction, we have Lipschitz, thus continuous. □

We can show by differentiation and definition of $(A(r), \psi(r))$, that

$$(\odot) \quad \frac{d\,CSD}{dr}(A(r), \psi(r)) = -\frac{1}{2}\mathcal{E}(A(r)).$$

We only need to show $\mathcal{E}$ bounded, by analyzing the growth rates in subsequences.

1.6. **Dichotomy.** We have the following dichotomy:

- $\exists r_n \to \infty$, $\mathcal{E}(A(r_n)) < C$ for all n.
- $\exists r_n \to \infty$, $\mathcal{E}(A(r_n)) \geq Cr_n$ and then $CS(A(r_n)) \geq Cr_n^2$.

Proof. Denote $CS(r)$, $\mathcal{E}(r)$ and $CSD(r)$ for the respective corresponding quantities with $(A(r), \psi(r))$ substituted in. Can assume $\mathcal{E}(r) > 1$ for $r >> 0$, or the first case holds. Fix $\epsilon_0 \in (0, 1/5)$.

Case A: $\exists r_n \to \infty$ with $CS(r_n) \geq \epsilon_0 r_n \mathcal{E}(r_n)$ for all n.

We claim as a blackbox: $(\oslash)$ $\exists C$, if (A, ψ) satisfies $(\ddagger)_r$ with $\mathcal{E}(A) > 1$, then $|CS(A)| \leq Cr^{2/3}\mathcal{E}(A)^{4/3}$.

From this and hypothesis of Case A, we are in the second case of the dichotomy.

Case B: we have $CS(r) < \epsilon_0 r\mathcal{E}(r)$ (labelled as (a)).

Define $v(r) := \mathcal{E}(r) - \frac{CS(r)}{r} = 2\frac{2CSD(r)}{r}$.

$(\odot)$ implies $\frac{dv}{dr} = \frac{CS}{r^2}$ (labelled (b)). (a) is equivalent to $\mathcal{E} < \frac{v}{1-\epsilon_0}$ (labelled (c)).

(a), (b) and (c) imply that $\frac{dv}{dr} < \frac{\epsilon v}{r}$ with $\epsilon := \frac{\epsilon_0}{1-\epsilon_0} < \frac{1}{4}$.

From these, we can deduce $v < C_1 r^{\epsilon}$ for some C_1 (labelled as (d)).

$(\oslash)$, (c) and (d) imply that $CS < C_2 r^{\frac{2}{3}+\frac{4}{3}\epsilon}$. Putting this into (b), we get that $\frac{dv}{dr} < C_2 r^{\frac{4}{3}(\epsilon-1)}$, and as $\epsilon < \frac{1}{4}$, the exponent $\frac{4}{3}(\epsilon - 1) < -1$. Integrating, we have v bounded from above.

Then by (c), $\mathcal{E} < \frac{v}{1-\epsilon_0} < C$. □

But then we have the following result:

Proposition 1.9. *$\exists\kappa > 0$ s.t. for $r >> 0$, (A, ψ) solution to $(\ddagger)$, we have $|\deg(A, \psi) - \deg(A_{triv}, \psi_{triv}) + \frac{1}{4\pi^2}CS(A)| < \kappa r^{31/16}$, where it is important that the exponent is less than* 2.

Suppose we are not in the first scenario in the dichotomy statement, then we have $CS(r_n) \geq Cr_n^2$ for some subsequence $r_n \to \infty$; however, by the above inequality where two degrees are constant, we have $CS(r_n)$ grows less than r_n^2, which is a contradiction.

We have obtained a sequence (A_{r_n}, ψ_{r_n}) with finite uniform energy bound (condition (2)) and nontrivial specified in condition (1), by the statement at the beginning of this section, we have a nontrivial orbit set, in particular a closed Reeb orbit.