SEIBERG-WITTEN FLOER HOMOLOGY LECTURE 14

DINGYU YANG

Please email yangding@math.hu-berlin.de if anything. Lecture notes up to now are available at www.mathematik.hu-berlin.de/∼yangding/monopole.html. Exercises sprinkled throughout lecture notes have been collected into an exercise sheet at www.mathematik.hu-berlin.de/∼yangding/Exercise_SWF.pdf.

1. Lecture 14: Detecting volume in 3d contact geometry

Let (Y, λ) be a closed connected contact 3-manifold. Recall from last time that, λ is a 1-form such that $\lambda \wedge d\lambda$ nowhere 0, thus a volume form.

For any $\Gamma \in H_1(Y)$, which (using the volume form/orientation) is under Poincaré duality corresponds to $PD(\Gamma) \in H^2(Y)$ which corresponds to a (iso class of) line bundle denoted still by $PD(\Gamma)$, and a spin^c structure $\mathfrak{s}_\xi \otimes PD(\Gamma)$. Here, recall the spin bundle of $\mathfrak{s}_\xi$ is $\underline{\mathbb{C}} \oplus \xi$ with $\xi = \ker \lambda$ the contact structure. We assume $c_1(\mathfrak{s}_\xi \otimes PD(\Gamma)) = c_1(\xi) + 2PD(\Gamma)$ is torsion. Thus we have absolute $\mathbb{Z}$ grading.

Let $\{\sigma_k\}_k$ be a sequence of nonzero homogeneous classes in $\widehat{HM}^{-*}$, $* \in \mathbb{Z}$ with $\lim_{k\to\infty} -\mathrm{gr}(\sigma_k) = \infty$. Here, in the monopole Floer cohomology, we count trajectories from right to left, namely, $\partial[\mathfrak{b}] = \sum_{[\mathfrak{a}]} \#\check{M}([\mathfrak{a}], [\mathfrak{b}])[\mathfrak{a}]$. Then, the topic of today's lecture is about the following volume detecting property:

$$\lim_{k\to\infty} \frac{c_{\sigma_k}(Y, \lambda)^2}{-\mathrm{gr}(\sigma_k)} = \mathrm{vol}(Y, \lambda) := \int_Y \lambda \wedge d\lambda.$$

Now we explain the undefined term c_σ for a nonzero class:

There is a filtered version of $\widehat{HM}^{-*}$ as follows: $\widehat{C}^{-*}_L := \{a \in \hat{C}^{-*} \mid \mathcal{E}(a) \leq L\}$, with $a = \sum_i n_i c_i = \sum_i n_i[(A_i, \psi_i)]$ and $\mathcal{E}(a) := \sum_i n_i(i \int_Y \lambda \wedge F_{A_i})$, is a subcomplex of $\widehat{C}^{-*}$, inducing $\iota_L : \widehat{HM}^{-*}_L \to \widehat{HM}^{-*}$.

Given $\sigma \in \widehat{HM}^{-*} \backslash \{0\}$, define

$$c_\sigma(Y, \lambda) = \mathrm{inf}_L\{L \mid \sigma \in \mathrm{im}(\iota_L)\}.$$

This volume detecting result is proved by Cristofaro-Gardiner–Hutchings–Ramos, in their "The asymptotics to ECH capacities" paper.

1.1. Proof of "$\leq$". $(\ddagger)_r$ $\begin{cases} *F_A = (r\langle c(\cdot)\psi, \psi\rangle - i\lambda) + (i * d\mu + \pi\overline{w}) \\ D_A\psi = 0. \end{cases}$

Here π is the universal constant.

Last time, $(\dagger)$ $|\mathrm{gr}(A, \psi) + \frac{1}{4\pi^2}CS(A)| < \kappa r^{31/16}$ for $r > r_*$. (We can shift the degree to get rid of one grading term from last time.)

Fix $\delta \in (0, \frac{1}{16})$. Given $j \in \mathbb{N}$, define r_j to be the largest real number s.t.

$$(\circledast) \quad j = \frac{1}{16\pi^2} r_j^2 \mathrm{vol}(Y, \lambda) - r_j^{2-\delta}.$$

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Since $r \gg 0$, no generators from reducibles (using a spectral flow estimate). So for j,

$$s_j := \{r \mid \exists \text{ a generator with grading } \geq -j \text{ associated to reducible solution to } (\ddagger)_r\} < \infty.$$

Claim: $s_j < r_j$ if $j \gg 0$. (This is one of significance of r_j.)

($\odot$) (Taubes): For (A,ψ) solution to $(\dagger)_r$, $\mathcal{E}(A) := i\int_Y \lambda \wedge F_A \leq \frac{r}{2}\mathrm{vol}(Y,\lambda) + C$.

$CSD(A,\psi) = \frac{1}{2}(CS(A) - r\mathcal{E}(A)) + I\int_Y \mu \wedge F_A + r\int_Y \langle D_A\psi, \psi\rangle d\mathrm{vol}$.

Fix $\sigma \in \widehat{HM}^{-*}$ nonzero with $\mathrm{gr}(\sigma) \geq -j$ (h defined last time using minmax for homology $\widehat{HM}_*$, here we have cohomolohgy, so maxmin)

$$h(r) = \max_{\sum_i n_i^r c_i^r \in \sigma} \min_i CSD(c_i^r) = CSD(c^r_{\mathrm{maxmin}}).$$

$(A(r),\psi(r)) := c^r_{\mathrm{maxmin}}$, and $h(r)$ piecewise smooth. Define $CSD(r) = CSD(A(r),\psi(r))$ and $\mathcal{E}(r) := \mathcal{E}(A(r))$.

$\frac{d}{dr}CSD(r) = -\frac{1}{2}\mathcal{E}(r)$ for all $r > s_j$ (no reducible) s.t. $\hat{C}^*$ is defined.

Proposition 1.1. $\lim_{r\to\infty} \mathcal{E}(r) = 2\pi c_\sigma(Y,\lambda)$.

Lemma 1.2 (a)**.** $r \gg 0$, $\begin{cases} \mathcal{E}(A) > 2\pi L_0 \text{ a universal threshold} \\ |CS(A)| \leq Cr^{2/3}\mathcal{E}^{4/3} \text{ appeared in the last lecture} \end{cases}$ *for (A,ψ) to $(\dagger)_r$ that is not (A_{triv},ψ_{triv}).*

Fix $\gamma \in (0,\frac{\delta}{4})$ and $\delta \in (0,\frac{1}{16})$.

Lemma 1.3 (b)**.** *For all j, $\exists$ ρ s.t. $r \gg \rho$ and $\hat{C}^{-*}$ defined, (A,ψ) non-trivial solution to $(\dagger)_r$ of grading $-j$, then $|CS(A)| \leq r^{1-\gamma}\mathcal{E}(A)$.*

Proof. (of Lemma (b)) If not, $\exists$ such a (A,ψ) with

$$r^{1-\gamma}\mathcal{E}(A) < |CS(A)| \leq Cr^{2/3}\mathcal{E}(A)^{4/3}$$

if $r \gg 0$. This implies that $r^{\gamma-1}|CS(A)| > \mathcal{E}(A) > C^{-3}r^{1-3\gamma}$, thus $|CS(A)| > C^{-3}r^{2-4\gamma}$ and this exponent is bigger than $\frac{31}{16}$ by choice ($4\gamma < \delta < \frac{1}{16}$) and and thus contradicts to $(\dagger)$ as $r \gg 0$. $\square$

Proof. (of Proposition) Assume $r \gg 0$, (a) and (b) ($\mathrm{gr}(\sigma) = -j$) and non-trivial solution in $\mathrm{gr}(\sigma)$ irreducible with energy $\mathcal{E} > 0$.

$\begin{cases} \int |F_A| \overset{\text{Taubes}}{\leq} \kappa(\mathcal{E}(A)+1) \\ \text{Lemma (a)} \end{cases} \Rightarrow$ ((a)) $|i\int_Y \mu \wedge F_A| \leq \kappa\mathcal{E}(A)$.

As $CSD(A,\psi) = \frac{1}{2}(CS(A) - r\mathcal{E}(A)) + i\int \mu \wedge F_A$, by Lemma (a) and Inequality (c), we have:

$\begin{cases} (1 - r^{-\gamma} - 2\kappa r^{-1})\mathcal{E}(A) \leq \frac{-2}{r}CSD(A,\psi) \leq (1 + r^{-\gamma} + 2\kappa r^{-1})\mathcal{E}(A) \\ \lim_{r\to\infty} \mathcal{E}(A_{triv}) = \lim_{r\to\infty} \frac{CSD(A_{triv},\psi_{triv})}{r} = 0. \end{cases}$

This implies the proposition by the definition of c_σ. $\square$

To see the "$\leq$" part of the statement, heuristically:

Suppose σ_j nonzero of gr $-j$. Then,

$$
\begin{aligned}
&\lim_j \frac{4\pi^2 c_{\sigma_j(Y,\lambda)^2}}{j} \\
\overset{\text{Prop}}{=}&\lim_j \frac{\mathcal{E}(A(r_j))^2}{j} \\
\underset{\circledast}{\overset{\odot}{\leq}}&\frac{(\frac{r_j}{2}\mathrm{vol}(Y,\lambda)+C)^2}{\frac{1}{16\pi^2}r_j^2\mathrm{vol}(Y,\lambda)+r_j^{2-\delta}} \\
=&4\pi^2\mathrm{vol}(Y,\lambda).
\end{aligned}
$$

1.2. **Proof of "$\geq$".** Let $(Y_\pm,\lambda_\pm)$ closed oriented contact 3-manifolds. A strong symplectic cobordism $(Y_+,\lambda_+)\to(Y_-,\lambda_-)$ is a compact symplectic 4-manifold (X,ω) (where 2-form ω satisfies $d\omega=0$ and ω^2 nowhere zero) with $\partial X = Y_+\sqcup \bar{Y}_-$ s.t. $\omega|_{Y_\pm}=d\lambda_\pm$.

A weakly exact symplectic cobordism is a strong symplectic cobordism with $\omega = d\lambda$ exact. This gives rise to a morphism

$$\Phi_L(X,\omega):\widehat{HM}_L^{-*}(Y_+,\lambda_+,0)\to\widehat{HM}_L^{-*}(Y_-,\lambda_-,0).$$

Let $A\in H_2(X,\partial)$, $\partial A=\Gamma_+-\Gamma_-$. For a strong cobordism, the filtration level might get shifted but between the limits is well-defined $\widehat{HM}^{-*} := \lim\widehat{HM}_L^{-*}$. Here Γ_+ is identified as $PD(\Gamma_+)$ the line bundle to tensor with $\mathfrak{s}_\xi$. Thus, we have $\Phi_L(X,\omega):\widehat{HM}^{-*}(\Gamma_+)\to\widehat{HM}^{-*}(\Gamma_-)$. This is defined by counting SW trajectories in the (completed) cobordism with limits in the $\pm$ pieces. We have the following properties:

- Trivial cobordism $\Rightarrow \Phi=\mathrm{Id}$.
- $\phi(X_r)\circ\phi(X_l)=\phi(X_l\circ X_r)$.
- $\phi(X_1\sqcup X_2)=\phi(X_1)\otimes\phi(X_2)$.
- We have U map increasing $\widehat{HM}^{-*}\to\widehat{HM}^{-*+2}$, corresponding to cup product with c_1 of tautological line bundle in coupled homology picture. We have $\phi(X)\circ U_+ = U_-\circ\phi(X)$.

Proof. (of "$\geq$") Step 1: $\widehat{HM}^{-*}$ finitely generated and U is isomorphism.

$*$ large enough, $\Rightarrow\exists$ finite collection of sequence satisfying (§) $U\sigma_{k+1}=\sigma_k$ for all k, s.t. every nonzero homogeneous σ_k of sufficiently large k is in one of these sequences in (§). Thus suffice to prove "$\geq$" for (§).

Step 2: $([-a,0]\times Y, d(e^s\lambda))$. For $\epsilon>0$, $B(r):=\{z\in\mathbb{C}^2\mid \pi|z|^2\leq r\}$ with standard ω_0. Let $\varphi_i: B(r_i)\to[-a,0]\times Y$, $i=1,\cdots,N$, with disjoint images, s.t. $\omega^2([-a,0]\times Y\setminus\bigsqcup_i\varphi_i(B(r_i)))<\epsilon$ (using Darboux charts). Denote $X := [-a,0]\times Y\setminus\bigsqcup_i\varphi_i(B(r_i))$.

Weakly exact cobordism $X:(Y,\lambda)\to(Y,e^{-a}\lambda)\sqcup\bigsqcup_i\partial B(r_j)$.

Step 3: $\Phi(X)$ formula¿

$\widehat{HM}^{-*}(\partial B(r_i))$ has basis $\{\zeta_k\}_{k\leq 0}$. $\zeta_0=[(A_{triv},\psi_{triv})]$ and $U\zeta_i=\zeta_{i+1}$.

For any $\sigma\in\widehat{HM}^{-*}$, $\Phi(\sigma)=\sum_{k\leq 0}\sum_{k_1+\cdots+k_N=k}U^k\sigma\otimes\zeta_{k_1}\otimes\cdots\otimes\zeta_{k_N}$.

Step 4: $\sigma_k, k\leq 0$ with $U\sigma_k=\sigma_{k+1}$.

$$
\begin{aligned}
c_{\sigma_k}(Y,\lambda) &\geq c_{\Phi(\sigma_k)}\big((Y,e^{-a}\lambda)\sqcup\bigsqcup_{i=1}^{N}\partial B(r_i)\big)\\
&= \max_{U^{\bar{k}}\sigma_k\neq 0}\max_{\bar{k}=k_1+\cdots+k_N}\big(c_{U^{\bar{k}}\sigma_k}(Y,e^{-a}\lambda)+\sum_{i=1}^{N}c_{\zeta_{k_i}(\partial B(r_i)}\big)\\
&\geq \max_{k_1+\cdots+k_N=k-1}\sum_{i=1}^{N}c_{\zeta_{k_i}}(\partial B(r_i)).
\end{aligned}
$$

Here $c_{\zeta_{k_i}}(\partial B(r_i)) = dr_i$ where d is the unique nonnegative integer s.t. $\frac{d^2+d}{2}\leq k_i\leq\frac{d^2+3d}{2}$.

This implies $\liminf\frac{c_{\sigma_k}^2}{k}\geq 4\sum_i \mathrm{vol}(B(r_i))^2=\frac{1-e^{-a}}{2}\mathrm{vol}(Y,\lambda)-\epsilon$, with $a>0$ and ϵ arbitrary. This immediately implies "$\geq$". $\square$