SEIBERG-WITTEN FLOER HOMOLOGY LECTURE 15

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Please email `yangding@math.hu-berlin.de` if you want to take exam to get credit for this course. The format is informal and involves a presentation of 20 mins of something covered or related in the lecture series. Lecture notes up to now are available at `www.mathematik.hu-berlin.de/~yangding/monopole.html`. Exercises sprinkled throughout lecture notes have been collected into an exercise sheet at `www.mathematik.hu-berlin.de/~yangding/Exercise_SWF.pdf`.

1. Lecture 15: Smooth closing Lemma + non-simplicity

1.1. **Smooth closing lemma in $3d$ contact dynamics.** Recall (Y, ξ) contact 3-manifold, where contact structure $\xi = \ker \lambda$ for some 1-form λ with $\lambda \wedge d\lambda$ nowhere zero. λ is called contact form. We also sometimes call (Y, λ) contact manifold. Can associate Reeb vector field X_λ by $\iota_{X_\lambda} d\lambda = 0$ and $\lambda(X_\lambda) = 1$.

For $f > 0$, $f\lambda$ is another contact form defining the same ξ, but dynamics (Reeb flow φ^t, periodic orbits etc) of $X_{f\lambda}$ can be quite different to X_λ. (Ex: write $X_{f\lambda}$ in terms of X_λ and f.) Here, we quickly recall the idea of flow: We associates a 1-parameter family $\{\varphi^t\}_{t\in\mathbb{R}}$ of diffeomorphism for a vector field X_λ on closed Y as follows: $\dot{\gamma} = X_\lambda \circ \gamma$ is an ODE and solution $\gamma_x : \mathbb{R} \to Y$ exists all time (as Y is closed) and unique upon specifying the initial condition $\gamma_x(0) = x$, and we define $\varphi^t(x) = \gamma_x(t)$.

Theorem 1.1 (Kei Irie)**.** *(Y, λ) closed contact 3-manifold.*

$\{f \in C^\infty(Y, \mathbb{R}^{>0}) \mid$ the union of periodic Reeb orbits of $X_{f\lambda}$ is dense in $Y\}$ is residual in $C^\infty(Y, \mathbb{R}^{>0})$ (with C^∞ topology). (Here, residual means it contains a countable intersection of open dense subsets.)

Denote $\mathcal{P}(Y, \lambda) = \{\gamma : \mathbb{R}/T_\gamma\mathbb{Z} \to Y \mid T_\gamma > 0, \dot{\gamma} = X_\lambda \circ \gamma\}$ the set of closed Reeb orbits.

$\mathcal{A} : \mathcal{P}(Y, \lambda) \to \mathbb{R}^{>0}$, $\gamma \mapsto \mathcal{A}(\gamma) := T_\gamma = \int \gamma^*\lambda$, action. $\mathcal{A}(\lambda) = \mathrm{Im}(\mathcal{A})$ action spectrum of (Y, λ).

Define $\mathcal{A}(\lambda)_+$ set of actions for orbits sets, namely, let $\mathcal{A}(\lambda)_0 = \{0\}$, $\mathcal{A}(\lambda)_m = \{\sum_{i=1}^m T_i \mid T_i \in \mathcal{A}(\lambda)\}$, and $\mathcal{A}(\lambda)_+ = \bigcup_{m\geq 0} \mathcal{A}(\lambda)_m$.

Fact: $\mathcal{A}(\lambda)_+$ is a closed set of (Lebesgue) measure 0 in $\mathbb{R}^{\geq 0}$.

($\mathcal{A}(\lambda) \subset \{$critical values of a fixed real smooth function$\} \Rightarrow \mathcal{A}(\lambda)_+$ is measure 0; $\mathcal{A}(\lambda)$ closed, $\min \mathcal{A} > 0 \Rightarrow \mathcal{A}(\gamma)_+$ is closed.)

Choose $\Gamma \in H_1(Y)$, s.t. $c_1(\xi) + 2PD(\Gamma)$ torsion like last time.

For a nonzero $\sigma \in \widehat{HM}^{-*}(\Gamma)$, define $c_\sigma(Y, \lambda) = \inf\{L \mid \sigma \in \mathrm{Im}(\iota_L)\}$ where $\iota_L : \widehat{HM}^{-*}_L \to \widehat{HM}^{-*}$.

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Here remark, although the filtration is defined as

$$\mathcal{E}(\sum_i n_i[(A_i,\phi_i)]) := \sum_i n_i(i\int_Y \lambda\wedge F_{A_i}) \leq L,$$

as $r\to\infty$, and $\phi^r = (\alpha^r_\phi,\beta^r_\phi)$ with $(\alpha^r_\phi)^{-1}(0)$ converges to a Reeb orbit set $\alpha := (\alpha^\infty_\phi)^{-1}(0)$. (Hope the overuse of α will not be confusing as we will denote α an orbit set and α_ϕ the first component of ϕ w.r.t. the splitting $PD(\Gamma)\oplus(PD(\Gamma)\otimes\xi)$.) And in this (subsequential) limiting process for solution (A^r,ϕ^r) to $(\ddagger)_r$ with bounded energy $\mathcal{E}$, $\mathcal{E}(A^r)$ converges to $\mathcal{A}(\alpha)$. So this the following which values $c_\sigma(Y,\lambda)$ makes sense.

$c_\sigma(Y,\lambda)$ is called spectral invariant, because:

- $f\geq 1$, $c_\sigma(Y,f\lambda)\geq c_\sigma(Y,\lambda)$.
- $a\in\mathbb{R}^{>0}$, $c_\sigma(Y,a\lambda) = ac_\sigma(Y,\lambda)$.
- $(f_j)_{j\in\mathbb{N}}$ in $C^\infty(Y,\mathbb{R}^{>0})$ with $f_j \underset{C^0}{\to} 1$, we have $\lim\limits_{j\to\infty} c_\sigma(Y,f_j\lambda) = c_\sigma(Y,\lambda)$.
- (Action selector, spectrality) $\sigma\in\widehat{HM}^{-*}$, $c_\sigma(Y,\lambda)\in\mathcal{A}_+$.

Before the proof, let me quickly recap a standard notion in contact geometry which we have not used/needed so far.

λ is nondegenerate if for all $x\in\mathrm{Fix}(\varphi^T)$ for all $T>0$, $d\varphi^T := \xi_x\to\xi_{\varphi^T(x)} = \xi_x$ (preserves the contact structure direction) has no eigenvalue 1, this implies that T-periodic orbits are isolated among such (T-periodic orbits). We also say $x\in\mathrm{Fix}(\varphi^T)$ is non-degenerate, where $(\varphi^t(x))_{t\in[0,T]}$ is an embedded periodic orbit, if $\varphi^T:\xi_x\to\xi_x$ and all the iterates do not have 1 as eigenvalue.

Proof. (spectrality) λ is non-degenerate. If $c_\sigma := c_\sigma(Y,\lambda)\notin\mathcal{A}(\lambda)_+$, as the latter being closed, $\exists\epsilon>0$ s.t. $(c_\sigma-\epsilon,c_\sigma+\epsilon)\cap\mathcal{A}(\lambda)_+=\emptyset$. $\Rightarrow \widehat{HM}^{-*}_{c_\sigma-\epsilon}(\Gamma) = \widehat{HM}^{-*}_{c_\sigma+\epsilon}$ (recall again, Taubes showed $\mathcal{E}(A^r)$ (which is tied to $\phi^r = (\alpha^r_\phi,\beta^r_\phi)$ through SW equation) converges to $\mathcal{A}(\alpha)$ with $\alpha := (\alpha^\infty_\phi)^{-1}(0)$.) $\Rightarrow \mathrm{im}(\iota_{c_\sigma-\epsilon}) = \mathrm{im}(\iota_{c_\sigma+\epsilon})$, which contradicts to the definition of c_σ.

λ degenerate, $\exists$ $(f_j)_{j\geq 1}$ with $f_j>0$, $f_j \underset{C^1}{\to} 1$ and $f_j\lambda$ non-degenerate. Each $c_\sigma(Y,f_j\lambda)\in\mathcal{A}(f_j\lambda)_{m(j)}$ where total multiplicity $m(j)$ is defined as $m(j) := \sum_i n_i$ (where $\alpha = \bigsqcup_i(\gamma_i,n_i)$ where γ_i is the underlying embedded closed Reeb orbit and n_i is its multiplicity). We have $\sup_j m(j)<0$, since $\inf_j\min\mathcal{A}(f_j\lambda)>0$ bounded away from zero (the number of max allocation to create $m(j)$ is bounded above). Thus, (up to subsequence) $m(j)=m$ constant, $c_\sigma(Y,f_j\lambda) = a^1_j+...+a^m_j$, $a^k_m = \lim\limits_{j\to\infty} a^k_j$ exists $1\leq k\leq m$ and $a^k_j\in\mathcal{A}(f_j\lambda)$. As $f_j \underset{C^1}{\to} 1 \underset{\text{Reeb orbit sets converge}}{\Rightarrow} a^k_\infty\in\mathcal{A}(Y,\lambda)$ for all k. Thus, $c_\sigma(Y,\lambda) = a^1_\infty+\cdots+\infty^m_\infty\in\mathcal{A}(\lambda)_m\subset\mathcal{A}(\lambda)_+$. $\square$

Theorem 1.2 (volume detecting, CG-H-R, covered last time)**.** *Let (Y,λ) closed connected contact 3-manifold. $\Gamma\in H_1(Y;\mathbb{Z})$ s.t. $c_1(\xi)+2PD(\Gamma)$ torsion. Let σ_k be sequence of nonzero homogeneous classes in $\widehat{HM}^{-*}(\Gamma)$ s.t. $-gr(\sigma_k)\to\infty$. (We always have such thanks to the existence result in monopole homology that we have covered.) Then the volume detecting property holds:*

$$\lim_{k\to\infty}\frac{c_{\sigma_k}(Y,\lambda)^2}{-gr(\sigma_k)} = \int_Y\lambda\wedge d\lambda =: vol(Y,\lambda).$$

As an important corollary:

Corollary 1.3. *Let λ, λ' be contact forms with same $\xi = \ker\lambda = \ker\lambda'$. Suppose for any $\Gamma \in H_1(Y;\mathbb{Z})$ s.t. $c_1(\xi) + 2PD(\Gamma)$ torsion and any $\sigma \in \widehat{HM}^{-*}(\Gamma)\backslash\{0\}$ thereof, we have $c_\sigma(Y,\lambda) = c_\sigma(Y,\lambda')$, then $vol(Y,\lambda) = vol(Y,\lambda')$.*

Proof. Due to the calculation 2 lectures ago, there exists σ_k nonzero with $-\mathrm{gr}(\sigma_{k+1}) = -\mathrm{gr}(\sigma_k) + 2$. With hypothesis applied to this sequence, the conclusion follows by volume detecting. □

Define $\|f\|_{C^\infty} := \sum_{l=0}^{\infty} 2^{-l} \frac{\|f\|_{C^l}}{1+\|f\|_{C^l}}$ which is always ≤ 2.

Lemma 1.4 ($\mathcal{C}^\infty$ closing lemma, Kei Irie)**.** *For any non-empty open set U in Y, $\epsilon > 0$, $\exists$ $f \in C^\infty(Y)$ s.t. $\|f-1\|_{C^\infty} < \epsilon$ and $\exists$ non-degenerate $\gamma \in \mathcal{P}(Y, f\lambda)$ which intersects U.*

Proof. Take any $h \in C^\infty(Y, \mathbb{R}^{\geq 0})$ s.t. $\mathrm{supp}(h) \subset U$, $\|h\|_{C^\infty} < \epsilon$ and $h \not\equiv 0$. Then a calculation shows ($\oslash$) $\mathrm{vol}(Y,(1+h)\lambda) > \mathrm{vol}(Y,\lambda)$.

Claim: $\exists$ $t \in [0,1]$ and $\gamma \in \mathcal{P}(Y,(1+th)\lambda)$ which intersects U.

Proof. (of Claim) Suppose not, then for any $t \in [0,1]$, for all $\gamma \in \mathcal{P}(Y,(1+th)\lambda)$, γ avoids U. Then $\mathcal{P}(Y,(1+th)\lambda) = \mathcal{P}(Y,\lambda)$ for all t, since closed Reeb orbits for $(1+th)\lambda$ and λ coincide in $Y\backslash U$. $\Rightarrow \mathcal{A}((1+th)\lambda)_+ = (A)(\lambda)_+$.

For any $\Gamma \in H_1(Y)$ with $c_1(\xi) + 2PD(\Gamma)$ torsion, and for all $\sigma \in \widehat{HM}^{-*}\backslash\{0\}$, $c_\sigma(Y,(1+th)\lambda) \in \mathcal{A}((1+th)\lambda)_+ = \mathcal{A}(\lambda)_+$, the latter is measure 0 in $\mathbb{R}^{\geq 0}$, and $c_\sigma(Y,(1+th)\lambda)$ is continuous in t, thus constant. $\underset{\text{Corollary}}{\Rightarrow}$ $\mathrm{vol}(Y,\lambda) = \mathrm{vol}(Y,(1+h)\lambda)$, which is contradicting to $\oslash$. [Claim] □

Take $g \in \mathcal{C}^\infty(Y)$ with $\|g\|_{C^\infty}$ sufficiently small, $g|_{\mathrm{im}\gamma} \equiv 0$, and $dg|_{\mathrm{im}\gamma} \equiv 0$, so $\gamma \in \mathcal{P}(Y, e^g(1+th)\lambda)$ s.t. γ is moreover nondegenerate ($\varphi^T : \xi_x \circlearrowleft$ is twisted by $\mathrm{Hess}(g)$). $f := e^g(1+th)$ is the desired. [Closing lemma] □

Proof. (Theorem of $\{f$ s.t. Reeb orbits of $X_{f\lambda}$ is dense$\}$ is residual)

Let $U \subset Y$ be nonempty open. Let

$$\mathcal{F}_U := \{f \in C^\infty(Y,\mathbb{R}^{>0}) \mid \exists \text{ nondegenerate } \gamma \in \mathcal{P}(Y, f\lambda) \text{ that intersects } U\}.$$

$\mathcal{F}_U$ is open (due to open condition) and dense (closing lemma) in $C^\infty(Y,\mathbb{R}^{>0})$.

Take a countable basis $(U_i)_{i\in\mathbb{N}}$ of open sets in Y. Then $\bigcap_i \mathcal{F}_{U_i}$ is residual and any f in it has Reeb orbits of $X_{f\lambda}$ dense in Y. (For any open neighborhood of any point, there will be a point on closed periodic orbits in it.) [Theorem] □

Remark 1.5. There is a variant (same argument) for closed geodesics on a closed Riemannian surface (as closed geodesics on (Σ, g) correspond to periodic Reeb orbits of the associated contact 3-manifold cosphere bundle).

1.2. **Simplicity conjecture.** Exercise session is converted into the final session of lecture to cover a recent exciting progress partly using ideas/tools in this lecture series.

Volume detecting $\lim\limits_{k\to\infty} \frac{c_{\sigma_k}(Y,\lambda)^2}{-\mathrm{gr}(\sigma_k)} = \int_Y \lambda \wedge d\lambda$.

"$\leq$" uses structure and intrinsic property of Seiberg-Witten $(\ddagger)_r$ very deeply via intricate analysis.

"$\geq$" needs Seiberg-Witten, but potentially replaceable. Because the other approach (e.g. embedded contact homology, which is more direct at set-up with orbit set as generators of homology) still needs to associate a map $\phi(\mathrm{cob})$ for a cobordism cob which is used essentially in the calculation for "$\geq$", and at the moment, this is not well-defined (except in Seiberg-Witten Floer homology and via isomorphism $ECH_*(\Gamma) \overset{\text{Taubes}}{\cong} \widehat{HM}^{-*}(\Gamma)$ through it) due to geometric transversality issue.

For the discussion in our "exercise session", we follow the "Simplicity conjecture" paper by Cristofaro-Gardiner–Humilière–Seyfaddini.

(S, ω) surface with area 2-form (symplectic 2-manifold).

A homeomorphism $\phi : S \to S$ is an area-preserving homeomorphism if it preserves the measure (from ω defined as $\omega(A) := \int_A \omega$): $\omega(\phi^{-1}(U)) = \omega(U)$.

(D, ω) a unit disk with boundary with standard area form. $\mathrm{Homeo}_c(D, \omega)$ group of area-preserving homeomorphisms of 2-disk that are identity near the boundary (i.e., compactly supported).

Definition 1.6. A group G is simple if any normal subgroup (denoted by $H \lhd G$, meaning that the subgroup H satisfies $gHg^{-1} \subset H$ for $g \in G$) is either $\{e\}$ or G. I.e., it does not have a non-trivial (not $\{e\}$) proper (not G) normal subgroup.

Question: (Fathi, '80) Is $\mathrm{Homeo}_c(D, \omega)$ simple?
Why is it a good/interesting question?

- dim ≥ 3 understood by Fathi, SIMPLE.
- ICM 2006. 4 different mathematicians in different areas ask about this.
- Motivation for C^0-symplectic topology.

Theorem 1.7. *(CG-H-S) $\mathrm{Homeo}_c(D, \omega)$ is NOT simple.*

Remark 1.8.
- No natural (possibly discontinuous) homeomorphism analogous to flux, Calabi, mass-flow.
- Le Roux fragmentation $\Rightarrow$ If not simple, then lots of proper normal subgroups. (This might partly explain why it is hard to find one.)

Why we end the course with this topic and what are the connections between various topics?

$$\begin{tikzcd}
\text{Thom conj. solved by KM} & \text{SW Floer homology } \widehat{HM} \text{ etc} & \\
\text{using SW on } W^4 \text{ w. } \partial = Y^3 \arrow[r, "\text{valued in}"] & \text{SW on } \mathbb{R} \times Y \arrow[d, "\text{calculation}"', "\text{and existence}"] & \\
\text{Taubes' equiv of counts} \arrow[r] & \text{4d SW on } \mathbb{R} \times Y : (A^r, (\alpha^r, \beta^r)) \overset{r \to \infty}{\to} & \\
\text{betw. 4d SW \& holo. curves} & (\alpha^\infty)^{-1}(0) \text{ holo. curve w. orbit sets as limits} \arrow[r] & \text{Weinstein conj.} \\
& \text{Hutchings' } ECH_* \cong \text{KM's } \widehat{HM}^{-*} & \text{Taubes' proof} \arrow[d, "\text{Hutchings's } c_\sigma"', "\text{Taubes' } ECH_*(\Gamma) \cong \widehat{HM}^{-*}(\Gamma)"] \\
C^\infty \text{ closing lemma} & \text{volume detecting} \arrow[l, "\text{Irie}"'] \arrow[d, "\text{analogue betw.}"', "PFH = ECH"] & \text{properties on spectral invariant} \arrow[l, "\text{CG-H-R}"', "\text{``}\leq\text{'' \& ``}\geq\text{''}"] \\
\text{non-simple } \mathrm{Homeo}_c(D, \omega) & \text{``} \geq \text{'' part of Calabi detecting} \arrow[l, "\text{CG-H-S}"'] &
\end{tikzcd}$$

1.2.1. *Stable Hamiltonian structure.* $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$, $\omega_{S^2} = \frac{1}{4\pi} d\theta \wedge dz$, $\mathrm{Area}(S^2) = 1$.

$D = \{(x, y) \mid x^2 + y^2 \leq 1\}$, $S^+ = \{(x, y, z) \in S^2 \mid z \geq 0\}$.

$\iota : D \to S^2$, $(r, \theta) \mapsto (\theta, 1 - r^2)$ with the image S^+.

$\omega = \iota^* \omega_{S^2} = \frac{1}{2\pi} r dr \wedge d\theta$. Disk (D, ω) has area $1/2$.

More general notion to contact structure (odd dimension in general, here we focus on 3dim)

Stable Hamiltonian structure (SHS) (α, Ω) on a 3-manifold Y is a pair, where α 1-form, Ω closed 2-form, and $\alpha \wedge \Omega$ nowhere zero, and (stability) $\ker \Omega \subset \ker d\alpha$ (equivalent to $d\alpha = g\Omega$ for some $g \in C^\infty(Y, \mathbb{R})$, here $\ker \Omega$ means kernel of the map $\iota_\bullet \Omega = \Omega(\bullet,) : TM \to T^*M$). Reeb vector field $X = X_{\alpha,\Omega}$ is defined by $\iota_X \Omega = 0$ and $\alpha(X) = 1$.

Example 1.9.
- Contact $(\lambda, d\lambda)$.
- Mapping torus. (S, ω) closed surface with area 2-form, φ area-preserving diffeomorphism $\varphi^* \omega = \omega$. Mapping torus $Y_\varphi := \frac{S \times [0,1]_r}{(x,1) \sim (\varphi(x), 0)}$. $\alpha = dr$ and $\Omega = \mathrm{pr}_1^* \omega / \sim = \omega_\varphi$. SHS. Reeb $X = \frac{\partial}{\partial r}$. Reeb orbits=periodic orbits of φ.

1.2.2. *Periodic Floer homology and* c_d. Floer homology (has the spirit of HM) $PFH(\varphi, h)$ periodic Floer homology. Here $h \in H_1(Y_\varphi) \backslash \{0\}$.

- Generators (over $\mathbb{Z}/2$) $\alpha = \{(\alpha_i, m_i)\}_i$ orbit set. α_i embedded periodic Reeb orbits, m_i positive integer, 1 if α_i hyperbolic (linearized return map $\varphi_x^T : \xi_x \to \xi_x$ has real eigenvalues). $h = \sum_i m_i [\alpha_i]$.
- There is relative grading $I(\alpha, \beta, Z)$ for orbit sets α, β and Z a spanning surface homology class from α to β, which calculate the dimension of the moduli space below.
- $\mathbb{R}_s \times Y_\varphi$. The tangent bundle $TY_\varphi = \mathbb{R}X \oplus \xi$ with X Reeb and $\xi = \ker \alpha$. Almost complex structure (90^o rotation) $J : T(\mathbb{R} \times Y_\varphi) \to T(\mathbb{R} \times Y_\varphi)$, $J : \frac{\partial}{\partial s} \mapsto X$, and $\xi \to \xi$ with $\Omega(\cdot, J \cdot)$ metric.

  $u : (\Sigma, j) \to (\mathbb{R} \times Y_\varphi, J)$ mapping from a Riemann surface Σ with j (almost) complex structure is (J-)holomorphic if $du \circ i = J(u) \circ du$, namely, du intertwines almost complex structures. This equation gives a Fredholm problem and forms a moduli space.
- $M_J^1(\alpha, \beta)$ denotes the space of holomorphic embedded curve/current C (already mod out domain reparametrization) of dimension $I(\alpha, \beta, [C]) = 1$ with asymptotics to α as $s \to +\infty$ and to β as $s \to -\infty$, then modulo $\mathbb{R}$ translation in $\mathbb{R} \times Y_\varphi$. This index $I(\alpha, \beta, [C])$ has the property that C in the moduli space with $I(\alpha, \beta, [C]) = 1$ (if transverse as a point in the moduli space) is automatically embedded.

  (Taubes 4d SW on $\mathbb{R} \times Y_\varphi$ with first component of spinor having the zero set $(\alpha^r)^{-1}(0) \overset{r \to \infty}{\to}$ holomorphic curves with ends Reeb orbit sets.)
- $\partial \alpha = \sum_\beta \#_2 M_J^1(\alpha, \beta) \beta$. We have $\partial^2 = 0$ (this is super non-trivial involving elaborate definition and argument of gluing). The homology is called PFH. (This is an SHS variant of embedded contact homology for Y contact.)

Specialize to $(S, \omega) = (S^2, \omega_{S^2})$, $\varphi \in \mathrm{Diff}(S^2, \omega_{S^2})$ supported in S^+. $Y_\varphi \cong S^2 \times S^1$. $h \in H_1(Y_\varphi) = \mathbb{Z}$. $\pi : Y_\varphi \to S^1$, $d = \deg(h) = \#_{\mathrm{inters}}([\mathrm{fiber}], h)$.

Choose a cycle γ_0 in Y_φ s.t. $\pi|_{\gamma_0} : \gamma_0 \to S^1$ is an orientation preserving.

- generator for $\widetilde{PFC}$ which comes with an action filtration. (α, Z) with Z spanning surface homology class between α and $d\gamma_0$.
- absolute grading: $I(\alpha, Z)$ with $I(\alpha, \beta, Z - Z') = I(\alpha, Z) - I(\beta, Z')$.
- differential counts J-holomorphic current C from α (from generator (α, Z)) to β (from generator (β, Z')) s.t. $Z' + [C] = Z$, as follows: $M^1_J((\alpha, Z), (\beta, Z'))$ is the moduli spaces of such with index $I(\alpha, \beta, [C]) = I(\alpha, beta, Z - Z') = I(\alpha, Z) - I(\beta Z') = 1$, then quotiented by $\mathbb{Z}$ translation in $\mathbb{R} \times Y_\varphi$. Define $\partial(\alpha, Z) = \sum_{(\beta, Z')} \#_2 M^1_J((\alpha, Z), (\beta, Z'))(\beta, Z')$.

  $\partial^2 = 0$ gives rise to $\widetilde{PFH}$.
- Action $\mathcal{A}(\alpha, Z) = \int_Z \omega_\varphi$. $\widetilde{PFC}^L = \{(\alpha, Z) \mid \mathcal{A}(\alpha, Z) \leq L\}$. $\partial : \widetilde{PFC}^L \circlearrowleft$. $\Rightarrow \iota_L : \widetilde{PFH}^L \to \widetilde{PFH}$.
- $c_\sigma(\varphi) = \inf\{L \mid \sigma \in \mathrm{im}(\iota_L)\}$.

$\widetilde{PFH}_*(Y_\varphi, d)$ depends on the homotopy class of φ.

We calculate $\widetilde{PFH}_*(Y_\varphi, d) = \begin{cases} \mathbb{Z}/2 \text{ if } * = d \bmod 2 \\ 0 \text{ otherwise.} \end{cases}$ by using φ irrational rotation $(z, \theta) \mapsto (z, \theta + \alpha)$ with α irrational.

For every (d, k), $k = d (\mathrm{mod} 2)$, there exists a unique class $\sigma_{d,k}$ in grading k. Define $c_{d,k}(\varphi) := c_{\sigma_{d,k}}(\varphi)$, $c_d(\varphi) := c_{d,-d}(\varphi)$.

1.2.3. *Finite energy homeomorphism.* Define $H \in C^\infty_c(S^1_t \times D)$ time-dependent function on D^2 supported in $\mathring{D}$.

There exists a unique X_H such that $\omega(X_H, \cdot) := dH$ as ω is an area form. Flow of X_H, φ^t_H, Hamiltonian flow.

Every area preserving diffeomorphism $\phi \in \mathrm{Diff}_c(D, \omega)$ is φ^1_H for some $H \in C^\infty_c(S^1 \times D)$.

Hofer norm/energy: $\|H\|_{(1,\infty)} = \int_0^1 \big(\max_{x \in D} H(t, x) - \min_{x \in D} H(t, x)\big) dt$.

Definition 1.10. $\phi \in \mathrm{FHomeo}_c(D, \omega)$ is called finite energy homeomorphism if $\phi \in \mathrm{Homeo}_c(D, \omega)$ and $\exists\ H_i \in C^\infty_c(S^1 \times D)$ s.t. $\|H_i\|_{(1,\infty)} \leq C < \infty$ and $\varphi^1_{H_i} \underset{C^0}{\to} \phi$.

One can show that the $\mathrm{FHomeo}_c(D, \omega)$ is a normal subgroup.

The main focus of the lecture is to show:

Theorem 1.11. *$FHomeo_c(D, \omega)$ is a proper normal subgroup of $Homeo_c(D, \omega)$. (Namely, $\exists\ \phi \in Homeo_c(D, \omega) \backslash FHomeo_c(D, \omega)$ s.t. any $\{H_i\}$ with $\varphi^1_{H_i} \underset{C^0}{\to} \phi$ has $\|H_i\|_{(1,\infty)} \to \infty$.) Therefore, $Homeo_c(D, \omega)$ is not simple.*

We will construct an example of such ϕ:

Let $f : (0, 1] \to \mathbb{R}$ smooth, vanishes near 1, decreasing. $\lim_{r \to 0} f(r) = \infty$.

Define ϕ_f with $\phi_f(0) = 0$, $\phi_f(r, \theta) = (r, \theta + 2\pi f(r))$. $\phi_f \in \mathrm{Homeo}_c(D, \omega)$ called ∞-twist.

We first define Calabi invariant to motivate the condition below. Let $\theta \in \mathrm{Diff}_c(D, \omega)$, $\exists\ H \in C^\infty_c(S^1 \times D)$ s.t. $\theta = \varphi^1_H$. Define $\mathrm{Cal}(\theta) = \int_{S^1} \int_D H \omega dt$. $\mathrm{Cal} : \mathrm{Diff}_c(D, \omega) \to \mathbb{R}$ thus defined is a non-trivial group homomorphism independent of such H.

We can construct ∞-twist with a f such that $\int_0^1 \int_r^1 s f(s) ds r dr = \infty$.

Remark 1.12. $\omega = \frac{1}{2\pi} r dr \wedge d\theta$, ϕ_f smooth, defined by $f : [0,1] \to \mathbb{R}$ (defined at 0 now) as in the above formula. Then $\mathrm{Cal}(\phi_f) = \int_0^1 \int_r^1 sf(s) ds r dr$, which explains the above quantity.

1.2.4. *Spectral invariant c_d and Calabi detecting for monotone twist.* Claim: c_d defined above has the following properties:

- $c_d(\mathrm{Id}) = 0$.
- $H \leq G$ in $C_c^\infty(S^1 \times D) \Rightarrow c_d(\varphi_H^1) \leq c_d(\varphi_G^1)$ for all d.
- Hofer-continuous: $|c_d(\varphi_H^1) - c_d(\varphi_G^1)| \leq d\|H - G\|_{(1,\infty)}$.
- Spectrality: $\varphi_d(\varphi_H^1) \in \mathrm{Spec}_d(H)$ analogous to $\mathcal{A}_+$, which is the image of critical points of the action functional in the current setting.
- (C^0) $c_d : \mathrm{Diff}_c(D, \omega) \subset \mathrm{Diff}_c(S^2, \omega_{S^2}) \to \mathbb{R}$ continuous w.r.t. C^0 on $\mathrm{Diff}_c(D,\omega)$ extends continuously to $\mathrm{Homeo}_c(D,\omega)$.

Theorem 1.13. *For a monotone twist $\varphi = \phi_f$, namely, ϕ_f defined by the same formula above but with smooth function $f : [0,1] \to \mathbb{R}$. Then $\lim\limits_{d\to\infty} \frac{c_d(\varphi)}{d} = Cal(\varphi)$. (We only need the "$\geq$" part.)*

1.2.5. *∞-twist is not of finite energy and proof of non-simplicity.*

Lemma 1.14 (($\odot$), linear growth for FHomeo_c)**.** *$\psi \in FHomeo_c(D,\omega) \Rightarrow \exists\ C = C(\psi)$ s.t. $\frac{c_d(\psi)}{d} \leq C$ for all d.*

Proof. By definition, $\exists\ H_i \in C_c^\infty(S^1 \times D)$ with $\|H_i\|_{(1,\infty)}$ bounded by C.

Hofer continuous and $c_d(\mathrm{Id}) = 0$, $\varphi_{H_i}^1 \underset{C^0}{\to} \psi$. $\Rightarrow c_d(\varphi_{H_i}^1) \leq d\|H_i\|_{(1,\infty)} \leq dC$.

c_d extends to $\mathrm{Homeo}_c(D,\omega)$ continuously $\Rightarrow c_d(\psi) = \lim\limits_{i\to\infty} c_d(\varphi_{H_i}^1) \leq dC\ \forall d$. □

Lemma 1.15 (($\circledast$), ∞-twist has super-linear growth)**.** *$\exists\ \phi_{f_i} \in Diff_c(D,\omega)$.*

- *$\phi_{f_i} \underset{C^0}{\to} \phi_f$ ∞-twist.*
- *$\exists\ F_i$ supported in $\mathring{D}$ s.t. $\varphi_{F_i}^1 = \phi_{f_i}$ and $F_i \leq F_{i+1}$.*
- *$\lim\limits_{i\to\infty} Cal(\phi_{f_i}) = \infty$.*

Proof. Choose smooth $f_i : [0,1] \to \mathbb{R}$. $f_i = f$ on $[\frac{1}{i}, 1]$ and $f_i \leq f_{i+1}$.

ϕ_{f_i} defined using twist. $\phi_{f_i} = \phi_f$ outside $D(\frac{1}{i})$, so $\phi_f^{-1}\phi_{f_i} \underset{C^0}{\to} \mathrm{Id}$.

ϕ_{f_i} is $\varphi_{F_i}^1$ for $F_i(r,\theta) = \int_r^1 sf_i(s)ds$, $F_i \leq F_{i+1}$.

By definition of Cal,

$$\begin{aligned}
\mathrm{Cal}(\phi_{f_i}) &= \int_0^1 \int_r^1 sf_i(s) ds r dr \\
&\geq \int_{\frac{1}{i}}^1 \int_r^1 sf_i(s) ds r dr \\
&= \int_{\frac{1}{i}}^1 \int_r^1 sf(s) ds r dr \to \infty.
\end{aligned}$$

So, $\lim\limits_{i\to\infty} \mathrm{Cal}(\phi_{f_i}) = \infty$. □

Proof. (∞-twist is not a finite energy homeomorphism, thus $\mathrm{FHomeo}_c(D,\omega)$ is proper, and $\mathrm{Homeo}_c(D,\omega)$ is not simple.)

$c_d(\phi_{f_i}) \leq c_d(\phi_{f_{i+1}})$.

$\phi_{f_i} \underset{C^0}{\to} \phi$, $c_d(\phi) = \lim\limits_{i\to\infty} c_d(\phi_{f_i})$.
So $c_d(\phi_{f_i}) \le c_d(\phi_f)$ $\forall i$ $\forall d$.
Thus, $\lim\limits_{d\to\infty} \frac{c_d(\phi_f)}{d} \ge \lim\limits_{d\to\infty} \frac{c_d(\phi_{f_i})}{d} \underset{\text{“}\ge\text{'' of Calabi detecting}}{\ge} \mathrm{Cal}(\phi_i) \underset{\circledast}{\to} \infty$.
Thus, $\lim\limits_{d\to\infty} \frac{c_d(\phi_f)}{d} = \infty$ and $(\odot) \Rightarrow \phi_f \notin \mathrm{FHomeo}_c(D,\omega)$. $\square$

The end of the lecture series. Thank you very much for following along. If you want to take the example to get credit, please email me.