SEIBERG-WITTEN FLOER HOMOLOGY LECTURE 2

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1. Lecture 2

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1.1. **Variational origin of 3d SW.** We continue from last time. Let Y be an oriented closed Riemannian 3-manifold. (C.f. P 18 corollary 2.45 of John Morgan's Seiberg-Witten equation and application to topology of smooth four manifolds to arrive at our spin^c from general definition.)

For a spin^c structure $\mathfrak{s} = (S, \rho)$, fix a reference spin^c connection B_0 as a base point. Consider the configuration space of pairs (B, Ψ), where B is a spin^c connection and a spinor $\Psi \in \Gamma(S)$. We introduced the CSD functional

$$\mathcal{L}(B, \Psi) := -\frac{1}{8}\int_Y (B^t - B_0^t) \wedge (F_{B^t} + F_{B_0^t}) + \frac{1}{2}\int_Y h(D_B\Psi, \Psi) d\mathrm{vol},$$

called Chern-Simons-Dirac functional. Later, we also denote $h(\, , \,)$ by $\langle \, , \, \rangle$.

Write $B = B_0 + b$ with $b \in \Omega^1(Y, i\mathbb{R})$, then $B^t = B_0^t + 2b$, $F_{B^t} = F_{B_0^t} + 2\nabla_{B_0^t} b = F_{B_0^t} + 2\nabla_{B^t} b$ (where $b \wedge b$ term is identically 0, in this Abelian case). We have

$$\mathcal{L}(B+b, \Psi+\psi) - \mathcal{L}(B, \Psi) = \int_Y \big(\langle b, \frac{1}{2}{*F_{B^t}} + \rho^{-1}(\Psi\Psi^*)_0\rangle + \mathrm{Re}\langle \psi, D_B\Psi\rangle\big) d\mathrm{vol} + o(b, \psi).$$

Here the first $\langle \, , \, \rangle$, which is the natural metric on Lie algebra valued 1-form, which in this case is the metric on 1-forms in front of i; and ρ is an isometry.

Using the L^2 metric $\langle (b, \psi), (b', \psi')\rangle_{L^2} = \int_Y \big(\langle b, b'\rangle + \mathrm{Re}\langle \psi, \psi'\rangle\big) d\mathrm{vol}$, define (formal) gradient $\nabla\mathcal{L}$ via $\langle \nabla\mathcal{L}, v\rangle_{L^2} = d\mathcal{L}(v)$. $\nabla\mathcal{L} = 0$ corresponds 3d SW equation.

1.2. **Gauge-invariance of S^1-valued functional.** Recall symmetry/Gauge group is $\mathcal{G}_Y := \{u : Y \to S^1\}$, acting by $u : (B, \Psi) \mapsto (u(B), u\Psi)$, where $u(B) := u \circ \nabla_B \circ u^{-1} = B - u^{-1}du$, and $u\Psi$ is the fiberwise scalar multiplication.

We remark that $[u] \in [Y, S^1] \cong [Y, K(1, \mathbb{Z})] \cong H^1(Y, \mathbb{Z})$, and de Rham representative for $[u]$ is $\frac{1}{2\pi i}u^{-1}du$.

We have $\mathcal{L}(u(B, \Psi)) - \mathcal{L}(B, \Psi) = 2\pi^2([u] \cup c_1(S))[Y]$. So $\mathcal{L}$ descends to a $\mathbb{R}/2\pi^2\mathbb{Z} \cong S^1$-valued functional that is invariant under $\mathcal{G}_Y$.

1.3. **Negative gradient flow equation as a 4d SW equation.** Consider a path $\mathbb{R} \to$ configuration space of pairs, $t \mapsto (B(t), \Psi(t))$, satisfying the negative gradient flow equation for $\mathcal{L}$. $\frac{\partial}{\partial t}B = -(\frac{1}{2} * F_{B^t} + \rho^{-1}(\Psi\Psi^*)_0)$ (here we omit the $\otimes\mathrm{Id}_S$ on RHS, which induces an equation for B^t) and $\frac{\partial}{\partial t}\Psi = -D_B\Psi$.

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We can construct 4-manifold $\mathbb{R}_t \times Y =: Z$ and spin^c structure (S_Z, ρ_Z), where $S_Z = S^+ \oplus S^- = S \oplus S$, and $\rho_Z : TZ \to \mathrm{Hom}(S_Z, S_Z)$ defined as

$$\rho_Z(\frac{\partial}{\partial t}) = \begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix} \quad \text{and} \quad \rho_Z(v) := \begin{pmatrix} 0 & -\rho(v)^* \\ \rho(v) & 0 \end{pmatrix} \text{ for } v \in TY.$$

Time-dependent spin^c connection $B(t)$ on S gives a spin^c connection A on S_Z. $\nabla_A := \frac{\partial}{\partial t} + \nabla_B$ is in temporal gauge, namely trivial in $\mathbb{R}_t$ factor.

We have $D_{A^t} = \frac{\partial}{\partial t} + D_{B^t}$ and $F_{A^t} = dt \wedge (\frac{\partial}{\partial t} B^t) + F_{B^t}$.

Exercise 1.1. Recall the Hodge star $*_n$, and check $*_4 F_{A^t} = *_3(\frac{\partial}{\partial t} B^t) + dt \wedge *F_{B^t}$.

From the above, we see that the native gradient flow equation for $\mathcal{L}$ on (the configuration space of pairs on) Y is 4d SW equation on Z. The converse is also true up to gauge transformation (action of gauge group), which is left as an exercise.

1.4. **Morse theory for manifold without boundary.** We quickly review the case without boundary. Let (B, g_B) be a smooth closed Riemannian manifold. (C.f. K-M, Hutchings and Schwarz.) We are discussing the Morse-Witten picture, not the classical handlebody picture, which can generalize appropriately.

For $f : B \to \mathbb{R}$ real valued, can define its gradient ∇f as above via $g_B(\nabla f, v) = df(v)$. Consider the negative gradient flow equation $\dot{x} = -\nabla f(x)$ with $x = x(t)$ and dot means time differentiation; or in the flow notation $\phi_t(x)$ (with initial condition x at $t = 0$) satisfying $\dot{\phi}_t(x) = -\nabla f(\phi_t(x))$.

Definition 1.2. f is Morse, if at each critical point a (exactly where $\nabla f(a) = 0$, and we denote the set of critical points of f as $\mathrm{Crit}(f)$), its (self-adjoint) Hessian $\nabla(\nabla f) : T_a B \to T_a B$ has no kernel. Thus $T_a B = K_a^+ \oplus K_a^-$, into positive and negative eigenspaces. Its index $i(a) := \dim K_a^-$.

We use intersection theoretic instead of functional analytic approach in this introduction.

For critical point a, denote the stable manifold $S_a := \{x \mid \lim\limits_{t\to\infty} \phi_t(x) = a\}$ and the unstable manifold $U_a := \{x \mid \lim\limits_{t\to-\infty} \phi_t(x) = a\}$. Note that they are smooth at a due to exponential convergence. We also have $T_a U_a = K_a^-$. Denote

$$\begin{aligned} M(a,b) : &= \{\text{points along the flow lines from } a \text{ to } b\} \\ &= \{x \mid \phi_t(x) \text{ is a flow line from } a \text{ to } b\} \\ &= U_a \cap S_b \text{ in } B. \end{aligned}$$

Definition 1.3. $-\nabla f$ is Morse-Smale, if all U_a and S_b intersect transversely (meaning $T_y U_a + T_y S_b = T_y B$ for all $y \in U_a \cap S_b$) or all $a, b \in \mathrm{Crit}(f)$.

Then $\dim M(a,b) = \dim U_a + \dim S_b - \dim B = \dim U_a - \dim U_b = i(a) - i(b)$.

In the second description, $\mathbb{R}$ acts on $M(a,b)$ via $\phi_t(\cdot)$.

If $a \neq b$, then $\check{M}(a,b) := M(a,b)/\mathbb{R}$, the space of unparametrized flow lines, is a Hausdorff manifold.

We define the Morse chain complex (C_*, d) as follows:

$C_k := \bigoplus\limits_{a \in \mathrm{Crit}(f), i(a)=k} \mathbb{Z}/2\mathbb{Z} e_a.$

For a, b with $i(a) - i(b) = 1$, we have $\check{M}(a,b)$ 0-dimensional compact manifold, and we can count number of points mod 2 and denoted by $n(a,b)$.

Define the diffeential $\partial e_a := \sum_{b \in \mathrm{Crit}(f), i(b)=k-1} n(a,b) e_b$.

We have $\partial^2 = 0$, being a chain complex, because for a, c with $i(a) - i(c) = 2$, the coefficient of e_c in $\partial\partial e_a$, $\sum_{b \in \mathrm{Crit}(f), i(b)=k-1} n(a,b)n(b,c)$, and it is 0 mod 2.

The last claim follows because we can compactify $\check{M}(a,c)$ into $\check{M}^+(a,c)$ by adding broken unparametrized flow lines. So $\check{M}^+(a,c)$ is compact 1-manifold with boundary exactly being broken flow lines from a to b to c for some b. Such manifold has boundary counted 0 (mod 2) $=\sum_{b \in \mathrm{Crit}(f), i(b)=k-1} n(a,b)n(b,c)$.

Its homology $H_*(C_*, \partial) = H_*(B; \mathbb{Z}/2\mathbb{Z})$.

1.5. **Morse theory for manifold with vertical boundary.** As will be for the most of this course, we follow closely Kronheimer-Mrowka's.

As we have see, the ambient configuration space modulo gauge group to define SW equation/flow equation of $\mathcal{L}$ has singularity at (B, Ψ) with $\Psi = 0$. We will see that a resolution will produce a manifold with boundary whose lift from $\nabla\mathcal{L}$ is tangent to the boundary. So we need to look at Morse theory in this case. The smoothness is best addressed using a doubling construction.

Let B be a manifold with boundary ∂B. To talk about smoothness and etc, let us consider its double, namely, a manifold $\tilde{B}$ without boundary and with a smooth involution $\iota : \tilde{B} \to \tilde{B}$ with fixed point codimension 1 and $\tilde{B}/\iota$ identified with B (thus fixed point set of ι with ∂B). We only consider Riemannian metric (resp. function f) on B that is restricted from (or extendable to) an ι-invariant Riemannian metric on $\tilde{B}$ (resp. $\tilde{f}$). In particular, $\nabla f|_{\partial B} \subset T\partial B$. We suppress this in the background.

Let f be a Morse function on B, then it has critical point in $B \backslash \partial B$, denoted by $\mathfrak{c}^o$ and critical point in ∂B, denoted by $\mathfrak{c}^\partial$. The normal vector (does not matter inwards or outwards as it does not change eigenvalue) ν to ∂B at $a \in \mathfrak{c}^\partial$ is a eigenvector of Hessian. To see this, note that at critical point $a \in \partial B$,

$$g_B(\nabla_\nu \nabla f, w) \overset{\text{self-adjoint}}{=} g_B(\nabla_w \nabla f, \nu) = 0 \text{ for all } w \in T_a\partial B,$$

as ∇f is along $T\partial B$, which means $\nabla_\nu \nabla f$ is a (non-zero being Morse) multiple of ν. Therefore ν either lies in K_a^+, then we denote $a \in \mathfrak{c}^s$ and call a boundary-stable, or it lies in K_a^-, then we denote $a \in \mathfrak{c}^u$, and call a boundary-unstable.

Draw a diagram illustrating the above, which can be made into higher dimensional. The diagram will be updated soon.

Remark 1.4. For $a \in \mathfrak{c}^s$ and $b \in \mathfrak{c}^u$, we have $U_a \subset \partial B$ and $S_b \subset \partial B$, so U_a and S_b cannot have transverse intersection in B. But it makes sense and we can ask the next best thing, transverse in ∂B.

Definition 1.5. $-\nabla f$ is regular, if for $a \in \mathfrak{c}^s$ and $b \in \mathfrak{c}^u$, we have U_a and S_b intersect transversely in ∂B, otherwise, U_a and S_b intersect transverse n B.

Then $M(a,b)$ is a manifold of dimension $i(a) - i(b) + 1$ in the first case (as we subtract 1 dimension less), usual formula otherwise.

In this setting, we can have a broken flow line configuration that is not a limiting flow from smooth flow lines from a to c. Draw a picture of a broken flow line from a to b to c, where $S_c \subset \partial B$ but U_a with $a \in B\backslash\partial B$. A smooth flow line from a to c has to be both in ∂B and $B\backslash\partial B$. Not a pathology, as we need to examine a more complete picture, and include only either kind of boundary-critical points if including interior critical points, to be seen next.

Draw a complete picture with index difference 2 but 2-time broken flow line. $a, d \in \mathfrak{c}^o$, $b \in \mathfrak{c}^s$, $c \in \mathfrak{c}^u$ for the unparametrized broken configuration to exist in dimension 0, we need to have $i(a) - i(b) = 1$, $i(b) - i(c) = 0$ (recall i is defined in B) and $i(c) - i(d) = 1$.

To be continued in the next lecture.