SEIBERG-WITTEN FLOER HOMOLOGY LECTURE 3

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1. Lecture 3

Please email yangding@math.hu-berlin.de to be added to the mail list for possible future (last minute) announcement. Lectures start at 9:15 AM.

This is a slightly updated version fixing some obvious typos to an earlier version.

1.1. Broken flow lines not being limited from smooth ones.

We have seen:

- In proving $\partial^2 = 0$, it is crucial to have broken flow lines being limited to by smooth flow lines. There, 1-broken flow lines are boundary (points) of the compact 1-dimensional "flow space" manifolds (thus counted as 0 mod 2). Namely, starting from a 1-broken flow line and moving inside the flow space, flow lines become smooth until reaching the boundary of the flow space, which is another 1-broken one, thus 1-broke flow lines exist in pairs.
- Notation ∂^2 is a succinct way to keep track of broke flow lines.
- In the case where the background B has vertical boundary ∂B, we can have broken flow lines not limits of smooth ones. For example, there is no smooth flow lines limiting to a 1-broken flow line from $a \in \mathfrak{c}^o$ to $b \in \mathfrak{c}^s$ to $c \in \mathfrak{c}^u$ (because $U_c \subset \partial B$, the smooth flow lines from a stays in $B \backslash B$, thus the limiting smooth flow lines would have to be in both $B \backslash \partial B$ and ∂B).

Remark 1.1. Due to bullet points 1 and 3, to have a chain complex with both $\mathfrak{c}^o$ and $\mathfrak{c}^\partial$, cannot have both $\mathfrak{c}^s$ and $\mathfrak{c}^u$ in the same complex.

1.2. Examining the boundary combinatorial types of broken flow lines.

Lemma 1.2. *Consider B with ∂B with metric and Morse function respecting the doubling. Let $a \in \mathfrak{c}^o_k$, $c \in \mathfrak{c}^o_{k-2}$. Recall $\check{M}(a,c) = M(a,c)/\mathbb{R}$ denotes the space of unparametrized smooth flow lines, and $\check{M}^+(a,c)$ is the compactification of it by adding broken flow lines (limits). Then $\check{M}^+(a,c) \backslash \check{M}(a,c)$ consists of*

$$\text{either } (\check{x}_1, \check{x}_2) \in \check{M}(a,b) \times \check{M}(b,c) \text{ for some } b \in \mathfrak{c}^o,$$
$$\text{or } (\check{x}_1, \check{x}_2, \check{x}_3) \in \check{M}(a,b_1) \times \check{M}(b_1,b_2) \times \check{M}(b_2,c) \text{ for } b_1 \in \mathfrak{c}^s_{k-1} \text{ and } b_2 \in \mathfrak{c}^u_{k-1}.$$

Proof. If broken once, then the intermediate critical point $\notin \mathfrak{c}^\partial$, because a critical point at the boundary cannot be both forwards and backwards limiting point of flow lines in the interior.

If broken twice with three flow lines, the middle flow line has to be exceptional case in the definition of regularity (we call it "obstructed" for short, in the view of transversality), from $\mathfrak{c}^s$ to $\mathfrak{c}^u$, without dropping in index. Constraints of limiting end points also mean that we cannot have adjacent obstructed flow lines.

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Cannot be broken 3+ times, (due to the last line in the last paragraph), as we would need to have at least three flow lines connecting points in $\mathfrak{c}^o$, which would involve a factor with negative dimension (not possible due to being manifold). $\square$

1.3. **Definition of various operators $\bar{\partial}^*_*$ and ∂^*_*.** (Not to be confused with similarly looking differential operators.) Another notational remark: In the lecture for ease of writing we wrote c in the place of $\mathfrak{c}$ and $\mathscr{C}_*$ in the place of C_* below.

For $a, b \in \mathfrak{c}^\partial$, consider $M(a,b) = U_a \cap S_b$.

- for $a \in \mathfrak{c}^u, b \in \mathfrak{c}^s$, both $M(a,b)$ and $M^\partial(a,b) =: M(a,b) \cap \partial B$ are manifolds and they are distinct, in fact $\partial M(a,b) = M^\partial(a,b)$. (Note that the nature of critical points implies that $U_a \cap \partial B$ intersects transversely with $S_b \cap \partial B$ in ∂B.)
- for the other three cases $\big($(i) $a \in \mathfrak{c}^u, b \in \mathfrak{c}^u$, (ii) $a \in \mathfrak{c}^s, b \in \mathfrak{c}^s$, $a \in \mathfrak{c}^s, b \in \mathfrak{c}^u\big)$, we have $M(a,b) = M^\partial(a,b)$ being manifolds, as the flow lines have to lie in ∂B. Here, in case (iii), the obstructed case, $M(a,b)$ is manifold due to "being regular".

Denote $\check{M}^\partial(a,b) := M^\partial(a,b)/\mathbb{R}$. We define the following four operations $\bar{\partial}^*_*$ counting points (mod 2) in 0-dimensional manifold $\check{M}^\partial(a,b)$. Here $\bar{\cdot}$ signifies boundary ∂B. The index difference between the domain and the target in each case ensures the spaces to be counted are 0 dimensional.

Denote $C^u_k := \bigoplus_{a \in \mathfrak{c}^u_k} (\mathbb{Z}/2\mathbb{Z}) e_a$, where e_a denotes the generator labelled by a. Similarly for C^s_k.

Define $\bar{\partial}^u_s : C^u_k \to C^s_{k-2}$ (super/subscripts indicate flow lines flowing from top to bottom) by defining on generators and extending by linearity:

$\bar{\partial}^u_s(e_a) = \sum_{b \in \mathfrak{c}^s_{k-2}} |\check{M}^\partial(a,b)| e_b$, here $|\cdot|$ counts points of a 0-dimensional space mod 2, and index drops two because we mod by $\mathbb{R}$-action and restricting to the boundary of a manifold with boundary.

We contrast it with $\partial^u_s : C^u_k \to C^s_{k-1}$ defined by $e_a \mapsto \sum |\check{M}(a,b)| e_b$.

We can similarly define $\bar{\partial}^s_s : C^s_k \to C^s_{k-1}$, $\bar{\partial}^u_u : C^u_k \to C^u_{k-1}$ and $\bar{\partial}^s_u : C^s_k \to C^u_k$. Note that last preserves the index, $\bar{\partial}^s_u(e_a) = \sum_{b \in \mathfrak{c}^u_k} |\check{M}^\partial(a,b)| e_b$. (Here, recall that $\dim \check{M}^\partial(a,b) = \dim M^\partial(a,b) - 1 \overset{\text{regularity}}{=} \dim U_a + \dim S_b - \dim \partial B - 1 = i(a) - i(b)$.) We also have:

$$
\begin{aligned}
&\partial^o_o : C^o_k \to C^o_{k-1} \\
&\partial^o_s : C^o_k \to C^s_{k-1} \\
&\partial^u_o : C^u_k \to C^o_{k-1} \\
&\partial^u_s : C^u_k \to C^s_{k-1}, \text{the last of which we have seen,}
\end{aligned}
$$

as the only 4 possibilities counting dimension-0 space of unparametrized flow lines in $B \backslash \partial B$. Other combinations will lie inside ∂B and have been covered in above $\bar{\partial}^*_*$.

1.4. **Recasting boundary combinatorial types into equations.** The above lemma says ① $\partial^o_o \partial^o_o + \partial^u_o \bar{\partial}^s_u \partial^o_s = 0$ mod 2. Note that we work in $\mathbb{Z}/2\mathbb{Z}$, and $-$ is $+$.

By considering boundary of 1-dimensional compactified space we have:

For $a \in \mathfrak{c}^o_k$, $c \in \mathfrak{c}^s_{k-2}$, analogously consider configurations of broken flow lines in $\check{M}^+(a,c) \backslash \check{M}(a,c)$, we have ② $\partial^o_s \partial^o_o + \bar{\partial}^s_s \partial^o_s + \partial^u_s \bar{\partial}^s_u \partial^o_s = 0$.

For $a \in \mathfrak{c}^u_k$, $c \in \mathfrak{c}^o_{k-2}$, ③ $\partial^o_o \partial^u_o + \partial^u_o \bar{\partial}^u_u + \partial^u_o \bar{\partial}^s_u \partial^u_s = 0$.

For $a \in \mathfrak{c}^u_k$, $c \in \mathfrak{c}^s_{k-2}$, interesting case ④ $\bar\partial^u_s + \partial^o_s\partial^u_o + \bar\partial^s_s\partial^u_s + \partial^u_s\bar\partial^u_u + \partial^u_s\bar\partial^s_u\partial^u_s = 0$.

Note that in the case ④, the first term counts the dimensional 0 space, as we mod out by $\mathbb{R}$ and take the boundary. Draw a picture for the last term.

Remark 1.3. Here the magic is that a sequence of smooth flow lines can break in a limit into allowable types of broken flow lines, and broken flow lines can be glued back to smooth flow lines, and those are captured exactly as the situation near a point at the boundary in a 1-dimensional manifold with boundary. We have suppressed this crucial detail now. Proving such a statement is more accessible from the functional analytic viewpoint (to be seen later) than the intersection theoretic approach above, which might be easier to meet and visualize at the first.

1.5. **Three variants of chain complexes.** We try to build chain complexes using C^o_*, C^s_* and C^u_*.

Recall if we use C^o_*, can only use one of C^s_* and C^u_*, due to Remark 1.1.

If we do not use C^o_*, we can use both boundary critical points and define

$$\bar C_k := C^s_k \oplus C^u_{k+1},$$

the last summand has an index shift because i is defined in B, and if defined in ∂B, $i^\partial = i - 1$. Define

$$\bar\partial = \begin{pmatrix} \bar\partial^s_s & \bar\partial^u_s \\ \bar\partial^s_u & \bar\partial^u_u \end{pmatrix}.$$

The off-diagonal operators $\bar\partial^u_s : C^u_{k+1} \to C^s_{k-1} \subset \bar C_{k-1}$ and $\bar\partial^s_u : C^s_k \to C^u_k \subset \bar C_{k-1}$ indeed counts the dimensional 0 spaces, as we discussed before. The complex $(\bar C_*, \bar\partial)$ is none other than the Morse chain complex for ∂B, just with different (boundary) critical points distinguished. Arguing as in the last lecture, we have $\bar\partial^2 = 0$. We write this out into

$$\bar\partial^s_s\bar\partial^s_s + \bar\partial^u_s\bar\partial^s_u = 0. \quad (\ddagger)$$
$$\bar\partial^s_u\bar\partial^s_s + \bar\partial^u_u\bar\partial^s_u = 0. \quad (\star)$$
$$\bar\partial^s_s\bar\partial^u_s + \bar\partial^u_s\bar\partial^u_u = 0$$
$$\bar\partial^s_u\bar\partial^u_s + \bar\partial^u_u\bar\partial^u_u = 0$$

After Morse theory for ∂B, now we want to consider C^o_* as well, and we can define two versions:

$\check C_k := C^o_k \oplus C^s_k$ with differential $\check\partial = \begin{pmatrix} \partial^o_o & \partial^u_o\bar\partial^s_u \\ \partial^o_s & \bar\partial^s_s + \partial^u_s\bar\partial^s_u \end{pmatrix}$. Check is also pronounced as "to", as the interior flow lines flowing to the bundary-stable critical points here, the overhead arrow also points to C. The boundary operator may look complicated, but it just includes all counts of dimension 0 spaces of (broken) unparametrized flow lines between appropriate critical points.

$\hat C_k := C^o_k \oplus C^u_k$ with differential $\hat\partial = \begin{pmatrix} \partial^o_o & \partial^u_o \\ \bar\partial^s_u\partial^o_s & \bar\partial^u_u + \bar\partial^s_u\partial^u_s \end{pmatrix}$. Hat is also pronounced as "from", as the interior flow lines flowing from the bundary-unstable critical points here and the overhead arrow also points away from C.

We now show $\check\partial^2 = 0$. Composing the matrix with itself, we want to show:

- The (1,1) entry of $\check\partial^2$ is 0, namely $\partial^o_o\partial^o_o + \partial^u_o\bar\partial^s_u\partial^o_s = 0$ which is just ①.
- $\partial^o_s\partial^o_o + \bar\partial^s_s\partial^o_s + \partial^u_s\bar\partial^s_u\partial^o_s = 0$, which is just ②.
- $\partial^o_o\partial^u_o\bar\partial^s_u + \partial^u_o\bar\partial^s_u\bar\partial^s_s + \partial^u_o\bar\partial^s_u\partial^u_s\bar\partial^s_u$ cannot factorize, but changing the second term into $\partial^u_o(\bar\partial^u_u\bar\partial^s_u)$ according to $(\star)$, it reads now (LHS of ③)$\bar\partial^s_u = 0$.

- $\partial_s^o\partial_o^u\bar{\partial}_u^s + \bar{\partial}_s^s\bar{\partial}_s^s + \partial_s^u\bar{\partial}_u^s\bar{\partial}_s^s + \bar{\partial}_s^s\partial_s^u\bar{\partial}_u^s + \partial_s^u\bar{\partial}_u^s\partial_s^u\bar{\partial}_u^s$ will be of the form of (LHS of ④)$\bar{\partial}_u^s = 0$, after replacing the second term by $\bar{\partial}_s^u\bar{\partial}_u^s$ due to (‡) and the third term by $\partial_s^u(\bar{\partial}_u^u\bar{\partial}_u^s)$ due to ($\star$).

Thus $\check{\partial}^2 = 0$.

Exercise 1.4. $\hat{\partial}^2 = 0$.

Remark 1.5. $(\check{C}_*, \check{\partial})$ calculates $H_*(B;\mathbb{Z}/2\mathbb{Z})$, $(\hat{C}_*, \hat{\partial})$ calculates $H_*(B, \partial B;\mathbb{Z}/2\mathbb{Z})$, and $(\bar{C}_*, \bar{\partial})$ calculates $H(\partial B;\mathbb{Z}/2\mathbb{Z})$.

1.6. **LES.** SES $0 \to C_*(\partial B) \to C_*(B) \to C_*(B, \partial B) \to 0$ leads to the long exact sequence. Can homologies of those chain models $(\bar{C}_*, \bar{\partial}), (\check{C}_*, \check{\partial}), (\hat{C}_*, \hat{\partial})$ fit into LES with induced morphisms from natural maps between these chain models?

The answer is yes. Define

$$i : \bar{C}_k := C_k^s \oplus C_{k+1}^u \to \check{C}_k := C_k^o \oplus C_k^s \text{ by } i = \begin{pmatrix} 0 & \partial_o^u \\ 1 & \partial_s^u \end{pmatrix}.$$

$$j : \check{C}_k := C_k^o \oplus C_k^s \to \hat{C}_k := C_k^o \oplus C_k^u \text{ by } j = \begin{pmatrix} 1 & 0 \\ 0 & \bar{\partial}_u^s \end{pmatrix}.$$

$$p : \hat{C}_k := C_k^o \oplus C_k^u \to \check{C}_{k-1} := C_{k-1}^s \oplus C_k^u \text{ by } p = \begin{pmatrix} \partial_s^o & \partial_s^u \\ 0 & 1 \end{pmatrix}.$$

Exercise 1.6. Check i, j and p are chain maps. (p is only a chain map up to a sign when working over $\mathbb{Z}$ after taking care of orientations of spaces).

Proposition 1.7. *There is an LES* $\cdots \to \check{H}_* \xrightarrow{j_*} \hat{H}_* \xrightarrow{p_*} \bar{H}_{*-1} \xrightarrow{i_*} \check{H}_{*-1} \to \cdots$.

This respects the LES from the above SES, which reads

$$\cdots \to H_*(B) \to H_*(B, \partial B) \to H_{*-1}(\partial B) \to H_{*-1}(B) \to \cdots.$$

The p_* is exhibited at the center of the repeated pattern because it plays a role in the proof where we want to identify $\check{C}$ under a quasi-isomorphism to

$$\mathrm{Cone}(p) := (\hat{C} \oplus \bar{C}, \begin{pmatrix} \hat{\partial} & 0 \\ p & \bar{\partial} \end{pmatrix}).$$

We will prove the proposition next time and discuss the compactness of solutions to SW equation.