SEIBERG-WITTEN FLOER HOMOLOGY LECTURE 4

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1. Lecture 4

Please email `yangding@math.hu-berlin.de` to be added to the mail list for possible future announcement. Lectures start at 9:15 AM. Lecture notes up to now are available at `www.mathematik.hu-berlin.de/ yangding/monopole.html`

1.1. **LES (continued).** SES $0 \to C_*(\partial B) \to C_*(B) \to C_*(B, \partial B) \to 0$ leads to the long exact sequence. As $(\bar{C}_*, \bar{\partial}), (\check{C}_*, \check{\partial}), (\hat{C}_*, \hat{\partial})$ calculate respective homologies in the aforementioned LES, we can see the LES using natural maps between these chain models. Define

$$i : \bar{C}_k := C^s_k \oplus C^u_{k+1} \to \check{C}_k := C^o_k \oplus C^s_k \text{ by } i = \begin{pmatrix} 0 & \partial^u_o \\ 1 & \partial^u_s \end{pmatrix}.$$

$$j : \check{C}_k := C^o_k \oplus C^s_k \to \hat{C}_k := C^o_k \oplus C^u_k \text{ by } j = \begin{pmatrix} 1 & 0 \\ 0 & \bar{\partial}^s_u \end{pmatrix}.$$

$$p : \hat{C}_k := C^o_k \oplus C^u_k \to \check{C}_{k-1} := C^s_{k-1} \oplus C^u_k \text{ by } p = \begin{pmatrix} \partial^o_s & \partial^u_s \\ 0 & 1 \end{pmatrix}.$$

Proposition 1.1. *There is an LES* $\cdots \to \check{H}_* \xrightarrow{j_*} \hat{H}_* \xrightarrow{p_*} \bar{H}_{*-1} \xrightarrow{i_*} \check{H}_{*-1} \to \cdots$.

Proof. Define $\check{E} := \mathrm{Cone}(p) := (\hat{C} \oplus \bar{C}, \check{e} := \begin{pmatrix} \hat{\partial} & 0 \\ p & \bar{\partial} \end{pmatrix})$. (Anti-)chain map property of $p : \hat{C} \to \bar{C}$ is incorporated into $\check{e}^2 = 0$. By construction of mapping cone, we have SES $\bar{C} \xrightarrow{\bar{i}} \check{E} \xrightarrow{\bar{j}} \hat{C}$. Now we want to establish a quasi-isomorphism (map inducing isomorphism between homologies) between $\check{E}$ and $\check{C}$ (respecting the maps on homology), then we are done. Indeed, define

$$(C^0 \oplus C^s) \oplus (C^s \oplus C^u_{*+1}) = \check{E} \underset{k}{\overset{l}{\rightleftarrows}} \check{C} = C^o \oplus C^s,$$

where $k : (x, y) \mapsto (x, \bar{\partial}^s_u y, y, 0)$, and $l : (e, f, g, h) \mapsto (e + \partial^u_o h, g + \partial^u_s h)$. We have $l \circ k = \mathrm{Id}$ and $k \circ l = \mathrm{Id} + \check{e} \circ K + K \circ \check{e}$ for chain homotopy $K : (e, f, g, h) \mapsto (0, h, 0, 0)$. Moreover, $j_* = \bar{j}_* \circ k_*$ and $\bar{i}_* = k_* \circ i_*$.

□

1.2. **Weizenböck, 4d-3d expression, and energies.** We follow [KM], also c.f. [Morgan].

Let $X = (X, g_X)$ be a compact oriented Riemannian 4-manifold with $\partial X = Y$, where the metric is cylindrical metric near Y (as $[-\epsilon, 0] \times Y$). A spin^c structure $\mathfrak{s}_X = (S_X, \rho_X)$ induces $s = (S, \rho)$ along the boundary Y as follows. Denote $n \in$

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$\Gamma(Y)$ the outward unit normal vector field, then $\rho_X(n) : S^+|_Y \overset{\cong}{\to} S^-|_Y$, and define $S := S^+|_Y$. Let $v \in TY$, then $\rho(v)$ is $S^+|_Y \overset{\rho_X(v)}{\to} S^-|_Y \overset{\rho(n)^{-1}}{\to} S^+|_Y$.

Let A (or ∇_A) be a spin^c connection in temporal gauge, so restricts to Y to a spin^c connection on Y.

We have Weitzenböck formula $D_A^- D_A^+ \Phi = \nabla_A^* \nabla_A \Phi + \frac{1}{2}\rho_X(F_{A^t}^+)\Phi + \frac{1}{4}s\Phi$, where s is the scalar curvature for Levi-Civita connection ∇ on X recalled below:

For curvature $F(X,Y)Z := \nabla_X \nabla_Y \tilde{Z} - \nabla_Y \nabla_X \tilde{Z} - \nabla_{[\tilde{X},\tilde{Y}]|}\tilde{Z}$, where $\tilde{\cdot}$ denotes any extension to a vector field and $\cdot|$ denotes restriction to a point (well-defined independent of choices), we have $g_X(F(X,Y)Z,W)$ anti-symmetric in X and Y, anti-symmetric in Z and W, and symmetric in (X,Y) and (Z,W). Define Ricci curvature $\mathrm{Ric}(X,Y) := \sum_i g_X(F(e_i,X)Y,e_i)$ for any orthonormal basis e_i (this order of summing reproduces Gaussian curvature in 2d), and define $s := \mathrm{trRic}$.

Note that adjoint operator is defined using $C_c^\infty(X\backslash\partial X)$ (smooth function of compact support away from boundary) in L^2 metric, so having boundry or not does not affect the formula. As $D_A = \begin{pmatrix} 0 & D_A^- \\ D_A^+ & 0 \end{pmatrix})$ is self-adjoint, D_A^- is adjoint to D_A^+. ρ_X maps imaginary valued self-adjoint 2-form to Hermitian (self-adjoint). So all four operators in front of Φ are self-adjoint.

Adjoint expression involving terms on Y (omitting $d$vol):
$\int_Z \langle \Phi, D_A^- D_A^+ \Phi\rangle = \int_X |D_A^+\Phi|^2 - \int_Y \langle \rho_X(n)\Phi, D_A^+\Phi\rangle$, and
$\int_Z \langle \Phi, \nabla_A^* \nabla_A \Phi\rangle = \int_X |\nabla_A \Phi|^2 - \int_Y \langle \Phi, (\nabla_A)_n \Phi\rangle$.
One can derive from the definition that

$$D_B(\Phi|_Y) = (\rho_X(n)^{-1} D_A^+ \Phi - (\nabla_A)_n \Phi)|_Y + \frac{H}{2}\Phi|_Y,$$

where H-term can be dropped if using cylindrical metric near Y.

Take $\langle \Phi, \cdot\rangle$ to the Weitzenböck formula and integrate and using the above expression about adjoint (involving boundary terms), we have:
$\|\mathcal{F}(A,\Phi)\|_{L^2}^2 := \int_X (|\frac{1}{2}\rho_X(F_{A^t}^+) - (\Phi\Phi^*)_0|^2 + |D_A^+\Phi|^2) = \mathcal{E}^{\mathrm{an}} - \mathcal{E}^{\mathrm{top}}$, where
$\mathcal{E}^{\mathrm{an}} := \frac{1}{4}\int_X |F_{A^t}|^2 + \int_X |\nabla_A \Phi|^2 + \frac{1}{4}\int_X (|\Phi|^2 + \frac{s}{2})^2 - \int_X \frac{s^2}{16}$, and
$\mathcal{E}^{\mathrm{top}} := \frac{1}{4}\int_X F_{A^t} \wedge F_{A^t} - \int_Y \langle \Phi|_Y, D_B(\Phi|_Y)\rangle + \int_Y \frac{H}{2}|\Phi|^2$.

In the cylindrical situation $[t_1,t_2]\times Y$ (which we currently have near Y) (denoting $\gamma = (A,\Phi)$ in 4d as $\gamma(t)$ in 3d), $\mathcal{E}^{\mathrm{an}} = \int_{t_1}^{t_2} (|\dot{\gamma}|^2 + |\nabla \mathcal{L}(\gamma)|^2)dt$ (see [KM] (4.20) for a gauge invariant expression), and $\mathcal{E}^{\mathrm{top}} = 2(\mathcal{L}(t_1) - \mathcal{L}(t_2))$.

For SW equation solution (A,Φ), $\mathcal{E}^{\mathrm{an}} = \mathcal{E}^{\mathrm{top}}$.

Exercise 1.2. Show $\int |F_{A^t}|^2 - \int F_{A^t} \wedge F_{A^t} = 2\int_X |F_{A^t}^+|^2$, and use this and the above to show for $s \geq 0$, we have $\Phi = 0$ for SW solution.

1.3. Compactness theorem.

Theorem 1.3. *Let us be in the above setting.*

(1) For any constant C, only finitely many $\mathfrak{s}_X$'s admit solution (A,Φ) to SW with $\mathcal{E}^{top}(A,\Phi) \leq C$.

(2) Let (A_n,Φ_n) be a sequence of smooth SW solution with $\mathcal{E}^{top}$-bound C. Then exist smooth gauge transformations $u_n : X \to S^1$ such that

(a) a subsequence of $u_n(A_n,\Phi_n) \overset{\text{weakly in } L_1^2}{\to} (A,\Phi)$ for some $(A,\Phi) \in L_1^2$ (explained below);

(b) *if the same subsequence (denoted with same index) satisfies*

$$\text{limsup}\, \mathcal{E}^{top}(A_n, \Phi_n) = \mathcal{E}^{top}(A, \Phi),$$

then convergence $u_n(A_n, \Phi_n)$ to (A, Φ) in L^2_1 is strong; and

(c) *the same subsequence (without need to satisfying hypothesis in (b)) converges in C^∞ on every $X' \subset\subset X\backslash\partial X$.*

Here L^p_k is Sobolev space, completion of smooth functions/sections in $\|f\|_{L^p_k} := (\sum_{0\leq i\leq k}\int_X |\nabla^i f|^p d\text{vol})^{1/p}$, with $1 < p < \infty$. Finite regularity but complete.

Let H be a Banach space, with dual H^*, $a_n \to a$ weakly, if for all $f \in H^*$, $f(a_n) - f(a) \to 0$. If H is Hilbert with inner product $\langle\cdot,\cdot\rangle$, $a_n \to a$ weakly if for all $f \in H$, $\langle a_n - a, f\rangle \to 0$. $a_n \to a$ (strongly) in H, if $\|a_n - a\|_H \to 0$. As an example, a orthonormal countable basis converges weakly to 0 but not strongly.

Proof. For SW solution, we have $\mathcal{E}^{\text{an}} = \mathcal{E}^{\text{top}}$. We then have $\int_X |F_{A^t_n}|^2 \leq C_1$, $\int_X |\Phi_n|^4 \leq C_2$ (can also seen from first SW equation), and $\int_X |\nabla_{A_n}\Phi_n|^2 \leq C_3$ (as we can bound s due to compactness). The first gives that $c_1(\mathfrak{s}_X)$ lies in a compact set. spin^c structure in 4d is also affine over H^2, which gives conclusion (i).

Therefore, can restrict to a fixed spin^c structure and fix a base spin^c connection A_0, we want to choose $u'_n : X \to S^1$ such that

$$d^*(A^t_n - A^t_0 - 2(u'_n)^{-1}du'_n) = 0 \text{ in } X$$

$$\langle A^t_n - A^t_0 - 2(u'_n)^{-1}du'_n, n\rangle = 0 \text{ at } \partial X$$

where non-subscript n is the unit outward normal.

u'_n can be of the form e^{ξ_n} for $\xi_n : X \to i\mathbb{R}$, if we can solve

$$2\Delta\xi_n = d^*(A^t_n - A^t_0) \quad \text{in } X$$

$$2\langle d\xi_n, n\rangle = \langle A^t_n - A^t_0, n\rangle \quad \text{at } \partial X.$$

This is Neumann boundary value problem. e^{ξ_n} is unique up to multiplying by constant for this trivial homotopy class case $[u'_n] = 0$.

$A_n - (u'_n)^{-1}du'_n =: u'_n(A_n)$ is said to be in Coulomb-Neumann gauge if the above pair of conditions for $u'_n = e^{\xi_n}$ holds.

For non-trivial homotopy class $a \in [X, S^1]$, there exists $v : X \to S^1$ with $[v] = a$ satisfying the homogeneous equation (thus the Coulomb-Neumann gauge condition can be solved for any homotopy class)

$$d^*(v^{-1}dv) = 0 \quad \text{in } X$$

$$\langle v^{-1}dv, n\rangle = 0 \quad \text{at } \partial X.$$

We have uniqueness if asking further $i\int \beta_r \wedge (A^t_n - A^t_0 - 2u_n^{-1}du_n) \in [0, 2\pi)$, where $\{\beta_r\}$ represents basis of $H^3(X;\mathbb{R})$ (this can be viewed as period condition on loops via Poincaré duality).

We need a lemma whose proof is delegated to the exercise session (see [KM] 5.1.2, 5.1.3 and the paragraph that follows):

Lemma 1.4. *For any imaginary-valued 1-form a satisfying $\langle a, n\rangle = 0$ at ∂X and $i\int \beta_r \wedge a \in [0, 2\pi)$, we have*

$$\|a\|^2_{L^2_1} := \int_X (|\nabla a|^2 + |a|^2)dvol \leq K_1 \int_X (|d^*a|^2 + |da|^2)dvol + K_2$$

for K_i constant.

To see (2)(a), write $(\tilde{A}_n, \tilde{\Phi}_n) := (u_n(A_n), u_n\Phi_n)$. Apply Lemma 1.4 to $\tilde{A}^t_n - A^t_0$, then $|d^*a|^2$-term on RHS is 0 due to Coulomb gauge, and $\int_X |da|^2$ term is bounded due to curvature bound (and finiteness of $F_{A^t_0}$ in L^2), so we get a L^2_1 bound for $\tilde{A}^t_n - A^t_0$.

We have Sobolev embedding $L^p_1 \hookrightarrow L^{p^*}$, where $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{\dim X} = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$. (Useful when $\dim X > p$, so that $p^* > 0$. Note that $p^* > p > 1$.) So we have L^4 bound for $\tilde{A}^t_n - A^t_0$.

We have $\|\nabla_{\tilde{A}_n}\tilde{\Phi}_n\|_{L^2}$ bounded at the start of the proof.

Then $\nabla_{A_0}\tilde{\Phi}_n = \nabla_{\tilde{A}_n}\tilde{\Phi}_n - (\tilde{A}_n - A_0)\tilde{\Phi}_n$ is L^2 bounded as the last term has both factors L^4 bounded, thus itself L^2 bounded (by Cauchy-Schwarz inequality) and the first term is $\nabla_{\tilde{A}_n}\tilde{\Phi}_n = u_n(\nabla_{A_n}(u_n^{-1}(u_n(\Phi_n)))) = u(\nabla_{A_n}\Phi_n)$ has the same norm as $\nabla_{A_n}\Phi_n$ which is L^2 bounded at the start of the proof.

We also have $\tilde{\Phi}_n$ L^2 bounded (due to L^4 bounded and compactness of X via Cauchy-Schwarz), thus $\|\tilde{\Phi}_n\|_{L^2_1}$ is uniformly bounded.

$(L^2_1)^* \cong L^2_1$. As unit ball in $(L^2_1)^*$ is weakly compact. We have a subsequence $\tilde{\Phi}_n$ weakly converging to a limit. This completes (2)(a).

To be continued. $\square$