SEIBERG-WITTEN FLOER HOMOLOGY LECTURE 5

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1. Lecture 5

Please email `yangding@math.hu-berlin.de` to be added to the mail list for possible future announcement. Lectures start at 9:15 AM. Lecture notes up to now are available at `www.mathematatik.hu-berlin.de/~yangding/monopole.html`

We continue to follow [KM] closely and at a few places flesh out some details.

1.1. **Recall where we were at from last time.** We have Seiberg-Witten equation for (A, Φ) over a compact Riemannian 4-manifold X with $\partial X = Y$:

$$\frac{1}{2}\rho_X(F^+_{A^t}) - (\Phi\Phi^*)_0 = 0$$
$$D^+_A\Phi = 0$$

We have two notions of energies:

$\mathcal{E}^{\mathrm{top}}(A,\Phi) = \frac{1}{4}\int_X F_{A^t}\wedge F_{A^t} - \int_Y\langle\Phi|_Y, D_B(\Phi|_Y) + \int_Y \frac{H}{2}|\Phi|^2$, where $N(V,W)n := (\nabla_V\tilde{W})^\perp$ and mean curvature $H := \mathrm{tr}_Y N$.

$\mathcal{E}^{\mathrm{an}}(A,\Phi) := \frac{1}{4}\int_X |F_{A^t}|^2 + \int_X |\nabla_A\Phi|^2 + \frac{1}{4}\int_X (|\Phi|^2 + \frac{s}{2})^2 - \int_X \frac{s^2}{16}$, where $s = \mathrm{tr}_X\mathrm{Ric}$ and $\mathrm{Ric}(V,W) = \mathrm{tr}_X g_X(R(\cdot,V)W,\cdot)$.

For SW solution (A,Φ), $\mathcal{E}^{\mathrm{top}}(A,\Phi) = \mathcal{E}^{\mathrm{an}}(A,\Phi)$.

We stated and proved the (1) and (2)(a) of the following compactness theorem (interior compactness up to gauge transformation under finite topological energy):

Theorem 1.1. *(1) Finiteness of spin^c structures admitting SW solutions under a given $\mathcal{E}^{top}$-bound.*

(2) Sequential compactness up to gauge transformation under finite $\mathcal{E}^{top}$-bound: Let (A_n,Φ_n) be a sequence of SW solution with $\mathcal{E}^{top}(A_n,\Phi_n) \le C < \infty$. There exists $u_n : X \to S^1$ such that

(a) a subsequence of $(\tilde{A}_n,\tilde{\Phi}_n) := u_n(A_n,\Phi_n) \underset{\text{in } L^2_1}{\overset{\text{weakly}}{\to}} (A,\Phi) \in L^2_1$;

(b) If $\mathrm{limsup}_n\mathcal{E}^{top}(A_n,\Phi_n) = \mathcal{E}^{top}(A,\Phi)$, then the same subsequence in (a) converges to (A,Φ) (strongly) in L^2_1; and

(c) the subsequence in (a) converges in $C^\infty_{loc}(X\backslash\partial X)$ (namely, in $C^\infty(X')$ for any open domain $X' \subset\subset X\backslash\partial X$).

We recalled again what Banach L^k_p and weak convergence for Banach/Hilbert space are.

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1.2. **Proof of (2)(b), norm preserving plus weak convergence imply strong convergence.** We prove (2)(b).

Note that $\mathcal{E}^{\mathrm{top}}(A_n, \Phi_n) = \mathcal{E}^{\mathrm{an}}(A_n, \Phi_n) = \mathcal{E}^{\mathrm{an}}(\tilde{A}_n, \tilde{\Phi}_n)$ as

$$|\nabla_{A_n}\Phi_n| = |u_n(\nabla_{A_n}\Phi_n)| = |(u_n \circ \nabla_{A_n} \circ u_n^{-1})(u_n\Phi_n)| = |\nabla_{\tilde{A}_n}\tilde{\Phi}_n|.$$

Then the hypothesis of (2)(b) means that we have uniform L^2-bound of the following $(F^+_{\tilde{A}_n})$, $\nabla_{\tilde{A}_n}\tilde{\Phi}_n$, and $(\tilde{\Phi}_n\tilde{\Phi}_n^*)_0$. Recall that L^2 norm of the last is a constant factor of $\|\tilde{\Phi}_n\|_{L^4}$ as we have seen.

$L^2 \cong (L^2)^*$, by Banach-Alaoglu which says that the unit/bounded ball in dual space is weakly compact, we have a common subsequence of triples weakly converges in L^2 to some limit. We also have $(\tilde{A}_n, \tilde{\Phi}_n)$ converges strongly in L^2.

We want to establish the weak limit of the triples:

- L^2 weak limit of $F^+_{\tilde{A}_n}$ is F^+_A.

  Indeed, we have $\langle \tilde{A}_n - A, d^*b\rangle = \langle \tilde{A}_n - dA, b\rangle \to 0$ for any smooth b compactly supported away from ∂X. Its self-dual projection says that $\langle F^+_{\tilde{A}_n} - F^+_A, b\rangle \to 0$ for all b.
- Weak limit of $\nabla_{\tilde{A}_n}\tilde{\Phi}_n$ is $\nabla_A\Phi$.

  Let $\tilde{A}_n = A_0 + a_n$ and $A = A_0 + a$ for a base connection A_0. We have $\nabla_{\tilde{A}_n}\tilde{\Phi}_n = \nabla_{A_0}\tilde{\Phi}_n + a_n\tilde{\Phi}_n$. The first term on RHS converges weakly in L^2 to $\nabla_{A_0}\Phi$ by an argument similar to the previous item. The second term on RHS converges in L^1 to $a\Phi$ in particular weakly converges to $a\Phi$. To see the L^1 convergences, note that both factors converge in L^2 to a and Φ, and we use Cauchy-Schwarz (CS) $\int|\alpha\beta| \le (\int|\alpha|^2)^{1/2}(\int|\beta|^2)^{1/2}$. So we have $\nabla_{\tilde{A}_n}\tilde{\Phi}_n$ converges weakly in L^2 to $\nabla_{A_0}\Phi + a\Phi = \nabla_A\Phi$.
- Similarly, weak limit of $(\tilde{\Phi}_n\tilde{\Phi}_n^*)$ in L^2 is $(\Phi\Phi^*)_0$.

In particular, (A, Φ) is an SW solution.

Recall a lemma, for Hilbert space (here we look at L^2), if $x_n \to x$ weakly in L^2 and $\lim_n \|x_n\|$ exists and equals to $\|x\|_{L^2}$, then x_n converges strongly to x in L^2. Proof is one line,

$$\|x_n - x\|^2 = \langle x_n - x, x_n - x\rangle = \|x_n\|^2 + \|x\|^2 - 2\langle x_n, x\rangle \to 2\|x\|^2 - 2\langle x, x\rangle = 0.$$

We have the norm preserving statement for three terms together, we can separate them because we have $\|x\| \le \limsup_n \|x_n\|$.

L^2 norm preserving in limit for $F^+_{\tilde{A}_n}$, $\nabla_{\tilde{A}_n}\tilde{\Phi}_n$ and $(\tilde{\Phi}_n\tilde{\Phi}_n^*)_0$ respectively means strong convergence in L^2 to F^+_A, $\nabla_A\Phi$ and $(\Phi\Phi^*)_0$ respectively.

First strong convergence says $\tilde{A}_n \to A$ in L^2_1 thus in L^4, recall that Sobolev embedding we went over $2^* = 4$ in this case. Third strong convergence means $\tilde{\Phi}_n \to \Phi$ in L^4. Putting both together and using CS again, we have $(A_0 - \tilde{A}_n)\tilde{\Phi}_n$ to $(A_0 - A)\Phi$ in L^2 strongly. Together with $\nabla_{\tilde{A}_n}\tilde{\Phi}_n \to \nabla_A\Phi$ strongly in L_2 before, $\nabla_{A_0}\tilde{\Phi}_n = \nabla_{\tilde{A}_n}\tilde{\Phi}_n + (A_0 - \tilde{A}_n)\tilde{\Phi}_n$ converges strongly in L^2 to $\nabla_A\Phi + (A_0 - A)\Phi = \nabla_{A_0}\Phi$. This finishes (2)(b).

1.3. **(2)(c) Two claims and abstract SW with gauge fixing.** We can prove (2)(c) if the following two claims hold.

Claim 1: L^2_1-converging sequence of smooth solutions in Coulomb gauge converges in C^∞ on every interior domain $X' \subset\subset X\backslash\partial X$.

Claim 2: On any interior domain, hypothesis in (2)(b) holds.

So basically, Claim 1 says that we can get the conclusion of (2)(c) from conclusion of 2(b); and Claim 2 says that we prove starting point of (2)(b) on any interior domain. Validity of both claims immediately gives (2)(c). (So (2)(b) was a tool in the proof.)

To prove Claim 1 using elliptic estimate, we first render SW into abstract form, so that the argument is instructive and transferable to other similar settings.

Denote $A = A_0 + a$. SW+Coulomb gauge fixing is

$$\begin{aligned} \frac{1}{2}\rho_X(F_{A_0^+}) + \rho_X(d^+a) - (\Phi\Phi^*)_0 &= 0 \\ D_{A_0}\Phi + a\Phi &= 0 \\ d^*a &= 0 \end{aligned}$$

Terms are collected into $D : \Gamma(iT^*X \oplus S^+) \to \Gamma(i\mathbb{R} \oplus i\mathfrak{su}(S^+) \oplus S^-)$,

$$(a, \Phi) \mapsto (d^*a, \rho(d^+a), D_{A_0}\Phi).$$

Write $\gamma := (a, \Phi)$.

Terms can be written as $Q(\gamma, \gamma)$ for a symmetric bilinear form

$$Q(\gamma, \hat{\gamma}) := (-\frac{1}{2}(\Phi\hat{\Phi}^* + \hat{\Phi}\Phi^*), \frac{1}{2}(a\hat{\Phi} + \hat{a}\Phi, 0)).$$

The leftover term is denoted by $-b := (\frac{1}{2}\rho_X(F_{A_0^+}), 0, 0)$. So we have the abstract expression of the SW under Coulomb gauge. $D\gamma + Q(\gamma, \gamma) = b$.

1.4. **Elliptic operator and estimate.** The key fact is that D is elliptic, which allows the following elliptic estimate (semi-Fredholm estimate):

Theorem 1.2. *(Gårding inequality) Let D be a first order elliptic operator. Let $X^{(1)} \subset\subset X$. Then there exists constant C and for any smooth γ, we have*

$$\|\gamma\|_{L^p_{k+1}(X^{(1)})} \le C(\|D\gamma\|_{L^p_k(X)} + \|\gamma\|_{L^p(X)}).$$

A differential operator $D : \Gamma(E) \to \Gamma(F)$ of order k (over $\mathbb{R}$ coefficient) between sections of bundles E and F over the same base X, if over trivializing neighborhood $U \subset \mathbb{R}^d$, $E|_U = U \times \mathbb{R}^m$ and $F|_U = U \times \mathbb{R}^n$ (so $\Gamma(E|_U) = (C^\infty(U))^m$ and $\Gamma(E|_U) = (C^\infty(U)^n)$), D is of the form

$$(f_1, \cdots, f_m) \mapsto (\sum_{i,|\alpha|\le k} a_{1i\alpha}\partial^\alpha f_i, \cdots, \sum_{i,|\alpha|\le k} a_{ni\alpha}\partial^\alpha f_i),$$

where $\partial^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}}$ for multi-index $\alpha = (\alpha_1, \cdots, \alpha_d)$ and $|\alpha| = \sum_j \alpha_j$.

The symbol of $D(x, \xi) : E_x \to F_x$ for $\xi \in T_x^*X$ in the above local coordinate is $(v_1, \cdots, v_m) \mapsto (\sum_{i,|\alpha|=k} a_{1i\alpha}\xi^\alpha v_i, \cdots, \sum_{i,|\alpha|=k} a_{ni\alpha}\xi^\alpha v_i)$, where $\xi^\alpha := \xi_1^{\alpha_1} \cdots \xi_d^{\alpha_d}$.

For coordinate-free way, $D(x, \xi)(v)$ is defined by choosing $f \in C^\infty(X)$ with $f(x) = 0$ and $d_x f = \xi$, and $e \in \Gamma(E)$ with $e(x) = v$, then $D(x, \xi)(v) := D(\frac{f^k}{k!}e)(x)$.

We will also write, for $\eta \in \Omega^1(X)$ and $V \in \Gamma(E)$,

$$\sigma(D, \eta)V := (x \mapsto D(x, \eta_x)V_x) \in \Gamma(F).$$

As an example, for $d : \Gamma(\wedge^p(T^*X)) \to \Gamma(\wedge^{p+1}(T^*X))$, $d(x, \xi)(\eta_x) = \xi \wedge \eta_x$.

A differential order is elliptic, if for any $\xi \in T_x^*X\backslash\{0\}$, $D(x, \xi) : E_x \to F_x$ is invertible.

Exercise 1.3. $D_{A_0} : \Gamma(S^+) \to \Gamma(S^-)$ is elliptic. For $\Gamma(E_1) \overset{D_1}{\to} \Gamma(E_2) \overset{D_2}{\to} \Gamma(E_3)$ where D_1 and D_2 are first order operators, such that for $\xi \in T_x^*X$ and for any $x \in X$,

$$(E_1)_x \overset{D_1(x,\xi)}{\to} (E_2)_x \overset{D_2(x,\xi)}{\to} (E_3)_x$$

is exact, then $\Gamma(E_2) \overset{D_1^*+D_2}{\to} \Gamma(E_1) \oplus \Gamma(E_3)$ is elliptic. Show $d^* + d^+$ is elliptic. Thus D in the SW with Coulomb gauge fixing is Elliptic.

1.5. **Proof of Claim 1.** For any interior domain $X' \subset\subset X \backslash \partial X$, choose cut off β with $\beta|_{X'} = 1$ and compactly supported in $X \backslash \partial X$.

In hypothesis, we have $\gamma_n \to \gamma$ in L_1^2. So for any $\epsilon > 0$, there exists i_0 such that $\|\gamma_i - \gamma_{i_0}\|_{L_1^2} \le \epsilon$ for all $i \ge i_0$.

From the abstract expression, we have

$$0 = D(\gamma_i - \gamma_j) + (Q(\gamma_i, \gamma_i) - Q(\gamma_j, \gamma_j)) = D(\gamma_i - \gamma_j) + Q(\gamma_i - \gamma_j, \gamma_i + \gamma_j).$$

Then (‡) $\|\beta(\gamma_i - \gamma_j)\|_{L_{k+1}^p(X)} \le C(\|D(\beta(\gamma_i - \gamma_j))\|_{L_k^p(X)} + \|\beta(\gamma_i - \gamma_j)\|_{L^p(X)})$ by Gårding.

First term on RHS inside the norm is $\beta D(\gamma_i - \gamma_j) + \sigma(D, d\beta)(\gamma_i - \gamma_j)$, see the notation for the second term in the previous subsection, which in L_k^p in particular is bounded multiple of $\|\gamma_i - \gamma_j\|_{L_k^p}$ (so is $\|\beta(\gamma_i - \gamma_j\|_{L^p}$).

We have

$$\begin{aligned} -\beta D(\gamma_i - \gamma_j) &= \beta Q(\gamma_i - \gamma_j, \gamma_i + \gamma_j) \\ &= Q(\beta(\gamma_i - \gamma_j), \gamma_i + \gamma_j - 2\gamma_{i_0}) + Q(\beta(\gamma_i - \gamma_j), 2\gamma_{i_0}). \end{aligned}$$

Recall Q here involves no differentiation and is just an algebraic bilinear form and can be regarded as (the projection with constant weight of) product of the factors.

Now we specialize to $L_{k+1}^p = L_1^3$ $(p = 3, k = 0)$, we use $L_1^3 \times L_1^2 \to L^3, (a, b) \mapsto ab$ is bounded/continuous. First Q term in $L^2 \le C\|\beta(\gamma_i - \gamma_j)\|_{L_1^3} \|\gamma_i + \gamma_j - 2\gamma_{i_0}\|_{L_1^2}$ whose second factor can be as small as we like ($\le \epsilon$) and this term can be moved to the LHS of (‡) at the expense of increasing C by a factor. Thus we get

$$\|\beta(\gamma_i - \gamma_j)\|_{L_1^3} \le C\|\gamma_i - \gamma_j\|_{L^3}.$$

We increase the regularity by 1 (here X' is arbitrary).

Specialize to $L_{k+1}^p = L_2^2$ and use $L_2^2 \times L_1^3 \to L_1^2$, we can L_2^2 bound in terms of L_1^2 bound.

Specialize to L_3^2 and use $L_3^2 \times L_2^2 \to L_2^2$, we can L_2^2 bound in terms of L_2^2 bound.

Specialize to L_2^{k+1}, for $k \ge 3$, we have the Banach algebra $L_k^2 \times L_k^2 \to L_k^2$, we get L_{k+1}^2 bound in terms of L_k^2 bound.

The above argument of getting increasingly better regularity is called elliptic bootstrapping.

Sobolve embedding $L_k^p \subset C^m$ for any $0 \le m < k - \frac{\dim X}{p} \overset{\text{our case}}{=} k - \frac{4}{2} = k - 2$. So we have $L_{k+3}^2 \subset C^k$. This finishes Claim 1.

Exercise 1.4. Using $L_k^2 \subset C^{k-3}$ for $k \ge 3$ to show Banach algebra property $L_k^2 \times L_k^2 \to L_k^2$ for $k \ge 3$.

1.6. **Proof of Claim 2.** Only show the cylindrical case near ∂X (namely, metrically $[-\epsilon, 0] \times Y$) which is sufficient for what follows. Denote $X_s := X \backslash (s, 0] \times Y$ with $s \in [-s, 0]$. Define $f_n(s) := \mathcal{S}_{X_s}^{\text{an}}(A_n, \Phi_n) : [-\epsilon, 0] \to \mathbb{R}$ by integrating over X_s only.

(f_n) has uniform bound from above and below with $f_n' \geq 0$ and with uniformly bounded integral. Thus, we must have $\mu(\{f_n' \leq M\}) \geq \delta > 0$ independent of n where μ is the Lebesgue measure. (Otherwise, for any i, there exists $n_i \to \infty$, $\delta_i \to 0$ such that $\mu(\{f_{n_i}' \leq i\}) < \delta_i$, so integral $\geq i(\epsilon - \delta_i) \to \infty$, which contradicts to the uniform boundedness of integral.)

We need a lemma: Let $\{S_\alpha\}_{\alpha \in A}$ where $S_\alpha \subset [a,b]$ and A is an infinite index set. If $\mu(S_\alpha) \geq \delta > 0$. Then there exists infinite $B \subset A$ such that $\bigcap_{\alpha \in B} S_\alpha \neq \emptyset$. (Exercise or see [KM] 5.1.6.)

Apply this lemma to $S_n := \{f_n' \leq M\} \subset [-\epsilon, 0]$. There exists $s_0 \in [-\epsilon, 0]$, $f_n'(s_0) \leq M$. LHS is $-\frac{d}{ds}\mathcal{L}(\gamma_n|_{\{s\}\times Y}(s_0) = \|\nabla_{\gamma_n(s_0)}\mathcal{L}\|_{L^2}^2$.

We need a 3d analogue of Lemma 1.4 in the last Lecture notes, which says: $(B_n, \Psi_n) := \gamma_n(s_0)$ on Y with $\|\nabla_{(B_n,\Psi_n)}\mathcal{L}\|_{L^2}^2 \leq M$. Then there exists v_n such that $v_n(B_n, \Psi_n)$ converges in $L^2_{1/2}$ norm to a L^2_1-limit. Here $L^2_{1/2}$-norm is defined using Laplacian, which can be taken as a black box or a reading assignment on a small chapter on pseudo-differential operator on e.g. Wells' Differential analysis on complex manifolds (or most books/notes on index theorem). For our purpose, it means $\int b_n \wedge db_n$ and $\int \langle D_{B_n}\tilde{\Phi}_n|_{\{s_0\}\times Y}, \tilde{\Phi}_n|_{\{s_0\}\times Y}\rangle$ are controlled (which is like 1/2 derivative in L^2).

Let $S := S^+|_{\{s_0\}\times Y}$ the spin bundle.

If $c_1(S)$ is torsion, then $\mathcal{L}$ is gauge-invariant and continuous in $L^2_{1/2}$ norm, which is the starting point of (2)(a).

If $c_1(S)$ not torsion, then $\mathcal{L}$ is a constant multiple of two $L^2_{1/2}$-terms above $+\frac{1}{4}\int b_n \wedge F_{B_0^+}$, as $b_n \in L^2_{1/2} \overset{\text{compact}}{\subset} L^2$, we have starting point of (2)(a) again. This finishes Claim 2, thus (2)(c) and compactness theorem.

We do not have bubbling phenomenon in the interior (which makes this theory drastically simpler, this is also why we spent some time on this part explaining some heavy lifting by analysis to go beyond just story telling), but the theorem does not discuss about what happens near the boundary Y, where (possibly several levels of) SW solutions on invariant cylinder break off. We will take a quick look at this after explaining how to deal with singular SW solution $(A, 0)$ which has stabilizer group S^1 in the configuration space quotiented by gauge group.