SEIBERG-WITTEN FLOER HOMOLOGY LECTURE 6

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1. Lecture 6

Please email `yangding@math.hu-berlin.de` to be added to the mail list for future announcement, or if having any questions, or to have contents added to the lectures to make it more self-contained or useful. Lecture notes up to now are available at `www.mathematik.hu-berlin.de/~yangding/monopole.html`

In $4d$, the configuration space $\mathcal{C}(X, \mathfrak{s}) = \mathcal{A} \times \Gamma(S^+) \ni (A, \phi)$. $\mathcal{A}$ is the space of spin^c connections which is an affine space over $\Gamma(iT^*X) = i\Omega^1(X)$, where we have suppressed $\cdot \otimes \mathrm{Id}_{S_X}$. The gauge group $\mathcal{G}_X = \{u : X \to S^1\}$ acts with the quotient $\mathcal{B}(X, \mathfrak{s}_X) := \mathcal{C}(X, \mathfrak{s}_X)/\mathcal{G}_X$.

(A, ϕ) is called irreducible if $\phi \neq 0$. The irreducible configurations are $\mathcal{C}^*(X, \mathfrak{s}) = \mathcal{A} \times (\Gamma(S^+)\backslash\{0\})$. $\mathcal{G}_X$ acts on $\mathcal{C}^* = \mathcal{C}^*(X, \mathfrak{s}_X)$.

We have $S^1 \overset{\text{constant functions}}{\hookrightarrow} \mathcal{G}_X \overset{(-u^{-1}du, u\cdot)}{\hookrightarrow} \mathcal{A} \times (\Gamma(S^+)\backslash\{0\}) \to \mathcal{B}^*(X, \mathfrak{s}_X)$, the latter two-arrow diagram is a principle bundle (with the middle arrow as inclusion of the fiber). This induces

$$S^1 \hookrightarrow P \to \mathcal{B}^*(X, \mathfrak{s}_X)$$

(namely, $P := \mathcal{A} \times (\Gamma(S^+)\backslash\{0\})/(\mathcal{G}_X/S^1))$, and this is an S^1 bundle over a manifold (the action being free on the irreducibles). More on this can be found in the next lecture. This part is to motivate why we are interested in S^1-action and the way of resolving singularity of this action in our setting.

1.1. **Toy example.** Ultimately, we want to deal with $(\Gamma(S^+), \langle\cdot,\cdot\rangle_{L^2})$ in $4d$ and $(\Gamma(S), \langle\cdot,\cdot\rangle_{L^2})$ in $3d$ in infinite dimensions. But we consider the toy model first: $(\mathbb{C}^n, \langle\ ,\ \rangle)$, where the latter is the standard inner product.

Let L be a Hermitian matrix on $\mathbb{C}^n$ (which plays the role of D_B in infinite dimension later).

Define function $\Lambda(z) := \frac{\langle z, Lz\rangle}{\|z\|^2}$ on $\mathbb{C}^n\backslash\{0\}$, and it is $\mathbb{C}^* := \mathbb{C}\backslash\{0\}$-invariant; and real-valued, as $\langle z, Lz\rangle = \langle L^*z, z\rangle = \langle Lz, z\rangle = \overline{\langle z, LZ\rangle}$.

So $\Lambda : \mathbb{C}^n\backslash\{0\} \to \mathbb{R}$ descends to $\mathbb{CP}^{n-1} = (\mathbb{C}^n\backslash\{0\})/\mathbb{C}^* \to \mathbb{R}$, mapping from the complex projective space. For $\mathbb{CP}^{n-1}$, we have another sphere model $\mathbb{CP}^{n-1} = S^{2n-1}/S^1$ where $S^{n-1} = \{\|\cdot\| = 1\}$.

Consider function $f(z) = \frac{1}{2}\langle z, Lz\rangle$ on $\mathbb{C}^n$.

The negative gradient flow equation for f is linear: $\frac{dz}{dt} = -Lz$ for $z : \mathbb{R} \to \mathbb{C}^n$.

Claim: The negative gradient flow for $\frac{1}{2}\Lambda$ on $\mathbb{CP}^{n-1}$ is $z : \mathbb{R} \to \mathbb{C}^n\backslash\{0\}$ satisfying $\frac{dz}{dt} = -Lz$ under the projection $\pi : \mathbb{C}^n\backslash\{0\} \to \mathbb{CP}^{n-1}$.

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To see this, switch to S^{2n-1} viewpoint where $\frac{1}{2}\Lambda = f$. For $w \in S^{2n-1}$, ∇f in $\mathbb{C}^n$ has normal component along w (recall $\|w\| = 1$), which is

$$\langle \frac{w}{|w|}, \nabla f \rangle \frac{w}{|w|} = \langle w, \nabla f \rangle w = \langle w, Lw \rangle w = \Lambda(w) w.$$

The tangent component is $\nabla f - \Lambda(w)w = Lw - \Lambda(w)w$ which is the gradient of $f|_{S^{2n-1}}$. The image of z satisfying $\frac{dz}{dt} = -Lz$ on S^{2n-1} is $\frac{dw}{dt} = -Lw + \Lambda(w)w$.

On S^{2n-1}, the critical point w is where $\nabla_w f = Lw$ is parallel with w, i.e. $Lw = \mu w$, from which we know $\mu = \langle w, Lw \rangle = \Lambda(w)$. The critical point w is precisely the eigenvector of L and its eigenvalue is $\Lambda(w)$.

For $w \in S^{2n-1}$ a critical point of ∇f (iff w is an eigenvector of L), $f = \frac{1}{2}\langle z, Lz \rangle$:

$$\begin{aligned}
&(\mathrm{Hessian}_w f)(v) \\
=&\nabla_w (\nabla f)(v) \\
=&\nabla (L - \langle z, Lz \rangle z)|_{z=w}(v) \quad \text{where } v \in T_w S^{2n-1} \text{ with } \langle v, w \rangle = 0 \\
=&Lv - \langle v, Lw \rangle w - \langle Lw, v \rangle w - \langle w, Lw \rangle v \quad \text{recall } Lw = \lambda_w w \text{ with eigenvector } \lambda_w \\
=&Lv - \lambda_w v \\
=&(L - \lambda_w) v.
\end{aligned}$$

Let us **assume** that the critical points of f are isolated, and the eigenspace for each eigenvalue is 1-dimensional. Order them $w_1, \cdots, w_n$ with corresponding eigenvectors $\lambda_1 < \cdots < \lambda_n$. The index $i(w_i) = \dim K^- = \dim T_{w_i} U_{w_i} = 2(i-1)$ and the unstable manifold U_{w_i} is the subspace in $\mathbb{CP}^{n-1}$ generated by $[w_1], \cdots, [w_{i-1}]$.

1.2. **Manifold situation with S^1 action.** Let P be a compact manifold with Riemannian metric (or a tame manifold with bounded geometry like $\mathbb{C}^n$). S^1 acts on the Riemannian manifold P by isometries.

$Q := P^{S^1}$ = fixed point set of S^1, which we assume is a manifold. Let S^1 acts freely $P \backslash Q$, and assume (actually a consequence of next paragraph) Q is of even codimension in P.

Let $N := N_Q P \to Q$ be the normal bundle with S^1 action, which then gives a complex vector bundle structure. Let $\mu : \mathbb{R}/2\pi\mathbb{Z} \times P \to P$ denote the S^1 action, then $re^{i\theta} \cdot v := r\mu(\theta)v$.

We want to define P^σ the (real oriented) blowup along Q with the blow-down map $\pi : P^\sigma \to P$ as follows:

For ϵ small, the disk bundle $N^\epsilon \overset{\exp}{\hookrightarrow} P$ diffeomorphic onto the image. Away from the zero section, $N^\epsilon \backslash \{(q,0)\}_{q \in Q} = (0, \epsilon) \times S(N) \overset{\Theta}{\hookrightarrow} P \backslash Q$ with $\Theta(r, v) := \exp(rv)$.

We have $P^\sigma := ([0, \epsilon) \times S(N)) \cup_\Theta P \backslash Q$, and we have projection $\pi : P^\sigma \to P$ which comes from gluing Θ and Id in two open parts.

π is diffeomorphism over $P \backslash Q$, and over $q \in Q$, the fiber of π is $S(N_q)$.

Exercise 1.1. $h : \mathbb{R}^m \to \mathbb{R}^n$ a smooth embedding with $h(0) = 0$. Then there exists smooth $h^\sigma : (\mathbb{R}^m)^\sigma \to (\mathbb{R}^n)^\sigma$ such that
$$\begin{tikzcd} (\mathbb{R}^m)^\sigma \arrow[r, "h^\sigma"] \arrow[d, "\pi"] & (\mathbb{R}^n)^\sigma \arrow[d, "\pi"] \\ \mathbb{R}^m \arrow[r, "h"] & \mathbb{R}^n \end{tikzcd}$$
commutes.

Construct h^σ, and show that h^σ is a smooth embedding.

(Hint: on $(\mathbb{R}^m)^\sigma = [0,\infty) \times S^{m-1}$, $h(rv) = r\tilde{h}(r,v)$ for a (unique) smooth and everywhere nonzero $\tilde{h}$, define $h^\sigma : (r,v) \mapsto (r\|\tilde{h}(r,v)\|, \tilde{h}(r,v)/\|\tilde{h}(r,v)\|)$, and show $\|\tilde{h}\|$ is smooth.)

S^1 action on P lifts to an S^1 action on P^σ, which is free. $P^\sigma \backslash Q = P \backslash Q$. S^1 acts on $\partial P^\sigma = S(N)$ freely.

Define $B^\sigma = P^\sigma / S^1 \xrightarrow{\pi} B$. Over Q, we have $\partial B^\sigma \xrightarrow{\pi} Q$ with fiber over q being $S(N_q)/S^1 = \mathbb{P}(N_q)$.

1.3. **Morse function on the blow-up.** $\tilde{f} : P \to \mathbb{R}$ invariant under S^1 has gradient $\tilde{V} = \nabla \tilde{f}$. $\tilde{V}$ defined on $P\backslash Q = P^\sigma \backslash \partial P^\sigma$ extends to smooth $\tilde{V}^\sigma$ on P^σ and we have $\tilde{V}^\sigma|_{\partial P^\sigma} \subset T\partial P^\sigma$. (Use the above exercise applied to the flow of $\tilde{V}$). $\tilde{V}^\sigma$ not a gradient, but as $\nabla \tilde{V}^\sigma$ at the zeros of $\tilde{V}^\sigma$ being symmetric with real eigenvalues, we can still define $K^\pm$ as before.

Example: $\tilde{f}(p) = \frac{1}{2}\langle p, Lp\rangle$, $p \in P = \mathbb{C}^n$, and $P^\sigma = [0,\infty) \times S^{2n-1} \ni (s,\phi)$.

The negative "gradient" equation $\dot{\phi} = -L\phi + \Lambda(\phi)\phi$, $\dot{s} = -\Lambda(\phi)s$.

Earlier, we have looked at $\mathbb{CP}^{n-1}$, while here we have $\mathbb{CP}^{n-1} \times [0,\infty)$.

$\text{Hessian}_w = (L - \lambda_w, \lambda_w)$, where w is critical point/eigenvector of L. So if $\lambda_w < 0$, $\text{index} = i_{\mathbb{CP}^{n-1}}(w) + 1$.

1.4. **4d SW.** We have $\mathcal{C}^\sigma(X, \mathfrak{s}_X) := \mathcal{A}(X, \mathfrak{s}_X) \times \mathbb{R}^{\geq 0} \times \mathbb{S}(\Gamma(S^+))$ the blow-up of $\mathcal{C}(X,\mathfrak{s}_X) = \mathcal{A}(X,\mathfrak{s}_X) \times \Gamma(S^+)$ along reducible configurations $\{(A,0)\}$, where $\mathbb{S}(\Gamma(S^+)) := \{\|\cdot\|_{L^2} = 1\}$.

$\mathcal{C}^\sigma(X,\mathfrak{s}_X) \to \mathcal{C}(X,\mathfrak{s})$, $(A,s,\phi) \mapsto (A, s\phi)$. The fiber over $(A,0)$ is $\{(A,0,\phi)\} \cong \mathbb{S}(\Gamma(S^+))$. Seiberg-Witten map $\mathcal{F} : \mathcal{C}(X,\mathfrak{s}_X) \to \Gamma(i\mathfrak{su}(S^+) \oplus S^-) =: \mathcal{V}$ as a section of the trivial bundle $\underline{\mathcal{V}} := \mathcal{C}(X,\mathfrak{s}) \times \mathcal{V}$.

The blowup section $\mathcal{F}^\sigma : \mathcal{C}^\sigma(X,\mathfrak{s}) \to \pi^* \underline{\mathcal{V}}$ is defined as

$$\mathcal{F}^\sigma : (A,s,\phi) \mapsto (\frac{1}{2}\rho_X(F^+_{A^t}) - s^2(\phi\phi^*)_0, D^+_A \phi).$$

This is not a pullback of $\mathcal{F}$. (The pullback section is not Fredholm.)

If $s \neq 0$, $\mathcal{F}^\sigma(A,s,\phi) = 0$ iff $\mathcal{F}(A,s\phi) = 0$. If $s = 0$, $\mathcal{F}^\sigma(A,0,\phi) = 0$ iff $\mathcal{F}(A,0) = 0$ and $D^+_A\phi = 0$. $\mathcal{G}_X$ acts on $\mathcal{F}^\sigma$ equivariantly.

1.5. **The restriction map and the blow-up flow equation.** Let $X^1 \subset\subset X$ be an open domain. $r : \mathcal{C}^\sigma(X,\mathfrak{s}_X) \dashrightarrow \mathcal{C}^\sigma(X', \mathfrak{s}_X|_{X'})$. The domain of this map is $\mathrm{dom}(r) := \{\phi|_{X'} \not\equiv 0\}$. Let $\gamma^\sigma = (A,s,\phi) \in \mathrm{dom}(r)$,

$$(A,s,\phi) \mapsto (A, s\|\phi\|_{L^2(X')}, \frac{\phi}{\|\phi\|_{L^2(X')}}).$$

The unique continuation (whose detail is covered in the exercise session) ensures $(\mathcal{F}^\sigma)^{-1}(0) \subset \mathrm{dom}(r)$. (If a SW solution restricts to X' and falls out of $\mathrm{dom}(r)$, then it is identically 0 which contradicts to our starting point.)

Exercise 1.2. In a temporal gauge, a solution $\mathcal{F}^\sigma(\gamma^\sigma) = 0$ on $X = I \times Y$ can be written as

$$\begin{cases} \frac{1}{2}\frac{d}{dt}B^t & = \frac{1}{2} * F_{B^t} - r^2\rho^{-1}(\psi\psi^*)_0 \\ \frac{d}{dt} r & = -\Lambda(B,r,\psi_r) \\ \frac{d}{dt}\psi & = -(D_B\psi - \Lambda(B,r,\psi)\psi)), \end{cases}$$

where $\Lambda(B,r,\psi) := \langle \psi, D_B\psi\rangle_{L^2(Y)}$. Here, $D_B\psi$ plays the role of Lz in the toy example.