SEIBERG-WITTEN FLOER HOMOLOGY LECTURE 7

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1. Lecture 7

Please email `yangding@math.hu-berlin.de` if anything. Lecture notes up to now are available at `www.mathematik.hu-berlin.de/~yangding/monopole.html`. Exercises sprinkled throughout lecture notes have been collected into an exercise sheet at `www.mathematik.hu-berlin.de/~yangding/Exercise_SWF.pdf`.

We will do the following in this lecture:

- Put the configuration space (and its blowup along irreducibles) and the bundle over it (where the section of SW expression lives), and the gauge group acting in this setting into Banach space setting (so that the finite dimensional intuition largely carries over and to ultimately apply Sard-Smale theorem).
- Using a slice to locally parametrize a quotient manifold using a submanifold through a point.
- Morse theory (i) needs ∇f non-degenerate at critical points (can be viewed as requiring $y \mapsto \nabla_y f$ as a transverse section of $TB \to B$), and (ii) needs $M(a, b)$ to be a manifold of correct dimension, by requiring
$$\big(t \mapsto x(t)\big) \mapsto \big(t \mapsto \dot{x}(t) + \nabla_{x(t)} f\big)$$
to be a transverse section of $\bigcup_{\gamma \in \mathrm{Map}(\mathbb{R},B)} \Gamma(\gamma^* TB) \to \mathrm{Map}(\mathbb{R}, B)$ in a suitable function space setting built upon point (i) (this can be generalized more readily instead of the formulation of stable and unstable submanifolds intersecting transversely). Analogously in the SW infinite dimensional picture, need (i) $\nabla\mathcal{L}$ non-degenerate, and (ii) need 4d SW solution space on a cylinder up to gauge transformation (flow lines of $-\nabla\mathcal{L}$ on 3d up to 3d gauge transformation) to a manifold with expected dimension. Each will be dealt with as a perturbation of a Fredholm defining section into a transverse section.

  We achieve by perturbing $\nabla\mathcal{L}$ into $\nabla\not{\mathcal{L}}$ by adding the gradient of a "generic" cylinder function, while keeping compactness.

1.1. **Functional space setting.**

Use M to denote either X (possibly with ∂X) in $4d$, or Y in $3d$ setting, compact (or bounded geometry such as $\mathbb{R} \times Y$) Riemannian with a spin^c structure $\mathfrak{s}$. Use W to mean the (positive) spin bundle S^+ for X, or S for Y.

The configure space $\mathcal{C}(M, \mathfrak{s}) = (A_0, 0) + \Gamma(iT^*M \oplus W) = \big(A_0 + \Gamma(iT^*M)\big) \times \Gamma(W)$, for example $\Gamma(iT^*X \oplus S^+)$ in $4d$, and $\Gamma(iT^*Y \oplus S)$ in $3d$.

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$\mathcal{C}_k(M,\mathfrak{s}) = \big(A_0 + L^2_k(iT^*M)\big) \times L^2_k(W)$. Banach/Hilbert manifold, a manifold based on local models of open sets in Banach/Hilbert space (which is a function space with finite regularity).

The blow-up with regularity

$$\mathcal{C}^\sigma_k(M,\mathfrak{s}) = \{(A,s,\phi) \in \big(A_0 + L^2_k(iT^*M)\big) \times \mathbb{R} \times L^2_k(W) \mid s \geq 0, \|\phi\|_{L^2} = 1\}.$$

The gauge group $\mathcal{G}_{k+1}(M) = \{u \in L^2_{k+1}(M;\mathbb{C}) \mid |u(p)| = 1\}$, here we ask $2(k+1) > \mathrm{dim}M$, so u is continuous and the condition makes sense by Sobolev embedding $L^p_k(M) \hookrightarrow C^r$ if $k - \frac{n}{p} > r$ (thus the function space without the condition is a Banach algebra).

$\mathcal{C}^\sigma_k(M,\mathfrak{s})$ Hilbert manifold with boundary with a Hilbert Lie group $\mathcal{G}_{k+1}$ acting smoothly and freely (always having $2(k+1) > \mathrm{dim}M$ in place).

The tangent space at $\gamma = (A_0, s_0, \phi_0)$ to $\mathcal{C}^\sigma_k := \mathcal{C}^\sigma_k(M,\mathfrak{s})$ is

$$\mathcal{T}_k|_\gamma := T_\gamma\mathcal{C}^\sigma_k := \{(a,s,\phi) \in L^2_k(iT^*M) \times \mathbb{R} \times L^2_k(W) \mid \mathrm{Re}\langle\phi_0,\phi\rangle_{L^2} = 0\}.$$

They fit into the tangent bundle $\mathcal{T}_k$, and one can also complete the fibers using weaker L^2_l norms and denote it as $\mathcal{T}_l$ for $l \leq k$, replacing k by l in the definition.

1.2. **Quotient.** We can form quotient $B_k(M,\mathfrak{s}) = \mathcal{C}_k(M,\mathfrak{s})/\mathcal{G}_{k+1}$, and the blow-up version $B^\sigma_k := \mathcal{C}^\sigma_k/\mathcal{G}_{k+1}$, where drop the dependence of manifold and spin^c structure for brevity. B^σ_k is a Hilbert manifold with boundary being a quotient of a free and smooth group action with closed orbit (the image of $\mathbb{d}_\gamma$ below is closed), and Hausdorff.

The group action $\mathrm{Act} : \mathcal{G}_{k+1} \times C^\sigma_k \to C^\sigma_k, (g,\gamma) \mapsto g\gamma$. The differential of this at the identity $g = e$ and a general configuration γ, is denoted by

$$\mathbb{d}_\gamma := d_{(e,\gamma)}\mathrm{Act} : T_e\mathcal{G}_{k+1} \to T_\gamma\mathcal{C}^\sigma_k.$$

We locally parametrize the quotient structure using a slice: if we choose any $\mathcal{S} \underset{\text{submanifold}}{\overset{\text{locally closed}}{\subset}} \mathcal{C}$ containing a given $\gamma \in \mathcal{G}\gamma$ such that $T_\gamma\mathcal{C} = \mathrm{im}\mathbb{d}_\gamma \oplus T_\gamma S$, then

$$\bar{\iota} : \mathcal{S} \to \mathcal{C}/\mathcal{G}$$

obtained as the composition $\mathcal{S} \overset{\text{inclusion } \iota}{\subset} \mathcal{C} \overset{\text{quotient}}{\subset} \mathcal{C}/\mathcal{G}$ is a diffeomorphism from an open neighborhood of γ onto the image, which is an open neighborhood of $\mathcal{G}\gamma$ in $\mathcal{C}/\mathcal{G}$ (by inverse function theorem).

1.3. **Construct a slice for the blow-up configuration space.** First consider the irreducible configuration space, then extend it to the blow-up.

$\gamma := (A_0, \Phi_0) \in \mathcal{C}^*_k(M,\mathfrak{s}) \subset \mathcal{C}_k(M,\mathfrak{s})$ with $\Phi_0 \neq 0$.

$\mathbb{d}_\gamma : L^2_{j+1}(i\mathbb{R}) \to \mathcal{T}_j|_\gamma, \xi \mapsto (-d\xi, \xi\Phi_0)$, for $j \leq k$.

Let $\mathcal{J}_j|_\gamma := \mathbb{d}_\gamma(L^2_{j+1}(i\mathbb{R}))$.

$\mathcal{K}_j|_\gamma$ denotes its L^2 orthogonal in $\mathcal{T}_j|_\gamma$, explicitly

$$\{(a,\phi) \mid -d^*a + i\mathrm{Re}\langle i\Phi_0,\phi\rangle = 0,\ \langle a|_{\partial M}, n\rangle = 0\},$$

where n is the outwards normal to ∂M. The first condition is $\mathbb{d}^*_\gamma(a,\phi) = 0$.

$\mathcal{J}_j = \bigcup_{\gamma\in\mathcal{C}^*} \mathcal{J}_j|_\gamma$ and similarly $\mathcal{K}_j$ are closed subbundles of $\mathcal{T}_j|_{\mathcal{C}^*}$ and they are orthogonal and complementary.

$\mathcal{J}_j$ extends (to the boundary of $\mathcal{C}^\sigma_k$) to $\mathcal{J}^\sigma_j$ over $\mathcal{C}^\sigma$ naturally.

For $\gamma = (A_0, s_0, \phi_0) \in \mathcal{C}^\sigma_k$, we define

$$\mathcal{K}^\sigma_j|_\gamma := \{(a,s,\phi) \mid -d^*a + is_0^2\mathrm{Re}\langle i\phi_0,\phi\rangle = 0, \langle a|_{\partial M}, n\rangle = 0, \mathrm{Re}\langle i\phi_0,\phi\rangle_{L^2} = 0\}.$$

They fit together $\mathcal{K}_j^\sigma$ subbundle of $\mathcal{T}_j^\sigma$ that is complementary to $\mathcal{J}_j^\sigma$ in $\mathcal{T}_j^\sigma$.

Want to find a closed submanifold $S_{k,\gamma}^\sigma \subset \mathcal{C}_k^\sigma$ based at $\gamma = (A_0, s_0, \phi_0)$ s.t. $T_\gamma S_{k,\gamma}^\sigma = \mathcal{K}_j^\sigma|_\gamma$. We define:

$$S_{k,\gamma}^\sigma := \{(A_0+a, s, \phi) \mid -d^*a + iss_0 \mathrm{Re}\langle i\phi_0, \phi\rangle = 0, \langle a|_{\partial M}, n\rangle = 0, \mathrm{Re}\langle i\phi_0, \phi\rangle_{L^2(M)} = 0\}.$$

$S_{k,\gamma}^\sigma$ has a well-defined limit as s_0 in $\gamma = (A_0, s_0, \phi_0)$ goes to 0, which defines $S_{k,\gamma}^\sigma$ for γ on the boundary.

Note that the construction of $\mathcal{S}_{k,\gamma}^\sigma$ is motivated from the proper transform of the slice $\mathcal{S}_{k,\gamma} := \{(A, \Phi) \mid -d^*a + i\mathrm{Re}\langle i\Phi_0, \Phi\rangle = 0, \langle a_{\partial M}, n\rangle = 0\}$ through any point $\gamma = (A_0, \Phi_0) \in \mathcal{C}_k$.

$\bar{\iota} :=$ quotient $\circ\, \iota : S_{k,\gamma}^\sigma \to B_k^\sigma$ with ι denoting the inclusion, called Coulomb-Neumann chart/slice (because as we shall see that to bring a general configuration into this slice based at a reducible configuration $(A_0, 0)$, one solves the same equation as in the Coulomb-Neumann gauge fixing previously in the argument of compactness).

1.4. SW map as a section from the blow-up space with finite regularity.

We have trivial bundle

$$\mathcal{V}_{k-1} := \mathcal{C}_k \times L_{k-1}^2(i\mathfrak{su}(S^+) \oplus S^-) \to \mathcal{C}_k$$

and the blow-down map $\mathcal{C}_k^\sigma \xrightarrow{\pi} \mathcal{C}_k$. Define $\mathcal{V}_{k-1}^\sigma := \pi^*\mathcal{V}_{k-1}$ for the compact manifold.

$F^\sigma(A, s, \phi) = (\frac{1}{2}\rho_X(F_{A^t}^+) - s^2(\phi\phi^*)_0, D_A^+\phi)$ is a section of $\mathcal{V}_{k-1}^\sigma \to \mathcal{C}_k^\sigma$ with $\mathcal{G}_{k+1}$ acting equivariantly and smoothly.

In $3d$, $\nabla\mathcal{L}$ smooth section $\mathcal{T}_{k-1} \to \mathcal{C}_k$, and $(\nabla\mathcal{L})^\sigma$ smooth section for $\mathcal{T}_{k-1}^\sigma \to \mathcal{C}_k^\sigma$.

1.5. As a preliminary, global slice: bring a general point in $\mathcal{C}_k$ to the slice $\mathcal{S}_{k,\gamma_0}$ at a reducible $\gamma_0 = (A_0, 0)$.

Recall that the defining condition for the slice $\mathcal{S}_{k,\gamma_0}$ is

$$\{-d^*a = 0, \langle a|_{\partial M}, n\rangle = 0\}.$$

To find a gauge transformation of the form $u = e^\xi$ to put $(A, \phi) = (A_0 + a, \phi)$ into the slice, one needs to solve (substituting into the above defining condition) $\Delta\xi = d^*a$, $\langle d\xi|_{\partial M}, n\rangle = \langle a|_{\partial M}, n\rangle$. ξ is unique if asking $\int_M \xi = 0$.

Define $\mathcal{G}_{k+1}^\perp = \{e^\xi \mid \int_M \xi = 0\}$.

We have a diffeomorphism

$$\mathcal{G}_{k+1}^\perp \times \mathcal{S}_{k,\gamma_0} \to \mathcal{C}_k, \quad (e^\xi, (a, \phi)) \mapsto (A_0 + (a - d\xi), e^\xi\phi),$$

restricted from the group action map Act.

B_k is quotient of $\mathcal{C}_k/\mathcal{G}_{k+1} := \iota(\mathcal{S}_{k,\gamma_0})/(\mathcal{G}_{k+1}/\mathcal{G}_{k+1}^\perp)$. Here $\mathcal{G}^h := \mathcal{G}_{k+1}/\mathcal{G}_{k+1}^\perp$ can be realized as the extension $S^1 \to \mathcal{G}^h \to H^1(M; \mathbb{Z})$, where the second map is taking the associated homotopy class, and $H^1(M;\mathbb{Z})$ is the components of gauge group. (The notation for $\mathcal{G}^h$ comes from an alternative realization as harmonic maps $u : M \to S^1$ with Neumann boundary condition $\Delta u = 0, \langle\nabla u, n\rangle = 0$.)

Exercise 1.1. From the above, show we have homotopy equivalences

$$B_k^\sigma \cong \iota(\mathcal{S}_{k,\gamma_{\gamma_0}} \cap \mathcal{C}_k^*)/\mathcal{G}^h \times (L_k^2(S)\backslash\{0\})/S^1 \cong H^1(M; i\mathbb{R})/2\pi H^1(M; i\mathbb{Z}) \times \mathbb{CP}^\infty.$$

1.6. **Tame perturbation.** In $3d$, we will take perturbation of the following form $f : \mathcal{C}(Y) \to \mathbb{R}$ invariant under $\mathcal{G}$. $\cancel{\mathcal{L}}:=\mathcal{L}+f$ perturbed CSD functional. The perturbation to the equation is $\mathfrak{q}=\nabla f$.

When $\mathfrak{q}$ having less regularity, we call it a **formal gradient** of some f if for all smooth $\gamma : [0,1] \to \mathcal{C}(Y)$, we have $(f\circ\gamma)(1)-(f\circ\gamma)(0)=\int_0^1\langle\dot\gamma,\mathfrak{q}\rangle_{L^2}dt$.

$\mathfrak{q}=(\mathfrak{q}^0,\mathfrak{q}^1)\in L^2(iT^*Y)\oplus L^2(S)$. $\nabla\cancel{\mathcal{L}}=\nabla\mathcal{L}+\mathfrak{q}$.

Lifted to the blow-up, $(\nabla\cancel{\mathcal{L}})^\sigma=(\nabla\mathcal{L})^\sigma+\mathfrak{q}^\sigma$. We will write out the LHS (which also then defines $\mathfrak{q}^\sigma$ in terms of $\mathfrak{q}$):

$$(\nabla\cancel{\mathcal{L}})^\sigma := \begin{pmatrix} \frac{1}{2} * F_{B^t} + r^2\rho^{-1}(\psi\psi^*)_0 + \mathfrak{q}^0(B,r,\psi) \\ \Lambda_{\mathfrak{q}}(B,r,\psi)r \\ D_B\psi + \tilde{\mathfrak{q}}^1(B,r,\psi) - \Lambda_{\mathfrak{q}}(B,r,\psi)r \end{pmatrix},$$

where $\tilde{\mathfrak{q}}^1(B,r,\psi)=\int_0^1\left(d_{(B,sr\psi)}\mathfrak{q}^1\right)(0,\psi)ds$, and

$$\Lambda_{\mathfrak{q}}(B,r,\psi)=\operatorname{Re}\langle\psi, D_B\psi+\tilde{\mathfrak{q}}^1(B,r,\psi)\rangle_{L^2}.$$

Exercise 1.2. Using the expression of $\nabla\mathcal{L}$ and the above to write down $\mathfrak{q}^\sigma$ in terms of $\mathfrak{q}=(\mathfrak{q}^0,\mathfrak{q}^1)$ explicitly.

(B,r,ψ) is a crtical point of $(\nabla\cancel{\mathcal{L}})^\sigma$ iff:
when $r\neq 0$, $(B,r\psi)$ is a critical point of $\nabla\cancel{\mathcal{L}}$; when $r=0$, $(B,0)$ is a critical point of $\nabla\cancel{\mathcal{L}}$ and ψ is an eignevector of $\phi\mapsto D_B\phi+\left(d_{(B,0)}\mathfrak{q}^1\right)(0,\phi)$.

We need tame perturbation (the following definition is a bit technical and one can proceed to cylinder functions right after it, and we hope to add a sketch of proof for the last assertion of this lecture which might make the definition more palatable):

Definition 1.3. (τ-model) Let $Z=[t_1,t_2]\times Y$. As the $4d$ blow-up model (with regularity) viewed as a path in $3d$ blow-up model, one needs to divide a factor to rescale the last variable so that it satisfies $\|\ \|_{L^2(Y)}=1$ (we need to invoke unique continuation theorem which implies that we never divide by 0), the rescaling factor needed to multiplied to the second variable, which now is a non-negative function $s(t)$ on $t\in[t_1,t_2]$. This is the τ-model and we use superscript τ to signify it, and we ask

$$s\in L^2_k([t_1,t_2])\cap\{s(t)\geq 0\}.$$

$\mathcal{C}^\tau_k$ is not a Hilbert manifold with boundary but a closed submanifold of an obvious Hilbert manifold (where the middle factor is $L^2_k([t_1,t_2])$).

Exercise 1.4. Work out the tangent bundle, and slice in τ-model.

Definition 1.5. (tame) Let $Z=[t_1,t_2]\times Y$. We want to use gauge invariant norm $L^2_{k,A}$, then we no longer have a trivial bundle $\mathcal{V}_k$ as a normed vector bundle. $L^2_{-k}:=(L^2_k)^*$.

Given $\gamma\in\mathcal{C}^\tau(Z)$, we view it as a path $\check\gamma(t)$ in $\mathcal{C}^\sigma_k(Y)$, then we have $\mathfrak{q}^\sigma(\check\gamma(t))\in L^2_k(Y;iT^*Y)\oplus\mathbb{R}\times L^2_k(Y;S)$, using Clifford multiplication to identify iT^*Y with $i\mathfrak{su}(S^+)$ and S with S^-. Then this becomes an element in $\mathcal{V}^\tau_k$, denoted by $\hat{\mathfrak{q}}^\tau(\gamma)$. One also has the non-blow-up version $\hat{\mathfrak{q}}$ of the above.

For an integer $k\geq 2$, a perturbation $\mathfrak{q}:\mathcal{C}(Y)\to\mathcal{T}_0$ is k-tame if it is the formal gradient of continuous $\mathcal{G}_Y$-invariant function on $\mathcal{C}(Y)$, such that

- $\gamma\mapsto\hat{\mathfrak{q}}(\gamma)$ is a smooth section of $\mathcal{V}_k(Z)\to\mathcal{C}_k(Z)$,
- for every integer $j\in[1,k]$, $\hat{\mathfrak{q}}$ is a continuous section of $\mathcal{V}_j(Z)\to\mathcal{C}_j(Z)$,

- for every integer $j \in [-k, k]$, $d\hat{\mathfrak{q}}$ which is a smooth section of
$$\mathrm{Hom}(T\mathcal{C}_k(Z), \mathcal{V}_k(Z)) \to \mathcal{C}_k(Z),$$
extends to a smooth section $D\mathfrak{q}$ of $\mathrm{Hom}(T\mathcal{C}_j(Z), \mathcal{V}_j(Z)) \to \mathcal{C}_k(Z)$,
- there exists constant m_2, $\|\mathfrak{q}(B,\psi)\|_{L^2} \le m_2(\|\psi\|_{L^2}+1)$ for $(B,\Psi) \in \mathcal{C}_k(Y)$,
- for any smooth connection A_0, there exists a real function μ_1 such that $\|\hat{\mathfrak{q}}(A,\Phi)\|_{L^2_{1,A}} \le \mu_1(\|A - A_0, \Phi\|_{L^1_{1,A_0}})$,
- $\mathfrak{q} : \mathcal{C}_1(Y) \to \mathcal{T}_0$ in Y is C^1.

If $\mathfrak{q}$ is tame for all $k \ge 2$, then $\mathfrak{q}$ is called **tame**.

This technical definition will become handy in the proof of achieving transversality while keeping compactness. One cannot help but notice some similarity with an ingredient in the definition of sc-smoothness in polyfold theory.

Readers can skip the above definition on the first reading and just proceed to the following construction of a class of functions called cylinder functions that will give tame perturbations.

1.7. **Cylinder functions.** We break up the construction into several bite-size pieces:

1.7.1. *Choose a simpler quotient representative.* We have $\mathcal{G}_{k+1} = \mathcal{G}^{\perp}_{k+1} \times \mathcal{G}^h$ from the above.

In $3d$ setting, at $\gamma_0 = (B_0, 0)$, we have difeomorphism $\mathcal{G}^{\perp}_{k+1} \times \mathcal{S}_{k,\gamma_0} \to \mathcal{C}_k(Y)$ which is the group action, where $\mathcal{S}_{k,\gamma_0} = \{d^*b = 0\}$ as above.

We want $\mathcal{G}_{k+1}$-invariant function. Due to the above splitting and identification, only need to construct functions on $\mathcal{S}_{k,\gamma_0}$ that are invariant under $\mathcal{G}^h$.

1.7.2. *Torus-valued function on configuration pairing with 1-forms.* Given a coclosed 1-form $c \in \Gamma(iT^*Y) = \Omega^1(i\mathbb{R})$ (namely, $d^*c = 0$), define
$$r_c : \mathcal{C}(Y) \to \mathbb{R}, \ \ (B_0 + b, \psi) \mapsto \int_Y b \wedge *\bar{c} = \langle b, c\rangle_Y.$$

$u \in \mathcal{G}_{k+1}$, $-u^{-1}du$ represents $h \in 2\pi i H^1(Y;\mathbb{Z})$, $r_c \circ u - r_c = (h \cup [*\bar{c}])[Y]$. If $c = d^*c'$ coexact, r_c is invariant under $\mathcal{G}_{k+1}$.

Denote $\mathbb{T} := H^1(Y; i\mathbb{R})/2\pi i H^1(Y;\mathbb{Z})$, we can choose i-valued harmonic 1-forms modulo those with $2\pi i$ periods as preferred representatives. Choose integral basis $w_1, \cdots, w_t \in H^1(Y, i\mathbb{R})$, so $\mathbb{T} \cong \mathbb{R}^t/2\pi\mathbb{Z}^t$.

$\mathcal{C}(Y) \to \mathbb{T}$, $(B_0 + b, \psi) \mapsto [b_{\mathrm{harm}}]$.

We have $(B,\psi) \mapsto (r_{w_1}(B,\psi), \cdots, r_{w_t}(B,\psi)) \bmod 2\pi\mathbb{Z}^t$.

1.7.3. *$\mathbb{C}$-valued equivariant function from pairing of the spinor direction.* We choose a splitting $S^1 \to \mathcal{G}^h \underset{v}{\rightleftarrows} H^1(Y;\mathbb{Z})$, e.g. using harmonic gauge transformations such that $u(x_0) = 1$ for some base point x_0.

Define $\mathcal{G}^0_{k+1}(Y) = \mathcal{G}^{h,0}(Y) \times \mathcal{G}^{\perp}_{k+1}(Y)$, where $\mathcal{G}^{h,0}(Y) = \mathrm{im}(v)$.

$\mathcal{G}^0_{k+1}(Y)$ acts freely on $\mathcal{C}_k(Y)$ with quotient B^0_k. The S^1 action on B^0_k induced by the $\mathcal{G}_{k+1} = S^1 \times \mathcal{G}^0_{k+1}$ action on $\mathcal{C}_k$ is $e^{i\theta} \in S^1$ acting as $(B,\psi) \mapsto (B, e^{i\theta}\psi)$. We have $B_k := \mathcal{C}_k/\mathcal{G}_{k+1} = B^0_k/S^1$.

$H^1(Y; i\mathbb{R}) \times S$ (with S spin bundle) acted by $\mathcal{G}^{h,0} = \mathrm{im}(v)$, with the quotient $\mathbb{T} \times S =: \mathbb{S}$ which fibers over $\mathbb{T} \times Y$.

For a smooth section Υ of $\mathbb{S} \to \mathbb{T} \times Y$, we can always choose a lift $\tilde{\Upsilon}$ which is a section of $H^1(Y; i\mathbb{R}) \times S \to H^1(Y; i\mathbb{R}) \times Y$. Denoting $\tilde{\Upsilon}_b(y) := \tilde{\Upsilon}(b, y)$, we have $\tilde{\Upsilon}_{\alpha+\kappa}(y) = \big(v(\kappa)\big)(y)\tilde{\Upsilon}_\alpha(y)$, where v is the splitting above.

Recall quickly on Hodge theory. In particular, $\Delta = d^*d$ has Green function $G : L^2_{k-1}(Y) \to L^2_{k+1}(Y)$, $\Delta \circ G = \mathrm{Id}$. Given Υ of $\mathbb{S}$, we define a $\mathcal{G}^0$-equivariant map $\Upsilon^\dagger : \mathcal{C}(Y) \to \Gamma(S), (B_0 + b, \psi) \mapsto e^{-Gd^*b}\tilde{\Upsilon}_{b_{\text{harm}}}$.

Define $q_\Upsilon : (B, \psi) \mapsto \int_Y \langle \psi, \Upsilon^\dagger(B, \psi)\rangle = \langle \psi, \tilde{\Upsilon}^\dagger\rangle_Y$. q_Υ is $\mathcal{G}^0$-invariant, and also S^1 equivariant.

1.7.4. *Invariant function from finitely many directions picked out.* Choose coclosed $c_1, \cdots, c_{n+t}$, where the first n coexact, and last t being basis w_i above. Choose m sections $\Upsilon_1, \cdots, \Upsilon_m$ of $\mathbb{S}$.

Define function $p : \mathcal{C}(Y) \to \mathbb{R}^n \times \mathbb{T} \times \mathbb{C}^m$,

$$(B, \psi) \mapsto (r_{c_1}(B, \psi), \cdots, r_{c_{n+t}}(B, \psi), q_{\Upsilon_1}(B, \psi), \cdots, q_{\Upsilon_m}(B, \psi)).$$

We have an S^1-action on $\mathbb{R} \times \mathbb{T} \times \mathbb{C}^m$ acting on the last factor $\mathbb{C}^m$. Choose an S^1-invariant compactly supported $g : \mathbb{R}^n \times \mathbb{T} \times \mathbb{C}^m \to \mathbb{R}$.

1.7.5. *Cylinder function and being large enough.* Define $f := g \circ p$ and $\mathfrak{q} := \nabla f$. A function of this form is called a **cylinder function**.

Exercise 1.6. Show that space of cylinder functions is 'large enough': for any

$$[B, \psi] \in B^*_k(Y) := \mathcal{C}^*_k / \mathcal{G}_{k+1},$$

any non-zero tangent vector $v \in T_{[B,\psi]}B^*_k(Y)$. There exists a cylinder function $f : \mathcal{C}_k(Y) \to \mathbb{R}$ with the quotiented restriction $\bar{f} := (f|_{\mathcal{C}^*_k})/\mathcal{G}_{k+1} : B^*_k(Y) \to \mathbb{R}$ such that $(d_{[B,\Psi]}\bar{f})(v) \neq 0$.

Make a countable collection of such choices of cylinder functions. The space $\mathcal{P}$ of countable sequence of constant weights used for linear combination of cylinder functions is the Banach space of perturbations.

1.8. **Transversality in $3d$.** Let $\mathcal{P}$ be the Banach space of weights for linear combinations for cylinder functions.

There exists a residual (complement of countable intersection of open dense sets, especially non-empty) subset $\mathcal{P}_{\text{res}}$ of $\mathcal{P}$, such that for any $\mathfrak{q} \in \mathcal{P}_{\text{res}}$, zeros of $(\nabla \pounds)^\sigma = (\nabla \mathcal{L})^\sigma + \mathfrak{q}^\sigma$, which is a section of $\mathcal{T}^\sigma_{k-1} \to \mathcal{C}^\sigma_k(Y)$, is non-degenerate (This means the following: We have for $j \leq k$, $\mathcal{T}^\sigma_j \cong \mathcal{J}^\sigma_j \oplus \mathcal{K}^\sigma_j$ as before, $(\nabla \pounds)^\sigma$ is transverse to the subbundle $\mathcal{J}^\sigma_{k-1}$, recalling it being invariant under group action).

This is proved using Sard-Smale theorem.

A sketch of the proof will be added shortly.