SEIBERG-WITTEN FLOER HOMOLOGY LECTURE 8

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Please email `yangding@math.hu-berlin.de` if anything. Lecture notes up to now are available at `www.mathematik.hu-berlin.de/∼yangding/monopole.html`. Exercises sprinkled throughout lecture notes have been collected into an exercise sheet at `www.mathematik.hu-berlin.de/∼yangding/Exercise_SWF.pdf`.

1. Lecture 8: Transversality in 3d making all zeros of perturbed SW map non-degenerate

1.1. **Perturbation of SW map / the gradient of CSD.**

No exercise session (so that the final lecture will have one).

$\mathcal{L}$ CSD functional. SW map $\nabla\mathcal{L}$.

Perturb it by $q = \nabla f = (q^0, q^1) \in L^2(iT^*Y) \oplus L^2(S)$.

$\tilde{q}^1(B, r, \psi) = \int_0^1 (d_{B,sr\psi} q^1)(0, \psi) ds$ for $(B, r, \psi) \in \mathcal{C}^\sigma$.

$\Lambda_q(B, r, \psi) = \mathrm{Re}\langle \psi, D_B\psi + \tilde{q}^1(B, r, \psi)\rangle_{L^2}$.

Let $\not{\mathcal{L}} := \mathcal{L} + f$, thus $\nabla \not{\mathcal{L}} = \nabla\mathcal{L} + q = \begin{pmatrix} \frac{1}{2} * F_{B^t} + r^2\rho^{-1}(\psi\psi^*)_0 + q^0(B, r, \psi) \\ \Lambda_q(B, r, \psi) r \\ D_B\psi + \tilde{q}^1(B, r, \psi) - \Lambda_q(B, r, \psi) r \end{pmatrix}$.

1.2. **Splitting the tangent bundle complementary to the group action.**

Last time $\mathcal{C}^\sigma_k$ has tangent bundle $T^\sigma_k = \mathcal{J}^\sigma_k \oplus \mathcal{K}^\sigma_k$, where the first factor is tangent to the gauge group orbit. We complete this in lower regularity to have $T^\sigma_j = \mathcal{J}^\sigma_j \oplus \mathcal{K}^\sigma_j$ for $j \leq k$ and most relevant case is $j = k - 1$.

A zero (perturbed SW solution) $\mathfrak{a} \in \mathcal{C}^\sigma_k(Y)$ of $(\nabla\not{\mathcal{L}})^\sigma$ is non-degenerate if $(\nabla\not{\mathcal{L}})^\sigma \in \Gamma(T^\sigma_{k-1})$ is transverse to $\mathcal{J}^\sigma_{k-1}$ (natural from quotient space viewpoint).

Want to show that: for $\mathcal{P}$ Banach space of tame perturbation, there exists a $\mathcal{P}^{\mathrm{res}} \subset \mathcal{P}$ (complemenet of countable intersection of open dense subsets, in particular, non-empty), such that $q \in \mathcal{P}^{\mathrm{res}}$, we have $(\nabla\not{\mathcal{L}})^\sigma = (\nabla\mathcal{L})^\sigma + q^\sigma$ has only non-degenerate zeros.

1.3. **Characterization of non-degeneracy.**

We want to abstractizing the Hessian $\nabla\big((\nabla\not{\mathcal{L}})^\sigma\big)$:

Definition 1.1 (k-ASAFOE)**.** An operator L acting on sections of a vector bundle $E \to Y$ is called k-almost self-adjoint first order elliptic (k-ASAFOE) if $L = L_0 + h$, where

- L_0 is SAFOE (self-adjoint first order elliptic) operator with smooth coefficients.
- $h : C^\infty(E) \to L^2(E)$ an operator extends to a bounded operator $L^2_j(E) \to L^2_j(E)$, for all $|j| \leq k$ (here, $L^2_{-m} = (L^2_m)^*$ with respect to L^2 inner product), but h is not necessarily self-adjoint.

Date: January 8, 2021.

It is called ASAFOE, if k-ASAFOE for all k.

Properties:

- L is k-ASAFOE, then regularizing: $u \in L^2_{-k}$, $Lu = v \in L^2_j$ with $|j| \leq k$, then $u \in L^2_{j+1}$.
- L is k-ASAFOE, then $L : L^2_j(E) \to L^2_{j-1}(E)$ is Fredholm of index 0 (due to self-adjoint) for $-k \leq j \leq k$.
- The previous item implies that $L : L^2_j \to L^2_{j-1}$ invertible iff injective. Moreover, invertible for one j implies invertible for all $|j| \leq k$.
- So λ is an eigenvalue iff $(L - \lambda) : L^2_j \to L^2_{j-1}$ not invertible (independent of j).
- $L : L^2_j \to L^2_{j-1}$ with $L = L_0 + h$ k-ASAFOE.

$$\text{Then: } \begin{cases} \text{If } h \text{ symmetric, then eigenvalues are real,} \\ \quad \text{there exists complete orthonormal eigenvectors in } L^2_{k+1}\text{, dense in } L^2. \\ \text{If } h \text{ non-symmetric, then imaginary parts of eigenvalues of} \\ \quad \text{complexification of } L \otimes 1_{\mathcal{C}} \text{ bounded by } L^2\text{-operator norm of } \frac{h-h^*}{2}. \end{cases}$$

Remark 1.2. *h symmetric, eigenvalues are unbounded in both directions.*

Denote the tangent bundle for $\mathcal{A}_k$ acted by $\mathcal{G}_{k+1}$ by $T^{\text{red}}_j = \mathcal{J}^{\text{red}}_j \oplus \mathcal{K}^{\text{red}}_j$ for $j \leq k$. Fibers are exact and coclosed 1-forms in $i\mathbb{R}$.

$\text{pr}_{T^{\text{red}}_{k-1}} \circ (\nabla \pounds|_{\mathcal{A}_k \times \{0\}})$ defines $(\nabla \pounds)^{\text{red}} : \mathcal{A}_k \to T^{\text{red}}_{k-1}$.

For $B \in \mathcal{A}_k$, $D_{q,B} : L^2_k(S) \to L^2_{k-1}(S)$, $\phi \mapsto D_B\phi + (d_{(B,0)}q^1)(0,\phi)$ is k-ASAFOE, S^1-equivariant, then it is complex linear operator.

Definition 1.3 (characterization)**.** A zero $\mathfrak{a} = (B, r, \psi) \in \mathcal{C}^\sigma_k$ is a non-degenerate

$$\text{zero of } (\nabla \pounds)^\sigma \text{ iff } \begin{cases} r \neq 0, (B, r\psi) \text{ is non-degenerate zero of } \nabla \pounds, \\ r = 0, \psi \text{ eigenvector of } D_{q,B} \text{ with simple eigenvalue } \lambda \neq 0 \\ \quad \text{(for } D_{q,B} \text{ as complex operator), B non-degenerate zero of } (\nabla \pounds)^{\text{red}}. \end{cases}$$

Remark 1.4. First, recall that $\text{Act}^\sigma : \mathcal{G}_{k+1} \times \mathcal{C}^\sigma_k \to \mathcal{C}^\sigma_k$, $\mathbb{d}^\sigma_{\mathfrak{a}} := d_{(\text{id},\mathfrak{a})}\text{Act}^\sigma : T_{\text{id}}\mathcal{G}_{k+1} \to T_{\mathfrak{a}}\mathcal{C}^\sigma_k$.

$(\nabla \pounds)^\sigma$ non-degenerate at a reducible $\mathfrak{a} = (B, 0, \psi)$ is equivalent to surjectivity of

$$\mathbb{d}^\sigma_{\mathfrak{a}} \oplus d_{(B,0,\psi)}(\nabla \pounds)^\sigma = \begin{pmatrix} -d & d_B(\nabla \pounds)^{\text{red}} & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ \psi & 0 & 0 & D_{q,B} - \lambda, \end{pmatrix}$$

where the last 3×3 matrix is $d_{\mathfrak{a}}(\nabla \pounds)^\sigma$.

1.4. h **non-symmetric in our cases, eigenvalues are real.** On irreducible $\mathcal{C}^*_k(Y)$, $T_j|_{\mathcal{C}^*_k(Y)} = \mathcal{J}_i \oplus \mathcal{K}_j$. The slice $\mathcal{S}_{k,\alpha} = \alpha + \mathcal{K}_{k,\alpha}$ through $\alpha = (B_0, \Psi_0)$, where $\{(b, \phi) \mid -d^*b + i\text{Re}\langle i\Psi_0, \phi\rangle = 0\}$.

$\text{Hess}_{q,\alpha} = \text{pr}_{\mathcal{K}_{k-1,\alpha}} \circ d_\alpha(\nabla \pounds)|_{\mathcal{K}_{k,\alpha}}$ is $\mathcal{G}_{k+1}$-equivariant.

Being symmetric implies that there exists a complete orthonormal basis $\{e_n\}$ in $\mathcal{K}_{0,\alpha}$ which are smooth with real eigenvalues λ_n. The span $\{e_n\}$ is dense in all $\mathcal{K}_{j,\alpha}$. The operator is Fredholm with index 0.

To show this, consider the extended Hessian

$$\begin{aligned}
&\left(\hat{\mathrm{Hess}}_{q,\alpha} : T_{k,\alpha} \oplus L^2_k(i\mathbb{R}) \to T_{k-1,\alpha} \oplus L^2_{k-1}(i\mathbb{R})\right) \\
=&\begin{pmatrix} d_\alpha \nabla \pounds & \mathbb{b}_\alpha \\ \mathbb{d}^*_\alpha & 0 \end{pmatrix} \\
=&\begin{pmatrix} D_{B_0} & 0 & 0 \\ 0 & *d & -d \\ 0 & d^* & 0 \end{pmatrix} + h \text{ with 3 coordinates being } L^2_k(S) \oplus L^2_k(iT^*Y) \oplus L^2_k(i\mathbb{R}) \\
=&\begin{pmatrix} 0 & x & \mathbb{d}_\alpha \\ x & \mathrm{Hess}_{q,\alpha} & 0 \\ \mathbb{d}_\alpha & 0 & 0 \end{pmatrix} \text{ with 3 coordinates being } \mathcal{J}_k \oplus \mathcal{K}_k \oplus L^2_k(i\mathbb{R}),
\end{aligned}$$

where $x = \mathrm{pr}_{\mathcal{J}_{k-1,\alpha}} \circ d_\alpha \nabla \pounds|_{\mathcal{K}_{k,\alpha}}$, is k-ASAFOE. $x = 0$ at zeros of $\nabla \pounds$.

It has $\mathrm{Hess}_{q,\alpha}$ as a block, so it has a complete orthonormal basis.

For blown-up, Hessian no longer the second derivative of a function. We have $\mathrm{Hess}^\sigma_{q,\mathfrak{a}} : \mathcal{K}^\sigma_{k,\mathfrak{a}} \to \mathcal{K}^\sigma_{k-1,\mathfrak{a}}$ similarly. $d_{\mathfrak{a}}(\nabla \pounds)^\sigma = \begin{pmatrix} 0 & x \\ y & \mathrm{Hess}^\sigma_{q,\mathfrak{a}}. \end{pmatrix}$

$x = 0$, $y = 0$ at zero of $(\nabla \pounds)^\sigma$.

Non-degenerate iff $\mathrm{Hess}^\sigma_{q,\mathfrak{a}}$ is surjective.

Definition 1.5. Extended $\hat{\mathrm{Hess}}^\sigma_{q,\mathfrak{a}} : T^\sigma_{k,\mathfrak{a}} \oplus L^2_k(i\mathbb{R}) \to T^\sigma_{k-1,\mathfrak{a}} \oplus L^2_{k-1}(i\mathbb{R})$ as a block matrix below:

$\mathbb{d}^\sigma_{\mathfrak{a}}$ is defined similar as before.

$\mathbb{d}^{\sigma,\dagger}_{\mathfrak{a}} : T^\sigma_{k,\mathfrak{a}} \to L^2_{k-1}(i\mathbb{R})$ s.t. $\ker \mathbb{b}^{\sigma,\dagger}_{\mathfrak{a}} = \mathcal{K}^\sigma_{k,\mathfrak{a}}$.

Let $\mathfrak{a} = (B_0, s_0, \psi_0)$, $\mathbb{d}^{\sigma,\dagger}_{\mathfrak{a}} : (b, s, \psi) \mapsto -d^*b + is_0^2 \mathrm{Re}\langle i\psi_0, \psi\rangle + i|\psi_0|^2 \mathrm{Re}(\frac{\int_Y \langle i\psi_0, \psi}{\int_Y 1})$.

For $\mathfrak{a}$ zero, we have $\hat{\mathrm{Hess}}^\sigma_{q,\mathfrak{a}} = \begin{pmatrix} d_{\mathfrak{a}}(\nabla \pounds)^\sigma & \mathbb{d}^\sigma_{\mathfrak{a}} \\ \mathbb{d}^{\sigma,\dagger}_{\mathfrak{a}} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \mathbb{d}^\sigma_{\mathfrak{a}} \\ 0 & \mathrm{Hess}^\sigma_{q,\mathfrak{a}} & 0 \\ \mathbb{d}^{\sigma,\dagger}_{\mathfrak{a}} & 0 & 0 \end{pmatrix}$.

This is not a perturbation of an elliptic operator on Y. ψ is orthogonal to ψ_0, r is not section unconstrainted. One way to remedy this is to define $\Psi := \psi + r\psi_0$. $(b, \Psi, c) \in L^2_j(iT^*Y \oplus S \oplus i\mathbb{R})$ in this coordinate above is $L_0 + h_{\mathfrak{a}}$, where $L_0 = \begin{pmatrix} *d & 0 & -d \\ 0 & D_{B_0} & 0 \\ -d^* & 0 & 0 \end{pmatrix}$ is k-ASAFOE. Not symmetric but spectrum is real (see [KM] Lemma 12.4.3).

1.5. **Transversality.**

Lemma 1.6. *$\mathcal{E}$, $\mathcal{F}$ and $\mathcal{P}$ separable (countable dense subset) Banach manifolds. $S \subset \mathcal{F}$ closed submanifold. $F : \mathcal{E} \times \mathcal{P} \to \mathcal{F}$, $F_p(e) = F(e, p)$. Suppose F is transverse to S and for all $(e, p) \in F^{-1}(S)$,*

$$T_e\mathcal{E} \overset{d_e F_p}{\to} T_f\mathcal{F} \overset{\pi}{\to} T_f F / F_f S,$$

where $f := F(e, p)$, is Fredholm. Then there exists a residual $\mathcal{P}^{res} \subset \mathcal{P}$ such that for any $p \in \mathcal{P}^{res}$, $F_p : \mathcal{E} \to \mathcal{F}$ is transverse to S.

1.5.1. *Irreducible case.* For irreducible case, suffice to look at $\nabla \text{\L}$ on $\mathcal{C}_k^*(Y)$. Consider $\mathfrak{g} : \mathcal{C}_k^* \times \mathcal{P} \to \mathcal{K}_{k-1}$, $(\alpha, q) \mapsto (\nabla \text{\L})(\alpha) = (\nabla \mathcal{L})(\alpha) + q(\alpha)$.

Claim: $\mathfrak{g}$ transverse to the zero section of $\mathcal{K}_{k-1}$. Namely, the surjectivity of

$$\big((b, \psi), \delta q\big) \mapsto \mathrm{Hess}_{q,\alpha}(b, \psi) + \delta q(\alpha)$$

at zero $\alpha = (b, \psi)$ and tangent δq.

Indeed, cokernerl of $\mathrm{Hess}_{q,\alpha}$ is finite dimensional, and L^2-orthogonal to the range $= \ker \mathrm{Hess}_{q,\alpha}$.

Suffice to show for $v \in \ker(\mathrm{Hess}_{q,\alpha})\backslash\{0\}$. There exists $\delta q \in \mathcal{P}$ s.t.

$$\langle \delta q(\alpha), v\rangle_{L^2} = \nabla(\delta f) = d(\delta f)(v) \neq 0.$$

True by tame perturbation via cylinder function being large enough and approxminated by dense $\mathcal{P}$. So $\mathfrak{g}^{-1}(0)$ Banach manifold.

$\mathcal{Z} = \mathfrak{g}^{-1}(0)/\mathcal{G}_{k+1} \subset B_k^*(Y) \times \mathcal{P}$, and we have $\mathcal{Z} \to \mathcal{P}$, restricted from pr_2, smooth Fredholm of index 0. The set of regular values is a residual set.

1.5.2. *Reducible case.* $\mathfrak{g}^{\mathrm{red}} : \mathcal{A}_k \times \mathcal{P} \to \mathcal{K}_{k-1}^{\mathrm{red}}$, $(B, q) \mapsto (\nabla \text{\L})^{\mathrm{red}}(B) = (\nabla \mathcal{L})^{\mathrm{red}}(B) + q(B, 0)$.

Same proof shows $\mathfrak{g}^{\mathrm{red}}$ transverse to the zero section of $\mathcal{K}_{k-1}^{\mathrm{red}}$. To achieve other conditions in the characterization, perturb in direction normal to the irreducibles.

$\mathcal{P}^\perp \subset \mathcal{P}$ that consists of q vanishing at reducible locus.

Let $\mathrm{Op}^{\mathrm{sa}}$ be the space of self-adjoint Fredholm map $L_k^2(S) \to L_{k-1}^2(S)$ of form $D_{B_0} + h$.

B_0 spin^c connection, h self-adjoint operator extendable to bounded $L_j^2 \to L_j^2$, $j \leq k$.

$\mathrm{Op}^{\mathrm{sa}}$ Banach, stratified by kernel dimension. Let $L \in \mathrm{Op}^{\mathrm{sa}}$ with $\ker L = V$, the tangent space to L in its stratum is $\ker(\mathrm{Op}^{\mathrm{sa}} \to \mathrm{Op}^{\mathrm{sa}}(V))$, $N \mapsto \mathrm{pr}|_V \circ N|_V$.

In $\mathrm{Op}^{\mathrm{sa}}$, the set of operators whose spectrum is not simple is countable union of the images of F_n, Fredholm operator of negative index. To define it, first denote $\mathrm{Op}_n^{\mathrm{sa}} \subset \mathrm{Op}^{\mathrm{sa}}$ the space of operators having 0 as eigenvalue of multiplicity exactly n. Then $F_n : \mathrm{Op}_n^{\mathrm{sa}} \times \mathbb{R} \to \mathrm{Op}^{\mathrm{sa}}$, $(L, \lambda) \mapsto L + \lambda$. F_n is local embedding, the normal bundle at $L + \lambda$ is isomorphic to the space of traceless self-adjoint $\ker L \to \ker L$.

D_{q,B_0} defined earlier.

Define $M : \mathcal{A}_k \times \mathcal{P}^\perp \to \mathrm{Op}^{\mathrm{sa}}$, $(B, q^\perp) \mapsto D_{q^\perp,B}$.

Claim: M is transverse to the stratification of $\mathrm{Op}^{\mathrm{sa}}$ and transverse to F_n for all n.

Proof. Let $q^\perp = \nabla f^\perp$. $V := \ker D_{q^\perp,B}$. Regard V as the subspace of normal bundle of $\mathcal{A}_k$ in $\mathcal{C}_k(Y)$.

Facts: Let K compact subset of finite dimensional C^1 submanifold $N \subset B_k^0(T) = \mathcal{C}_k(Y)/\mathcal{G}_{k+1}^0(Y)$, the latter of which has been introduced in the last lecture.

K, N invariant under S^1.

There exist a collection of coclosed forms c_ν and sections Υ of $\mathbb{S}$ and a neighborhood U of K in N s.t. $p : B_k^0(Y) \to \mathbb{R}^n \times \mathbb{T} \times \mathbb{C}^m$ gives an embedding of U.

Choose p s.t. dp embeds S^1-invariant $V = \ker D_{q^\perp,B}$ in $\mathbb{C}^m \subset T_{p(B,0)}(\mathbb{R}^n \times \mathbb{T} \times \mathbb{C}^m)$ from choices made to construct $\mathcal{P}$.

Choose S^1-invariant function δg on $\mathbb{R}^n \times \mathbb{T} \times \mathbb{C}^m$ s.t. $\delta f = \delta g \circ p$.

Hess of $\delta f|_V$ is any S^1-equivariant complex linear self-adjoin $V \to V$. Take δf approximated from $\mathcal{P}$.

Similar for showing transverse to F_n □

$(\mathfrak{g}^{\text{red}})^{-1}(0) \subset \mathcal{A}_k \times \mathcal{P}$.

$(\mathfrak{g}^{\text{red}})^{-1}(0)/\mathcal{G}_{k+1} \subset \mathcal{A}_k/\mathcal{G}_{k+1} \times \mathcal{P}$, we have a map from the former to $\mathcal{P}$ restricted from the pr_2 of the latter. This is Fredholm of index 0.

$\mathcal{P}^{\perp} \times (\mathfrak{g}^{\text{red}})^{-1}(0)/\mathcal{G}_{k+1} \overset{\text{Id}\times\text{or}_2}{\to} \mathcal{P}^{\perp} \times \mathcal{P}$.

Same argument shows $\mathcal{W} \subset \mathcal{P}^{\perp} \times (\mathcal{A}_k/\mathcal{G}_{k+1})$ consists of $(q^{\perp}, [B])$ where $D_{q^{\perp},B}$ either is non-simple or consists 0 as eigenvalue is a countable union of Banach submanifolds $\mathcal{W}_n$, n of finite positive codimension.

Indeed, at each $x \in \mathcal{W}$, there exists complement to $T_x\mathcal{W}$ contained in $TP^{\perp}$ direction.

Take product with $\mathcal{P}$, $\mathcal{W} \times \mathcal{P} \subset \mathcal{P}^{\perp} \times (\mathcal{A}_k/\mathcal{G}_{k+1}) \times \mathcal{P}$.

Similar statement implies that $\mathcal{P}^{\perp} \times (\mathfrak{g}^{\text{red}})^{-1}(0)/\mathcal{G}_{k+1}$ and $\mathcal{W} \times \mathcal{P}$ are transverse, so the intersection is a locally finite union of Banach submanifolds $U \subset \mathcal{P}^{\perp} \times (\mathfrak{g}^{\text{red}})^{-1}(0)/\mathcal{G}_{k+1}$ of finite positive codimension.

Projection to $\mathcal{P}^{\perp} \times \mathcal{P}$ for each component of U is Fredholm of negative index. Sard-Smale gives regular values. $\mathcal{P}^{\perp} \times \mathcal{P} \overset{\text{addition}}{\to} \mathcal{P}$ maps a residual set to a residual set.