SEIBERG-WITTEN FLOER HOMOLOGY LECTURE 9

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1. Lecture 9

Please email yangding@math.hu-berlin.de if anything. Lecture notes up to now are available at www.mathematik.hu-berlin.de/∼yangding/monopole.html. Exercises sprinkled throughout lecture notes have been collected into an exercise sheet at www.mathematik.hu-berlin.de/∼yangding/Exercise_SWF.pdf.

This lecture is about moduli spaces of trajectories and regularity. Exercise session fills some details skipped during the lecture and gives the proof for regularity.

We focus on shifting the functional analytic setting from $I \times Y$ with compact I to $\mathbb{R} \times Y =: Z$.

We saw in the lecture notes that $\mathcal{C}^\tau(I \times Y)$ the τ model (instead of σ model) adapted to the flow picture.

σ model: A typical point in the blow-up configuration space is (A, s, ψ) with constant $s \geq 0$ and $\|\psi\|_{L^2(Z)} = 1$ in 4d.

We restrict spinor $\check{\psi}(t) := \psi(t, \cdot)$ on each slice $Y \cong \{t\} \times Y$ (unique continuation property implies that if $\psi \neq 0$, then $\check{\psi}(t) \neq 0$ for all t.)

To make each slice a σ model (with last entry being of unit length), we need to divide the spinor by its norm and multiply this norm to the middle entry, $(\check{A}(t), s\|\check{\psi}(t)\|_{L^2(Y)}, \frac{\check{\psi}(t)}{\|\check{\psi}(t)\|_{L^2(Y)}})$, making it into a non-negative function $\mathbb{R} \to \mathbb{R}^{\geq 0}$. This is τ model.

$\mathcal{C}^\tau_k(Y)$ with the middle function asked to be in L^2_k. This is no longer a Hilbert/Banach manifold with boundary, but a closed subspace of a Hilbert manifold $\tilde{\mathcal{C}}^\tau_k(Y)$ where the middle function in the latter is $\mathbb{R} \to \mathbb{R}$ (no constraint).

For I compact, we have the correspondence between a point in $\mathcal{C}^\tau(I\times Y)/\mathcal{G}(I\times Y)$ in 4d and a smooth path in $\mathcal{C}^\sigma(Y)/\mathcal{G}(Y)$ in 3d.

But for $I = \mathbb{R}$, need care: $\tilde{\mathcal{C}}_{k,loc} :=$

$\{(A, s, \phi) \in A_0 + L^2_{k,loc}(iT^*Z) \times L^2_{k,loc}(\mathbb{R}, \mathbb{R}) \times L^2_{k,loc}(S^+) \mid \|\check{\psi}(t)\|_{L^2(Y;S)} = 1\}$, here $Y \cong \{t\} \times Y$ and $S \cong S^+|_{\{t\}\times Y}$. The subscript loc means L^2_k norm is bounded for function/section restricted to compact $I \times Y$.

We also consider the un-tilded version $\mathcal{C}^\tau_{k,loc}$ where $s(t) \geq 0$.

The gauge group $\mathcal{G}_{k+1,loc} = \{u \in L^2_{k+1,loc}(Z, \mathbb{C}) \mid |u(\cdot)| = 1\}$, with quotients $B^\tau_{k,loc} \subset \tilde{B}^\tau_{k,loc}$.

$q \in \mathcal{P}^{res}$ residual part such that zeros of $(\nabla \pounds)^\sigma$ non-degenerate.

4d SW map is a section $\mathcal{F}^\tau_q : \tilde{\mathcal{C}}^\tau_{k,loc}(\mathbb{R} \times Y) \to \mathcal{V}^\tau_{k-1,loc}(\mathbb{R} \times Y)$, the fiber of the latter at (A_0, s_0, ϕ_0) is $\{(a, s, \phi) \in$
$L^2_{k-1,loc}(i\mathfrak{su}(S^+)) \oplus L^2_{k-1,loc}(\mathbb{R}, \mathbb{R}) \oplus L^2_{k-1,loc}(S^-) \mid \mathrm{Re}\langle \check{\phi}_0(t), \check{\phi}(t)\rangle_{L^2(Y)} = 0 \text{ for all } t\}$.

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This bundle is not a locally trivial vector bundle, but ok if we choose some projection.

If $\mathfrak{b}$ is zero of $(\nabla\pounds)^\sigma$, $\mathfrak{b}$ corresponds to a translation-invariant $\gamma_\mathfrak{b} \in \mathcal{C}^\tau_{k,loc}(Z)$ s.t. $\mathcal{F}^\tau_q(\gamma_\mathfrak{b}) = 0$. So $\check{\gamma}_\mathfrak{b}(\cdot)$ is constant.

Definition 1.1. $[\gamma] \in \tilde{B}^\tau_{k,loc}(Z)$ is asymptotic to $[\mathfrak{b}]$ as $t \to \pm\infty$, if $[\tau_t^*\gamma] \to [\gamma_\mathfrak{b}]$ in $\tilde{B}^\tau_{k,loc}$, where $\tau_t^*\gamma := \gamma(\cdot + t)$. Written as $\underrightarrow{\lim}\gamma = [\mathfrak{b}]$ when $t \to +\infty$, and $\underleftarrow{\lim}[\gamma] = [\mathfrak{b}]$ when $t \to -\infty$.

Definition 1.2. A moduli of trajectories is
$M([\mathfrak{a}], [\mathfrak{b}]) = \{[\gamma] \in B^\tau_{k,loc}(Z) \mid \mathcal{F}^\tau_q(\gamma) = 0,\ \underleftarrow{\lim}\gamma = [\mathfrak{a}],\ \underrightarrow{\lim}\gamma = [\mathfrak{b}]\}$. It is independent of k due to elliptic regularity before. We also have $\tilde{M}([\mathfrak{a}][\mathfrak{b}])$, where we have $[\gamma] \in \tilde{B}^\tau_{k,loc}(Z)$ and with the same other constraints.

$[\gamma] \in M([\mathfrak{a}], [\mathfrak{b}])$ corresponds to $[\check{\gamma}(\cdot)]$ in $B^\sigma_k(Y)$ connecting from $[\mathfrak{a}]$ to $[\mathfrak{b}]$. It determines a relative homotopy class $z \in \pi_1(B^\sigma_k(Y); [\mathfrak{a}], [\mathfrak{b}])$ which is an affine space over $H^1(Y;\mathbb{Z})$, the components of gauge group, via action. So it decomposes into components, $M([\mathfrak{a}], [\mathfrak{b}]) = \bigsqcup_z M_z([\mathfrak{a}], [\mathfrak{b}])$. This is most natural way to describe the moduli space of trajectories.

But we need more direction version for transversality:

Choose lifts $\mathfrak{a}$ and $\mathfrak{b}$ in $\mathcal{C}^\sigma_k(Y)$ of zeros $[\mathfrak{a}]$ and $[\mathfrak{b}]$ of $(\nabla\pounds)^\sigma$. Choose smooth $\gamma_0 = (A_0, s_0, \phi_0) \in \mathcal{C}^\tau_{k,loc}(\mathbb{R} \times Y)$, which is $\gamma_\mathfrak{a}$ near $-\infty$ and $\gamma_\mathfrak{b}$ near $+\infty$ and $[\check{\gamma}_0] \in z$.

Define $\tilde{\mathcal{C}}^\tau_k(\mathfrak{a}, \mathfrak{b}) = \{\gamma \in \mathcal{C}^\tau_{k,loc}(Z) \mid \gamma - \gamma_0 \in L^2_k(iT^*Z) \times L^2_k(\mathbb{R}, \mathbb{R}) \times L^2_{k,A_0}(S^+)\}$, here L^2_k is global, and L^2_{k,A_0} means (higher) covariant derivative are defined using the connection A_0 from γ_0.

Sitting inside, we have $\mathcal{C}^\tau_k(\mathfrak{a}, \mathfrak{b})$ where we have the middle variable in $L^2_k(\mathbb{R}, [0, \infty))$.

The gauge group is defined as $\mathcal{G}_{k+1} := \{u \in \mathcal{G}_{k+1,loc} \mid u(\mathcal{C}^\tau_k(\mathfrak{a}, \mathfrak{b})) \subset \mathcal{C}^\tau_k(\mathfrak{a}, \mathfrak{b})\}$. A fact is $\mathcal{G}_{k+1}(Z) = \{u \in \mathcal{G}_{k+1,loc} \mid 1 - u \in L^2_{k+1}(Z; \mathbb{C})\}$.

Define $B^\tau_{k,z}([\mathfrak{a}], [\mathfrak{b}]) = \mathcal{C}^\tau_k(\mathfrak{a}, \mathfrak{b})/\mathcal{G}_{k+1}(Z)$. $B^\tau_k([\mathfrak{a}], [\mathfrak{b}]) := \bigsqcup_z B^\tau_{k,z}([\mathfrak{a}], [\mathfrak{b}])$. The tilde version are defined the same way. They are Hausdorff.

Theorem 1.3. *Let $[\gamma] \in M_z([\mathfrak{a}], [\mathfrak{b}])$. Choose any lift $\gamma \in \mathcal{C}^\tau_{k,loc}(Z)$, choose lifts $\mathfrak{a}, \mathfrak{b}$ and γ_0 such that $[\check{\gamma}_0] \in z$. Then there exists $u \in \mathcal{G}_{k+1,loc}$ s.t. $u(\gamma) \in \mathcal{C}^\tau_k(\mathfrak{a}, \mathfrak{b})$ (namely, there exists a gauge representative that lies in the direct description). Any two such u and u', we have $u'u^{-1} \in \mathcal{G}_{k+1}(Z)$. So*

$$\gamma \mapsto [(u(\gamma))] = [(u'u^{-1})(u(\gamma))] = [u'(\gamma)] \in B^\tau_{k,\tau}([\mathfrak{a}], [\mathfrak{b}])$$

is well-defined independent of choice u. Then actually this map descends to an injective map from $M_z([\mathfrak{a}], [\mathfrak{b}])$, and this map has the image

$$\{[\gamma] \in B^\tau_{k,\tau}([\mathfrak{a}, \mathfrak{b}]) \mid \mathcal{F}^\tau_q(\gamma) = 0\},$$

and this bijection is homeomorphism. Similarly, we have the statement for the tilde version.

1.0.1. *Local structure of moduli space of trajectories.* We have just realized $M_z([\mathfrak{a}], [\mathfrak{b}])$ as the zero set of $\mathcal{F}^\tau_q$. Now we want to show $\mathcal{F}^\tau_q$ to be locally non-linear Fredholm between Banach manifolds.

$L_0 + h$ k-ASAFOE on sections of $E \to Y$ as in the last lecture (L_0 SAFOE, $h : \mathcal{C}^\infty(E) \to L^2(E)$ extends to bounded $h : L^2_j(E) \to L^2_j(E)$ for all j with $|j| \leq k$). We pull $E \to Y$ back to $E \to Z = \mathbb{R} \times Y$.

Consider the translation-invariant $D = \frac{d}{dt} + L_0 + h$ is bounded $L^2_{j+1}(E) \to L^2_j(E)$ for $|j| \le k$.

Spectrum for an operator on a real Hilbert space means spectrum of its complexification.

Definition 1.4. Let $L_0 + h$ be a k-ASAFOE operator. It is hyperbolic, if spectrum is disjoint from the imaginary axis in $\mathbb{C}$.

Proposition 1.5. *$L_0 + h$ hyperbolic, then $D = \frac{d}{dt} + L_0 + h : L^2_{j+1}(E) \to L^2_j(E)$ is invertible (thus Fredholm).*

Now consider time-dependent h. $D := \frac{d}{dt} + L_0 + h : L^2_1 \to L^2$ (independent of k so we consider lowest k). Family $L_0 + h_t$, $t \in [0,1]$, which is a continuous path in $\{$bounded operator $L^2 \to L^2\}$ with $L_0 + h_0$ and $L_0 + h_1$ hyperbolic.

The spectral flow $\mathrm{sf}(L_0 + h_t) =$ "net number of eigenvalues whose real parts go from negative to positive". We make this precise in the exercise session as a genericity statement.

Proposition 1.6. *L_0 SAFOE on sections of $E \to Y$, h_t bounded $L^2(E) \to L^2(E)$ continuous in t in operator norm with $h_{\pm\infty} = h_\pm$, and $L_0 + h_\pm$ hyperbolic. Then $Q = \frac{d}{dt} + L_0 + h_t : L^2(\mathbb{R}\times Y; E) \to L^2(\mathbb{R}\times Y; E)$ is Fredholm with index $=$ sf$(L_0 + h_t)$.*

1.0.2. *Slice.* T^τ_j denotes the L^2_j fiber-completion of the tangent bundle of $\tilde{\mathcal{C}}^\tau_k(\mathfrak{a}, \mathfrak{b})$ (the latter of which has the constraint $\mathrm{Re}\langle \phi_0|_t, \phi|_t\rangle_{L^2(Y;S)} = 0$ in its definition), where ϕ_0 is from the base and ϕ is from the vector.

Write the derivative of the gauge group action as before as $\mathbb{d}^\tau_\gamma \xi = (-d\xi, 0, \xi\phi_0)$ with $\gamma = (A_0, s_0, \phi_0)$.

Define $\mathcal{S}^\tau_{k,\gamma} := \{(A = A_0 + a, s, \phi) \in \tilde{\mathcal{C}}^\tau_k(\mathfrak{a}, \mathfrak{b}) \mid \mathrm{Coul}^\tau_\gamma(A_0 + a, s, \phi) = 0\}$, where $\mathrm{Coul}^\tau_\gamma : \tilde{\mathcal{C}}^\tau_k(\mathfrak{a}, \mathfrak{b}) \to L^2_{k-1}(i\mathbb{R})$,

$$(A_0 + a, s, \phi) \mapsto -d^*a + iss_0\mathrm{Re}\langle i\phi_0, \phi\rangle + i|\phi_0|^2\mathrm{Re}(\frac{\int_Y \langle i\phi_0, \phi\rangle}{\int_Y 1}).$$

The point of this map is its linearization $d_\gamma \mathrm{Coul}^\tau_\gamma$ extends to $\mathbb{d}^{\tau,\dagger} : T^\tau_j \to L^2_{j-1}(i\mathbb{R})$ has the following property: $\mathcal{K}^\tau_{j,k} := \ker \mathbb{d}^{\tau,\dagger}_\gamma$ and $\mathcal{J}^\tau_{j,\gamma} = \mathrm{im}\mathbb{d}^\tau_\gamma$ are complementary closed subspaces spanning $T^\tau_{j,\gamma}$ which vary smoothly over the base.

Want to show that restricting to the slice denoted by $\cdot|$, the equation has Fredholm linearization.

$\mathcal{F}^\tau_q|$ a section of $\mathcal{V}^\tau_{k-1}| \to \mathcal{S}^\tau_{k,\gamma}$, where a tame perturbation q chosen s.t. zeros of $(\nabla \mathcal{L})^\sigma$ are non-degenerate.

$\mathcal{V}^\tau_{k-1}$ is not a trivial vector bundle along the path, and we need a projection Π^τ_γ to define linearization (just like differentiating on a sphere), where

$$\Pi^\tau_\gamma : L^2_j(i\mathfrak{su}(S^+)) \oplus L^2_j(\mathbb{R}, \mathbb{R}) \oplus L^2_{j,A_0}(S^-) \to \mathcal{V}^\tau_{j,\gamma},$$

$(\eta, r, \psi) \mapsto (\eta, r, \Pi^\perp_{\phi_0(t)}\psi)$ where $\Pi^\perp_{\phi_0(t)}\psi = \psi - \mathrm{Re}\langle \check{\phi}_0(t), \psi(t)\rangle_{L^2(Y;S)}\phi_0$.

$d\mathcal{F}^\tau_q$ is defined by taking derivative in the ambient Banach space, then projecting.

$a = b + cdt$ where b is in temporal gauge, and we denote

$$(a, r, \phi) = ((b, r, \psi), c) = (V, c).$$

$d_{\gamma_0}\mathcal{F}^\tau_q : (V, c) \mapsto \frac{d}{dt}V + d(\nabla \mathcal{L})^\sigma(V) + \mathbb{d}^\sigma_{\gamma_0(t)}c$. Here $\frac{d}{dt}V := \left(\frac{db}{dt}, \frac{dr}{dt}, \Pi^\perp_{\phi_0(t)}(\frac{d\phi}{dt})\right)$. (We used iT^*Y with $i\mathfrak{su}(S^+)$).

We impose the Coulomb gauge fixing condition $0 = \mathbb{d}_0^{\tau,\dagger}(V,c) = \frac{dc}{dt} + \mathbb{d}_{\gamma_0(t)}^{\sigma,\dagger}(V)$. We also assume γ_0 is in temporal gauge for convenience.

SW + gauge fixing, $Q_{\gamma_0} = d_{\gamma_0}\mathcal{F}_q^\tau \oplus \mathbb{d}_{\gamma_0}^{\tau,\dagger}$. In path notation, $(V,c) \mapsto \frac{d}{dt}(V,c) + L_{\gamma_0(t)}(V,c)$.

Here, if denoting $\mathfrak{c} := \gamma_0(t)$, $L_{\mathfrak{c}} = \begin{pmatrix} d_{\mathfrak{c}}(\nabla \pounds)^\sigma & \mathbb{d}_{\mathfrak{c}}^\sigma \\ \mathbb{d}_{\mathfrak{c}}^{\sigma,\dagger} & 0 \end{pmatrix} = \widehat{\mathrm{Hess}}^\sigma_{q,\mathfrak{c}}$ before.

Theorem 1.7. *Q_{γ_0} is Fredholm for $1 \le j \le k$ with index independent of j and satisfying the semi-Fredholm estimate/Gårding inequality $\|u\|_{L^2_j} \le C_1 \|Q_{\gamma_0} u\|_{L^2_{j-1}} + C_2 \|u\|_{L^2_{j-1}}$.*

$d(\mathcal{F}_q^\tau|_{\text{slice}}) : \mathcal{K}^\tau_{j,\gamma} \to \mathcal{V}^\tau_{j-1,\gamma}$ Fredholm with index same as that of Q_{γ_0}, also called relative grading

$$\mathrm{gr}_z([\mathfrak{a}],[\mathfrak{b}]) = \mathrm{gr}(\mathfrak{a},\mathfrak{b}) = \mathrm{sf}(\widehat{\mathrm{Hess}}^\sigma_{q,\hat{\gamma}_0(t)}) = \mathrm{sf}(\begin{pmatrix} 0 & \mathbb{d}^\sigma_{\gamma(t)} \\ \mathbb{d}^{\sigma,\dagger}_{\gamma_0(t)} & 0 \end{pmatrix} \oplus \mathrm{Hess}^\sigma_{q,\gamma_0(t)}) = \mathrm{sf}(\mathrm{Hess}^\sigma_{q,\gamma_0(t)}).$$

1.0.3. *Regularity.* $M_z([\mathfrak{a}],[\mathfrak{b}]) \subset \tilde{M}_z([\mathfrak{a}],[\mathfrak{b}]) \subset \tilde{B}^\tau_{k,z}([\mathfrak{a}],[\mathfrak{b}])$, where the first one has $s(t) \ge 0$.

A neighborhood of $[\gamma]$ in $M_z([\mathfrak{a}],[\mathfrak{b}])$ is the zero set of $\mathcal{F}_q^\tau|_{U_\gamma} : \mathcal{U}_\gamma \to \mathcal{V}^\tau_{k-1}$. If $d_\gamma \mathcal{F}_q^\tau| : \mathcal{K}^\tau_{k,\gamma} \to \mathcal{V}^\tau_{k-1,\gamma}$ is surjective, then $M_z([\mathfrak{a}],[\mathfrak{b}])$ is a manifold near $[\gamma]$ of dimension $\dim\ker d_\gamma \mathcal{F}_q^\tau| = \mathrm{ind} Q_\gamma = \mathrm{gr}_z([\mathfrak{a}],[\mathfrak{b}])$.

Unique continuation means a SW solution $\gamma = (A,s,\phi)$ has either $s \equiv 0$ or $s : \mathbb{R} \to \mathbb{R}\backslash\{0\}$.

In the second case if $s > 0$, $[\gamma] \in M_z([\mathfrak{a}],[\mathfrak{b}])$, so

$$M_z([\mathfrak{a}],[\mathfrak{b}]) = \tilde{M}_z([\mathfrak{a}],[\mathfrak{b}])/(i : [A,s,\phi] \mapsto [A,-s,\phi]).$$

In the flow form of $4d$ SW equation, we have appearance of $\Lambda_q(\mathfrak{a})$, which plays the eigenvalue role in the finite dimensional case. If $\mathfrak{a}$ is a reducible zero, $\mathfrak{a}$ is called boundary-stable if $\Lambda_q(\mathfrak{a}) > 0$, and boundary-unstable if $\Lambda_q(\mathfrak{a}) < 0$.

Lemma 1.8. *If $M_z([\mathfrak{a}],[\mathfrak{b}])$ conatins an irreducible trajectory, then $\mathfrak{a}$ is either irreducible or boundary-unstable, and $\mathfrak{b}$ is either irreducible or boundary-stable.*

Proof. $\frac{ds}{dt} = -\Lambda_q(\check{\gamma}(t))s$ and $s > 0$. We have $\Lambda_q(\check{\gamma}(t)) \to \Lambda_q(\mathfrak{a})$ and to $\Lambda_q(\mathfrak{b})$.

If $\mathfrak{a}$ reducible, $s \to 0$ at $-\infty$, then $\Lambda_q(\mathfrak{a}) < 0$.

If $\mathfrak{b}$ reducible, similarly, we have $\Lambda_q(\mathfrak{b}) > 0$. □

If γ reducible, $Q_\gamma = Q_\gamma^\partial \oplus Q_\gamma^\nu$.

$Q_\gamma^\partial = (d_\gamma \mathcal{F}_q^\tau)^\partial \oplus \mathbb{d}_\gamma^{\tau,\dagger}$ whose first factor is invariant under involution i, and $Q_\gamma^\nu : L^2_k(i\mathbb{R}) \to L^2_{k-1}(i\mathbb{R}), s \mapsto \frac{ds}{dt} + \Lambda(\check{\gamma})s$.

After calculation, one can see $(\dim\ker Q_\gamma^\nu, \mathrm{dimcok} Q_\gamma^\nu)$ is $(1,0)$ if $\mathfrak{a}$ is ∂-unstable and $\mathfrak{b}$ is boundary-stable; $(0,1)$ if $\mathfrak{a}$ is boundary-stable and $\mathfrak{b}$ is boundary-unstable (this case is said to be boundary-obstructed, this is still ok due to be constant dimension of cokernel); $(0,0)$ if both $\mathfrak{a}$ and $\mathfrak{b}$ are boundary-stable, or both $\mathfrak{a}$ and $\mathfrak{b}$ are boundary-unstable.

The definition of regular is stated.

The regularity theorem says that there exists a residual $q \in \mathcal{P}^{\mathrm{res}}$, (i) all zeros of $(\nabla \pounds)^\sigma$ non-degenerate; (ii) $M_z([\mathfrak{a}],[\mathfrak{b}])$ is regular.

The proof is provided in the exercise session.