SEIBERG-WITTEN FLOER HOMOLOGY LECTURES

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I have typed the lecture notes for all 15 sessions (as I went along), available now at the course webpage `www.mathematik.hu-berlin.de/∼yangding/monopole.html`

Please email `yangding@math.hu-berlin.de` for possibly taking an exam, or to be added to the mail list for possible future announcement, or if having any questions, or to have contents added to the lectures to make it more self-contained or useful (also feel free to look up certain concepts and notions in other textbooks and let me now if you need a pointer). I might make this notes more self-contained after the conclusion of the lecture course, providing some concise proof/intuition (or at least point towards another location to read) for certain statements.

Lectures always started at 9:15 AM every Friday Berlin time. Zoom log in detail is on the course webpage.

We follow Kronheimer-Mrowka's "Monopoles and Three-manifolds" [KM] closely throughout (please refer to that bible for more details and contexts and more applications on low dimensional topology), and towards the end of lecture series, we will cover a few relatively recent applications (Weinstein conjecture, contact volume detecting, smooth closing lemma and simplicity conjecture) based on four papers.

Some labels to refer to such as (⊛), (⊘) ... are only applicable within each subsection/lecture.

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Date: March 2, 2021.

1. Lecture 1

Let Y be a closed, orientable, Riemannian 3-manifold.

Definition 1.1. A spin^c structure on Y is a unitary rank 2 complex (namely $U(2)$) vector bundle $S \to Y$ (with Hermitian metric on S denoted by h, Riemannian metric on Y denoted by g) with Clifford multiplication $\rho : TY \to \mathrm{Hom}(S, S)$ which is a bundle map with the image $\mathfrak{su}(S) = \{a \mid \mathrm{tr} a = 0, a^* = -a\}$ (where a^* is defined by $h(ax, y) = h(x, a^* y)$), such that, denoting $\tilde{h}(a, b) = \frac{1}{2}\mathrm{tr}(a^* b)$,

$$\rho : (TY, g) \to (\mathfrak{su}(S), \tilde{h})$$

is a bundle isometry.

Exercise 1.2. More concretely, let $\sigma_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$. Then can choose orthonormal basis e_i of $T_y Y$ and basis for S_y such that $\rho(e_i) = \sigma_i$.

Exercise 1.3. Look up why for 3-manifold Y, TY is trivial. Then show a spin^c structure for Y always exists.

For any Hermitian line bundle $L \to Y$, any spin^c structure $\mathfrak{s}_0 = (S_0, \rho_0)$, define $\mathfrak{s} = (S, \rho)$ where $S := S_0 \otimes L$ and $\rho := \rho_0 \otimes \mathrm{Id}_L$. We remark (c.f. the main reference [KM]) up to isomorphism (bundle isomorphism intertwining the Clifford multiplications), any two spin^c structures are related this way.

Complex line bundle up to isomorphism via c_1 is $H^2(Y; \mathbb{Z})$, thus space of isomorphism classes of spin^c structures is an affine space over $H^2(Y; \mathbb{Z})$.

Let X be an oriented, Riemannian 4-manifold.

Definition 1.4. A spin^c structure on X is a Hermitian rank 4 (namely $U(4)$) vector bundle $S_X \to X$ with Clifford multiplication $\rho_X : TX \to \mathrm{Hom}(S_X, S_X)$, such that for each $x \in X$, we can find orthonormal basis e_0, e_1, e_2, e_3 such that $\rho(e_0) = \begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix}$, $\rho(e_i) = \begin{pmatrix} 0 & -\sigma_i^* \\ \sigma_i & 0 \end{pmatrix}$ for $i = 1, 2, 3$ for some orthonormal basis of $(S_X)_x$. Here, I_2 is 2 by 2 identity matrix.

Using the metric g, we can induce $\rho_X : T^*X \to Hom(S_X, S_X)$ with the same notation. We can extend to forms: $\rho(\alpha \wedge \beta) = \frac{1}{2}\big(\rho(\alpha)\rho(\beta) + (-1)^{\deg\alpha \deg\beta}\rho(\beta)\rho(\alpha)\big)$.

Exercise 1.5. $\rho(\mathrm{vol}) = \begin{pmatrix} -I_2 & 0 \\ 0 & I_2 \end{pmatrix}$.

Definition 1.6. Let S^+ denote the -1 eigenspace of $\rho(\mathrm{vol})$ and S^- the 1 eigenspace of $\rho(\mathrm{vol})$. They are called the positive and negative spin bundle with sections called spinors. So $S_X = S^+ \oplus S^-$.

For $e \in T_xX$, $\rho(e) : S_x^+ \to S_x^-$.

Hodge star operator $* : \Lambda^k X \to \Lambda^{\dim X - k} X$ is defined by $\alpha \wedge *\beta = g(\alpha, \beta)\mathrm{vol}$, where the metric on forms is induced from g and denoted by the same notation. Here X is a 4-manifold, and let $\Lambda^\pm$ denote the ± 1 eigenspace of $* : \Lambda^2 X \to \Lambda^2 X$, elements of which are called self-dual and anti-self-dual 2-forms respectively.

Exercise 1.7. $\rho : \Lambda^+ \to \mathfrak{su}(S^+)$ is a bundle isometry. $\rho : \Lambda^+ \to \mathrm{End}(S^-)$ is 0. For any unit vector $e \in T_xX$, $\det\rho(e) : \Lambda^2 S_x^+ \to \Lambda^2 S_x^-$ is independent of e.

Definition 1.8. For a spin^c structure (S_X, ρ), $\mathrm{Aut}(S_X, \rho)$ is the group of unitary bundle automorphisms of S_X that commute with ρ. This is precisely $\{u : X \to S^1\}$ called gauge group $\mathcal{G}_X$, which acts on S_X by scalar multiplication. The same is true for 3-manifold Y, $\mathrm{Aut}(S, \rho) = \{u : Y \to S^1\}$.

Definition 1.9. A connection ∇_A on a bundle $S_X \to X$ over Riemannian X is a $\mathbb{C}$-linear map $\nabla_A : \Gamma(S_X) \to \Gamma(T^*X \otimes S_X)$ such that $\nabla_A(fs) = f\nabla_A s + df \otimes s$. We can extend it using Levi-Civita connection ∇_X on X, and recall the curvature $F_A \in \Gamma(\Lambda^2 X \otimes \mathrm{End}(S_X))$ is defined via $\nabla_A(\nabla_A(s)) = F_A(s)$. A connection ∇_A on a unitary bundle S_X with a metric h is called unitary, if

$$d\big(h(s, \tilde{s})\big) = h(\nabla_A s, \tilde{s}) + h(s, \nabla_A \tilde{s}).$$

A connection ∇_A on an oriented Riemannian 4-manifold X with a spin^c structure $\mathfrak{s} = (S_X, \rho)$ is called a spin^c connection, if it is unitary and ρ is parallel, namely

$$(\nabla_A \rho)(v)(s) := \nabla_A\big(\rho(v)(s)\big) - \rho(\nabla_X v)(s) - \rho(v)(\nabla_A s) = 0$$

for all local sections v and s.

Remark 1.10. In particular, parallel transport via ∇_A preserves $S_X = S^+ \oplus S^-$.

Two spin^c connections $\tilde{A}$ and A differ by $a \otimes \mathrm{Id}_{S_X} = a$ for some $a \in \Omega^1(X; i\mathbb{R})$.

Definition 1.11. The Dirac operator $D_A : \Gamma(S_X) \to \Gamma(S_X)$ is defined as the composition $\Gamma(S_X) \xrightarrow{\nabla_A} \Gamma(T^*X \otimes S_X) \xrightarrow{\rho} \Gamma(S_X)$, where the second map is pointwise map $\big(\rho : T^*X \to \mathrm{End}(S_X)\big) = \big(\rho : T^*X \otimes S_X \to S_X\big)$ induced from ρ.

If $\nabla_{\tilde{A}} := \nabla_A + a$, then $\nabla_{\tilde{A}} = D_A + \rho(a)$.

Definition 1.12. As in Remark 1.10, we can decompose $D_A = D_A^+ + D_A^-$, where $D_A^\pm : \Gamma(S^\pm) \to \Gamma(S^\mp)$.

A spin^c connection A induces connection on $\Lambda^2 S^+$ and $\Lambda^2 S^-$ which are identified under $\det \rho(e)$ for any unit vector e, denoted by A^t. Note $\nabla_{\tilde{A}^t} = \nabla_{A^t} + 2a$.

Let $u : X \to S^1$ be an element of $\mathcal{G}_X$, it acts on connection via $u(A) = A - u^{-1}du$, due to S^1 is Abelian.

Definition 1.13. In dimension 3, spin^c connection B, Dirac operator D_B is defined the same way.

Let X be oriented Riemannian 4-manifold. $\mathfrak{s}_X = (S_X, \rho)$ a spin^c structure. Consider a pair (A, Φ) where A is a spin^c connection and $\Phi \in \Gamma(S^+)$.

Denote $(\Phi\Phi^*)_0$ the traceless part of $\Phi\otimes\Phi^*$ (omitting the tensor product), which is $\Phi\Phi^* - \frac{1}{2}\mathrm{tr}(\Phi\Phi^*)\mathrm{Id}_{S^+} = \Phi\Phi^* - \frac{1}{2}|\Phi|^2\mathrm{Id}_{S^+}$. It is Hermitian and traceless, thus is i times skew-Hermitian. Thus lies in $i\rho(\Lambda^+) = \rho(\Lambda^+ i)$. So it makes sense to write down the following Seiberg-Witten (SW) equation.

$$\begin{cases} \frac{1}{2}\rho(F^+_{A^t}) - (\Phi\Phi^*)_0 = 0 \\ D^+_A\Phi = 0 \end{cases}$$

Exercise 1.14. Denote the LHS of the above as $\mathcal{F}(A,\Phi)$. Check that

$$D_{(A,\Phi)}\mathcal{F} : (a,\phi) \mapsto (\rho(d^+a) - (\phi\Phi^* + \Phi\phi^*)_0, D^+_A\phi + \rho(a)\Phi).$$

For $u \in \mathcal{G}_X$, we have the action $u : (A,\Phi) \mapsto (u(A), u\Phi)$. $\mathcal{G}_X$ acts freely on

$$\{(A,\Phi) \mid \mathcal{F}(A,\Phi) = 0, \Phi \neq 0\}.$$

If $\Phi = 0$, SW equation reduces to $F^+_{A^t} = 0$, the anti-self-dual equation for A^t.

We return to 3-dimension, let Y be an oriented closed Riemannian 3-manifold.

Definition 1.15. For a spin^c structure $\mathfrak{s} = (S,\rho)$, fix a reference spin^c connection B_0. We define on the configuration space of pairs (B,Ψ), where B is a spin^c connection and $\Psi \in \Gamma(S)$ a spinor, a functional

$$\mathcal{L}(B,\Psi) := -\frac{1}{8}\int_Y (B^t - B^t_0)\wedge(F_{B^t} + F_{B^t_0}) + \frac{1}{2}\int_Y h(D_B\Psi,\Psi)d\mathrm{vol},$$

called Chern-Simons-Dirac functional. Later, we also denote $h(\ ,\ )$ by $\langle\ ,\ \rangle$.

2. Lecture 2

2.1. **Variational origin of 3d SW.** We continue from last time. Let Y be an oriented closed Riemannian 3-manifold. (C.f. P 18 corollary 2.45 of John Morgan's Seiberg-Witten equation and application to topology of smooth four manifolds to arrive at our spin^c from general definition.)

For a spin^c structure $\mathfrak{s} = (S,\rho)$, fix a reference spin^c connection B_0 as a base point. Consider the configuration space of pairs (B,Ψ), where B is a spin^c connection and a spinor $\Psi \in \Gamma(S)$. We introduced the CSD functional

$$\mathcal{L}(B,\Psi) := -\frac{1}{8}\int_Y (B^t - B^t_0)\wedge(F_{B^t} + F_{B^t_0}) + \frac{1}{2}\int_Y h(D_B\Psi,\Psi)d\mathrm{vol},$$

called Chern-Simons-Dirac functional. Later, we also denote $h(\ ,\ )$ by $\langle\ ,\ \rangle$.

Write $B = B_0 + b$ with $b \in \Omega^1(Y, i\mathbb{R})$, then $B^t = B^t_0 + 2b$, $F_{B^t} = F_{B^t_0} + 2\nabla_{B^t_0}b = F_{B^t_0} + 2\nabla_{B^t}b$ (where $b\wedge b$ term is identically 0, in this Abelian case). We have

$$\mathcal{L}(B+b,\Psi+\psi) - \mathcal{L}(B,\Psi) = \int_Y \left(\langle b, \frac{1}{2}*F_{B^t} + \rho^{-1}(\Psi\Psi^*)_0\rangle + \mathrm{Re}\langle\psi, D_B\Psi\rangle\right)d\mathrm{vol} + o(b,\psi).$$

Here the first $\langle\ ,\ \rangle$, which is the natural metric on Lie algebra valued 1-form, which in this case is the metric on 1-forms in front of i; and ρ is an isometry.

Using the L^2 metric $\langle(b,\psi),(b',\psi')\rangle_{L^2} = \int_Y \left(\langle b,b'\rangle + \mathrm{Re}\langle\psi,\psi'\rangle\right)d\mathrm{vol}$, define (formal) gradient $\nabla\mathcal{L}$ via $\langle\nabla\mathcal{L}, v\rangle_{L^2} = d\mathcal{L}(v)$. $\nabla\mathcal{L} = 0$ corresponds 3d SW equation.

2.2. **Gauge-invariance of S^1-valued functional.** Recall symmetry/Gauge group is $\mathcal{G}_Y := \{u : Y \to S^1\}$, acting by $u : (B, \Psi) \mapsto (u(B), u\Psi)$, where $u(B) := u \circ \nabla_B \circ u^{-1} = B - u^{-1}du$, and $u\Psi$ is the fiberwise scalar multiplication.

We remark that $[u] \in [Y, S^1] \cong [Y, K(1, \mathbb{Z})] \cong H^1(Y, \mathbb{Z})$, and de Rham representative for $[u]$ is $\frac{1}{2\pi i}u^{-1}du$.

We have $\mathcal{L}(u(B, \Psi)) - \mathcal{L}(B, \Psi) = 2\pi^2([u] \cup c_1(S))[Y]$. So $\mathcal{L}$ descends to a $\mathbb{R}/2\pi^2\mathbb{Z} \cong S^1$-valued functional that is invariant under $\mathcal{G}_Y$.

2.3. **Negative gradient flow equation as a 4d SW equation.** Consider a path $\mathbb{R} \to$ configuration space of pairs, $t \mapsto (B(t), \Psi(t))$, satisfying the negative gradient flow equation for $\mathcal{L}$. $\frac{\partial}{\partial t}B = -(\frac{1}{2} * F_{B^t} + \rho^{-1}(\Psi\Psi^*)_0)$ (here we omit the $\otimes \mathrm{Id}_S$ on RHS, which induces an equation for B^t) and $\frac{\partial}{\partial t}\Psi = -D_B\Psi$.

We can construct 4-manifold $\mathbb{R}_t \times Y =: Z$ and spin^c structure (S_Z, ρ_Z), where $S_Z = S^+ \oplus S^- = S \oplus S$, and $\rho_Z : TZ \to \mathrm{Hom}(S_Z, S_Z)$ defined as

$$\rho_Z(\frac{\partial}{\partial t}) = \begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix} \quad \text{and} \quad \rho_Z(v) := \begin{pmatrix} 0 & -\rho(v)^* \\ \rho(v) & 0 \end{pmatrix} \text{ for } v \in TY.$$

Time-dependent spin^c connection $B(t)$ on S gives a spin^c connection A on S_Z. $\nabla_A := \frac{\partial}{\partial t} + \nabla_B$ is in temporal gauge, namely trivial in $\mathbb{R}_t$ factor.

We have $D_{A^t} = \frac{\partial}{\partial t} + D_{B^t}$ and $F_{A^t} = dt \wedge (\frac{\partial}{\partial t}B^t) + F_{B^t}$.

Exercise 2.1. Recall the Hodge star $*_n$, and check $*_4 F_{A^t} = *_3(\frac{\partial}{\partial t}B^t) + dt \wedge *F_{B^t}$.

From the above, we see that the native gradient flow equation for $\mathcal{L}$ on (the configuration space of pairs on) Y is 4d SW equation on Z. The converse is also true up to gauge transformation (action of gauge group), which is left as an exercise.

2.4. **Morse theory for manifold without boundary.** We quickly review the case without boundary. Let (B, g_B) be a smooth closed Riemannian manifold. (C.f. K-M, Hutchings and Schwarz.) We are discussing the Morse-Witten picture, not the classical handlebody picture, which can generalize appropriately.

For $f : B \to \mathbb{R}$ real valued, can define its gradient ∇f as above via $g_B(\nabla f, v) = df(v)$. Consider the negative gradient flow equation $\dot{x} = -\nabla f(x)$ with $x = x(t)$ and dot means time differentiation; or in the flow notation $\phi_t(x)$ (with initial condition x at $t = 0$) satisfying $\dot{\phi}_t(x) = -\nabla f(\phi_t(x))$.

Definition 2.2. f is Morse, if at each critical point a (exactly where $\nabla f(a) = 0$, and we denote the set of critical points of f as $\mathrm{Crit}(f)$), its (self-adjoint) Hessian $\nabla(\nabla f) : T_aB \to T_aB$ has no kernel. Thus $T_aB = K_a^+ \oplus K_a^-$, into positive and negative eigenspaces. Its index $i(a) := \dim K_a^-$.

We use intersection theoretic instead of functional analytic approach in this introduction.

For critical point a, denote the stable manifold $S_a := \{x \mid \lim_{t\to\infty} \phi_t(x) = a\}$ and the unstable manifold $U_a := \{x \mid \lim_{t\to-\infty} \phi_t(x) = a\}$. Note that they are smooth at a due to exponential convergence. We also have $T_aU_a = K_a^-$. Denote

$$\begin{aligned} M(a, b) :&= \{\text{points along the flow lines from } a \text{ to } b\} \\ &= \{x \mid \phi_t(x) \text{ is a flow line from } a \text{ to } b\} \\ &= U_a \cap S_b \text{ in } B. \end{aligned}$$

Definition 2.3. $-\nabla f$ is Morse-Smale, if all U_a and S_b intersect transversely (meaning $T_yU_a + T_yS_b = T_yB$ for all $y \in U_a \cap S_b$) or all $a, b \in \mathrm{Crit}(f)$.

Then $\dim M(a,b) = \dim U_a + \dim S_b - \dim B = \dim U_a - \dim U_b = i(a) - i(b)$.

In the second description, $\mathbb{R}$ acts on $M(a,b)$ via $\phi_t(\cdot)$.

If $a \neq b$, then $\check{M}(a,b) := M(a,b)/\mathbb{R}$, the space of unparametrized flow lines, is a Hausdorff manifold.

We define the Morse chain complex (C_*, d) as follows:

$C_k := \bigoplus_{a \in \mathrm{Crit}(f), i(a)=k} \mathbb{Z}/2\mathbb{Z}e_a$.

For a, b with $i(a) - i(b) = 1$, we have $\check{M}(a,b)$ 0-dimensional compact manifold, and we can count number of points mod 2 and denoted by $n(a,b)$.

Define the diffeential $\partial e_a := \sum_{b \in \mathrm{Crit}(f), i(b)=k-1} n(a,b)e_b$.

We have $\partial^2 = 0$, being a chain complex, because for a, c with $i(a) - i(c) = 2$, the coefficient of e_c in $\partial\partial e_a$, $\sum_{b \in \mathrm{Crit}(f), i(b)=k-1} n(a,b)n(b,c)$, and it is 0 mod 2.

The last claim follows because we can compactify $\check{M}(a,c)$ into $\check{M}^+(a,c)$ by adding broken unparametrized flow lines. So $\check{M}^+(a,c)$ is compact 1-manifold with boundary exactly being broken flow lines from a to b to c for some b. Such manifold has boundary counted 0 (mod 2) $= \sum_{b \in \mathrm{Crit}(f), i(b)=k-1} n(a,b)n(b,c)$.

Its homology $H_*(C_*, \partial) = H_*(B; \mathbb{Z}/2\mathbb{Z})$.

2.5. **Morse theory for manifold with vertical boundary.** As will be for the most of this course, we follow closely Kronheimer-Mrowka's.

As we have see, the ambient configuration space modulo gauge group to define SW equation/flow equation of $\mathcal{L}$ has singularity at (B, Ψ) with $\Psi = 0$. We will see that a resolution will produce a manifold with boundary whose lift from $\nabla\mathcal{L}$ is tangent to the boundary. So we need to look at Morse theory in this case. The smoothness is best addressed using a doubling construction.

Let B be a manifold with boundary ∂B. To talk about smoothness and etc, let us consider its double, namely, a manifold $\tilde{B}$ without boundary and with a smooth involution $\iota : \tilde{B} \to \tilde{B}$ with fixed point codimension 1 and $\tilde{B}/\iota$ identified with B (thus fixed point set of ι with ∂B). We only consider Riemannian metric (resp. function f) on B that is restricted from (or extendable to) an ι-invariant Riemannian metric on $\tilde{B}$ (resp. $\tilde{f}$). In particular, $\nabla f|_{\partial B} \subset T\partial B$. We suppress this in the background.

Let f be a Morse function on B, then it has critical point in $B \backslash \partial B$, denoted by $\mathfrak{c}^o$ and critical point in ∂B, denoted by $\mathfrak{c}^\partial$. The normal vector (does not matter inwards or outwards as it does not change eigenvalue) ν to ∂B at $a \in \mathfrak{c}^\partial$ is a eigenvector of Hessian. To see this, note that at critical point $a \in \partial B$,

$$g_B(\nabla_\nu \nabla f, w) \overset{\text{self-adjoint}}{=} g_B(\nabla_w \nabla f, \nu) = 0 \text{ for all } w \in T_a\partial B,$$

as ∇f is along $T\partial B$, which means $\nabla_\nu \nabla f$ is a (non-zero being Morse) multiple of ν. Therefore ν either lies in K_a^+, then we denote $a \in \mathfrak{c}^s$ and call a boundary-stable, or it lies in K_a^-, then we denote $a \in \mathfrak{c}^u$, and call a boundary-unstable.

Draw a diagram illustrating the above, which can be made into higher dimensional. The diagram will be updated soon.

Remark 2.4. For $a \in \mathfrak{c}^s$ and $b \in \mathfrak{c}^u$, we have $U_a \subset \partial B$ and $S_b \subset \partial B$, so U_a and S_b cannot have transverse intersection in B. But it makes sense and we can ask the next best thing, transverse in ∂B.

Definition 2.5. $-\nabla f$ is regular, if for $a \in \mathfrak{c}^s$ and $b \in \mathfrak{c}^u$, we have U_a and S_b intersect transversely in ∂B, otherwise, U_a and S_b intersect transverse n B.

Then $M(a,b)$ is a manifold of dimension $i(a) - i(b) + 1$ in the first case (as we subtract 1 dimension less), usual formula otherwise.

In this setting, we can have a broken flow line configuration that is not a limiting flow from smooth flow lines from a to c. Draw a picture of a broken flow line from a to b to c, where $S_c \subset \partial B$ but U_a with $a \in B \backslash \partial B$. A smooth flow line from a to c has to be both in ∂B and $B \backslash \partial B$. Not a pathology, as we need to examine a more complete picture, and include only either kind of boundary-critical points if including interior critical points, to be seen next.

Draw a complete picture with index difference 2 but 2-time broken flow line. $a, d \in \mathfrak{c}^o$, $b \in \mathfrak{c}^s$, $c \in \mathfrak{c}^u$ for the unparametrized broken configuration to exist in dimension 0, we need to have $i(a) - i(b) = 1$, $i(b) - i(c) = 0$ (recall i is defined in B) and $i(c) - i(d) = 1$.

To be continued in the next lecture.

3. Lecture 3

3.1. **Broken flow lines not being limited from smooth ones.** We have seen:

- In proving $\partial^2 = 0$, it is crucial to have broken flow lines being limited to by smooth flow lines. There, 1-broken flow lines are boundary (points) of the compact 1-dimensional "flow space" manifolds (thus counted as 0 mod 2). Namely, starting from a 1-broken flow line and moving inside the flow space, flow lines become smooth until reaching the boundary of the flow space, which is another 1-broken one, thus 1-broke flow lines exist in pairs.
- Notation ∂^2 is a succinct way to keep track of broke flow lines.
- In the case where the background B has vertical boundary ∂B, we can have broken flow lines not limits of smooth ones. For example, there is no smooth flow lines limiting to a 1-broken flow line from $a \in \mathfrak{c}^o$ to $b \in \mathfrak{c}^s$ to $c \in \mathfrak{c}^u$ (because $U_c \subset \partial B$, the smooth flow lines from a stays in $B \backslash B$, thus the limiting smooth flow lines would have to be in both $B \backslash \partial B$ and ∂B).

Remark 3.1. Due to bullet points 1 and 3, to have a chain complex with both $\mathfrak{c}^o$ and $\mathfrak{c}^\partial$, cannot have both $\mathfrak{c}^s$ and $\mathfrak{c}^u$ in the same complex.

3.2. **Examining the boundary combinatorial types of broken flow lines.**

Lemma 3.2. *Consider B with ∂B with metric and Morse function respecting the doubling. Let $a \in \mathfrak{c}^o_k$, $c \in \mathfrak{c}^o_{k-2}$. Recall $\check{M}(a,c) = M(a,c)/\mathbb{R}$ denotes the space of unparametrized smooth flow lines, and $\check{M}^+(a,c)$ is the compactification of it by adding broken flow lines (limits). Then $\check{M}^+(a,c) \backslash \check{M}(a,c)$ consists of*

either $(\check{x}_1, \check{x}_2) \in \check{M}(a,b) \times \check{M}(b,c)$ *for some* $b \in \mathfrak{c}^o$,

or $(\check{x}_1, \check{x}_2, \check{x}_3) \in \check{M}(a,b_1) \times \check{M}(b_1,b_2) \times \check{M}(b_2,c)$ *for* $b_1 \in \mathfrak{c}^s_{k-1}$ *and* $b_2 \in \mathfrak{c}^u_{k-1}$.

Proof. If broken once, then the intermediate critical point $\notin \mathfrak{c}^\partial$, because a critical point at the boundary cannot be both forwards and backwards limiting point of flow lines in the interior.

If broken twice with three flow lines, the middle flow line has to be exceptional case in the definition of regularity (we call it "obstructed" for short, in the view of transversality), from $\mathfrak{c}^s$ to $\mathfrak{c}^u$, without dropping in index. Constraints of limiting end points also mean that we cannot have adjacent obstructed flow lines.

Cannot be broken 3+ times, (due to the last line in the last paragraph), as we would need to have at least three flow lines connecting points in $\mathfrak{c}^o$, which would involve a factor with negative dimension (not possible due to being manifold). $\square$

3.3. **Definition of various operators $\bar{\partial}^*_*$ and ∂^*_*.** (Not to be confused with similarly looking differential operators.) Another notational remark: In the lecture for ease of writing we wrote c in the place of $\mathfrak{c}$ and $\mathscr{C}_*$ in the place of C_* below.

For $a, b \in \mathfrak{c}^\partial$, consider $M(a,b) = U_a \cap S_b$.

- for $a \in \mathfrak{c}^u, b \in \mathfrak{c}^s$, both $M(a,b)$ and $M^\partial(a,b) =: M(a,b) \cap \partial B$ are manifolds and they are distinct, in fact $\partial M(a,b) = M^\partial(a,b)$. (Note that the nature of critical points implies that $U_a \cap \partial B$ intersects transversely with $S_b \cap \partial B$ in ∂B.)
- for the other three cases $\big($(i) $a \in \mathfrak{c}^u, b \in \mathfrak{c}^u$, (ii) $a \in \mathfrak{c}^s, b \in \mathfrak{c}^s$, $a \in \mathfrak{c}^s, b \in \mathfrak{c}^u\big)$, we have $M(a,b) = M^\partial(a,b)$ being manifolds, as the flow lines have to lie in ∂B. Here, in case (iii), the obstructed case, $M(a,b)$ is manifold due to "being regular".

Denote $\check{M}^\partial(a,b) := M^\partial(a,b)/\mathbb{R}$. We define the following four operations $\bar{\partial}^*_*$ counting points (mod 2) in 0-dimensional manifold $\check{M}^\partial(a,b)$. Here $\bar{\cdot}$ signifies boundary ∂B. The index difference between the domain and the target in each case ensures the spaces to be counted are 0 dimensional.

Denote $C^u_k := \bigoplus_{a \in \mathfrak{c}^u_k} (\mathbb{Z}/2\mathbb{Z}) e_a$, where e_a denotes the generator labelled by a. Similarly for C^s_k.

Define $\bar{\partial}^u_s : C^u_k \to C^s_{k-2}$ (super/subscripts indicate flow lines flowing from top to bottom) by defining on generators and extending by linearity:

$\bar{\partial}^u_s(e_a) = \sum_{b \in \mathfrak{c}^s_{k-2}} |\check{M}^\partial(a,b)| e_b$, here $|\cdot|$ counts points of a 0-dimensional space mod 2, and index drops two because we mod by $\mathbb{R}$-action and restricting to the boundary of a manifold with boundary.

We contrast it with $\partial^u_s : C^u_k \to C^s_{k-1}$ defined by $e_a \mapsto \sum |\check{M}(a,b)| e_b$.

We can similarly define $\bar{\partial}^s_s : C^s_k \to C^s_{k-1}$, $\bar{\partial}^u_u : C^u_k \to C^u_{k-1}$ and $\bar{\partial}^s_u : C^s_k \to C^u_k$. Note that last preserves the index, $\bar{\partial}^s_u(e_a) = \sum_{b \in \mathfrak{c}^u_k} |\check{M}^\partial(a,b)| e_b$. (Here, recall that $\dim \check{M}^\partial(a,b) = \dim M^\partial(a,b) - 1 \overset{\text{regularity}}{=} \dim U_a + \dim S_b - \dim \partial B - 1 = i(a) - i(b)$.) We also have:

$$\begin{aligned}
&\partial^o_o : C^o_k \to C^o_{k-1}\\
&\partial^o_s : C^o_k \to C^s_{k-1}\\
&\partial^u_o : C^u_k \to C^o_{k-1}\\
&\partial^u_s : C^u_k \to C^s_{k-1}, \text{the last of which we have seen,}
\end{aligned}$$

as the only 4 possibilities counting dimension-0 space of unparametrized flow lines in $B \backslash \partial B$. Other combinations will lie inside ∂B and have been covered in above $\bar{\partial}^*_*$.

3.4. **Recasting boundary combinatorial types into equations.** The above lemma says ① $\partial^o_o \partial^o_o + \partial^u_o \bar{\partial}^s_u \partial^o_s = 0$ mod 2. Note that we work in $\mathbb{Z}/2\mathbb{Z}$, and $-$ is $+$.

By considering boundary of 1-dimensional compactified space we have:

For $a \in \mathfrak{c}_k^o$, $c \in \mathfrak{c}_{k-2}^s$, analogously consider configurations of broken flow lines in $\check{M}^+(a,c) \backslash \check{M}(a,c)$, we have ② $\partial_s^o \partial_o^o + \bar{\partial}_s^s \partial_s^o + \partial_s^u \bar{\partial}_u^s \partial_s^o = 0$.

For $a \in \mathfrak{c}_k^u$, $c \in \mathfrak{c}_{k-2}^o$, ③ $\partial_o^o \partial_o^u + \partial_o^u \bar{\partial}_u^u + \partial_o^u \bar{\partial}_u^s \partial_s^u = 0$.

For $a \in \mathfrak{c}_k^u$, $c \in \mathfrak{c}_{k-2}^s$, interesting case ④ $\bar{\partial}_s^u + \partial_s^o \partial_o^u + \bar{\partial}_s^s \partial_s^u + \partial_s^u \bar{\partial}_u^u + \partial_s^u \bar{\partial}_u^s \partial_s^u = 0$.

Note that in the case ④, the first term counts the dimensional 0 space, as we mod out by $\mathbb{R}$ and take the boundary. Draw a picture for the last term.

Remark 3.3. Here the magic is that a sequence of smooth flow lines can break in a limit into allowable types of broken flow lines, and broken flow lines can be glued back to smooth flow lines, and those are captured exactly as the situation near a point at the boundary in a 1-dimensional manifold with boundary. We have suppressed this crucial detail now. Proving such a statement is more accessible from the functional analytic viewpoint (to be seen later) than the intersection theoretic approach above, which might be easier to meet and visualize at the first.

3.5. **Three variants of chain complexes.** We try to build chain complexes using C_*^o, C_*^s and C_*^u.

Recall if we use C_*^o, can only use one of C_*^s and C_*^u, due to Remark 3.1.

If we do not use C_*^o, we can use both boundary critical points and define

$$\bar{C}_k := C_k^s \oplus C_{k+1}^u,$$

the last summand has an index shift because i is defined in B, and if defined in ∂B, $i^\partial = i - 1$. Define

$$\bar{\partial} = \begin{pmatrix} \bar{\partial}_s^s & \bar{\partial}_s^u \\ \bar{\partial}_u^s & \bar{\partial}_u^u \end{pmatrix}.$$

The off-diagonal operators $\bar{\partial}_s^u : C_{k+1}^u \to C_{k-1}^s \subset \bar{C}_{k-1}$ and $\bar{\partial}_u^s : C_k^s \to C_k^u \subset \bar{C}_{k-1}$ indeed counts the dimensional 0 spaces, as we discussed before. The complex $(\bar{C}_*, \bar{\partial})$ is none other than the Morse chain complex for ∂B, just with different (boundary) critical points distinguished. Arguing as in the last lecture, we have $\bar{\partial}^2 = 0$. We write this out into

$$\bar{\partial}_s^s \bar{\partial}_s^s + \bar{\partial}_s^u \bar{\partial}_u^s = 0. \quad (\ddagger)$$
$$\bar{\partial}_u^s \bar{\partial}_s^s + \bar{\partial}_u^u \bar{\partial}_u^s = 0. \quad (\star)$$
$$\bar{\partial}_s^s \bar{\partial}_s^u + \bar{\partial}_s^u \bar{\partial}_u^u = 0$$
$$\bar{\partial}_u^s \bar{\partial}_s^u + \bar{\partial}_u^u \bar{\partial}_u^u = 0$$

After Morse theory for ∂B, now we want to consider C_*^o as well, and we can define two versions:

$\check{C}_k := C_k^o \oplus C_k^s$ with differential $\check{\partial} = \begin{pmatrix} \partial_o^o & \partial_o^u \bar{\partial}_u^s \\ \partial_s^o & \bar{\partial}_s^s + \partial_s^u \bar{\partial}_u^s \end{pmatrix}$. Check is also pronounced as "to", as the interior flow lines flowing to the bundary-stable critical points here, the overhead arrow also points to C. The boundary operator may look complicated, but it just includes all counts of dimension 0 spaces of (broken) unparametrized flow lines between appropriate critical points.

$\hat{C}_k := C_k^o \oplus C_k^u$ with differential $\hat{\partial} = \begin{pmatrix} \partial_o^o & \partial_o^u \\ \bar{\partial}_u^s \partial_s^o & \bar{\partial}_u^u + \bar{\partial}_u^s \partial_s^u \end{pmatrix}$. Hat is also pronounced as "from", as the interior flow lines flowing from the bundary-unstable critical points here and the overhead arrow also points away from C.

We now show $\check{\partial}^2 = 0$. Composing the matrix with itself, we want to show:

- The (1,1) entry of $\check{\partial}^2$ is 0, namely $\partial_o^o \partial_o^o + \partial_o^u \bar{\partial}_u^s \partial_s^o = 0$ which is just ①.

- $\partial_s^o\partial_o^o + \bar\partial_s^s\partial_s^o + \partial_s^u\bar\partial_u^s\partial_s^o = 0$, which is just ②.
- $\partial_o^o\partial_o^u\bar\partial_u^s + \partial_o^u\bar\partial_u^s\bar\partial_s^s + \partial_o^u\bar\partial_u^s\partial_s^u\bar\partial_u^s$ cannot factorize, but changing the second term into $\partial_o^u(\bar\partial_u^u\bar\partial_u^s)$ according to $(\star)$, it reads now (LHS of ③)$\bar\partial_u^s = 0$.
- $\partial_s^o\partial_o^u\bar\partial_u^s + \bar\partial_s^s\bar\partial_s^s + \partial_s^u\bar\partial_u^s\bar\partial_s^s + \bar\partial_s^s\partial_s^u\bar\partial_u^s + \partial_s^u\bar\partial_u^s\partial_s^u\bar\partial_u^s$ will be of the form of (LHS of ④)$\bar\partial_u^s = 0$, after replacing the second term by $\bar\partial_s^u\bar\partial_u^s$ due to (‡) and the third term by $\partial_s^u(\bar\partial_u^u\bar\partial_u^s)$ due to $(\star)$.

Thus $\check\partial^2 = 0$.

Exercise 3.4. $\hat\partial^2 = 0$.

Remark 3.5. $(\check C_*, \check\partial)$ calculates $H_*(B;\mathbb{Z}/2\mathbb{Z})$, $(\hat C_*, \hat\partial)$ calculates $H_*(B, \partial B;\mathbb{Z}/2\mathbb{Z})$, and $(\bar C_*, \bar\partial)$ calculates $H(\partial B;\mathbb{Z}/2\mathbb{Z})$.

3.6. **LES.** SES $0 \to C_*(\partial B) \to C_*(B) \to C_*(B, \partial B) \to 0$ leads to the long exact sequence. Can homologies of those chain models $(\bar C_*, \bar\partial), (\check C_*, \check\partial), (\hat C_*, \hat\partial)$ fit into LES with induced morphisms from natural maps between these chain models?

The answer is yes. Define

$$\begin{aligned} i :&\bar C_k := C_k^s \oplus C_{k+1}^u \to \check C_k := C_k^o \oplus C_k^s \text{ by } i = \begin{pmatrix} 0 & \partial_o^u \\ 1 & \partial_s^u \end{pmatrix}. \\ j :&\check C_k := C_k^o \oplus C_k^s \to \hat C_k := C_k^o \oplus C_k^u \text{ by } j = \begin{pmatrix} 1 & 0 \\ 0 & \bar\partial_u^s \end{pmatrix}. \\ p :&\hat C_k := C_k^o \oplus C_k^u \to \check C_{k-1} := C_{k-1}^s \oplus C_k^u \text{ by } p = \begin{pmatrix} \partial_s^o & \partial_s^u \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Exercise 3.6. Check i, j and p are chain maps. (p is only a chain map up to a sign when working over $\mathbb{Z}$ after taking care of orientations of spaces).

Proposition 3.7. *There is an LES* $\cdots \to \check H_* \xrightarrow{j_*} \hat H_* \xrightarrow{p_*} \bar H_{*-1} \xrightarrow{i_*} \check H_{*-1} \to \cdots$.

This respects the LES from the above SES, which reads

$$\cdots \to H_*(B) \to H_*(B, \partial B) \to H_{*-1}(\partial B) \to H_{*-1}(B) \to \cdots.$$

The p_* is exhibited at the center of the repeated pattern because it plays a role in the proof where we want to identify $\check C$ under a quasi-isomorphism to

$$\mathrm{Cone}(p) := (\hat C \oplus \bar C, \begin{pmatrix} \hat\partial & 0 \\ p & \bar\partial \end{pmatrix}).$$

We will prove the proposition next time and discuss the compactness of solutions to SW equation.

4. Lecture 4

4.1. **LES (continued).** SES $0 \to C_*(\partial B) \to C_*(B) \to C_*(B, \partial B) \to 0$ leads to the long exact sequence. As $(\bar C_*, \bar\partial), (\check C_*, \check\partial), (\hat C_*, \hat\partial)$ calculate respective homologies in the aforementioned LES, we can see the LES using natural maps between these chain models. Define

$$i : \bar{C}_k := C^s_k \oplus C^u_{k+1} \to \check{C}_k := C^o_k \oplus C^s_k \text{ by } i = \begin{pmatrix} 0 & \partial^u_o \\ 1 & \partial^u_s \end{pmatrix}.$$

$$j : \check{C}_k := C^o_k \oplus C^s_k \to \hat{C}_k := C^o_k \oplus C^u_k \text{ by } j = \begin{pmatrix} 1 & 0 \\ 0 & \bar{\partial}^s_u \end{pmatrix}.$$

$$p : \hat{C}_k := C^o_k \oplus C^u_k \to \check{C}_{k-1} := C^s_{k-1} \oplus C^u_k \text{ by } p = \begin{pmatrix} \partial^o_s & \partial^u_s \\ 0 & 1 \end{pmatrix}.$$

Proposition 4.1. *There is an LES* $\cdots \to \check{H}_* \xrightarrow{j_*} \hat{H}_* \xrightarrow{p_*} \bar{H}_{*-1} \xrightarrow{i_*} \check{H}_{*-1} \to \cdots$.

Proof. Define $\check{E} := \mathrm{Cone}(p) := (\hat{C} \oplus \bar{C}, \check{e} := \begin{pmatrix} \hat{\partial} & 0 \\ p & \bar{\partial} \end{pmatrix})$. (Anti-)chain map property of $p : \hat{C} \to \bar{C}$ is incorporated into $\check{e}^2 = 0$. By construction of mapping cone, we have SES $\bar{C} \xrightarrow{\bar{i}} \check{E} \xrightarrow{\bar{j}} \hat{C}$. Now we want to establish a quasi-isomorphism (map inducing isomorphism between homologies) between $\check{E}$ and $\check{C}$ (respecting the maps on homology), then we are done. Indeed, define

$$(C^0 \oplus C^s) \oplus (C^s \oplus C^u_{*+1}) = \check{E} \underset{k}{\overset{l}{\rightleftarrows}} \check{C} = C^o \oplus C^s,$$

where $k : (x, y) \mapsto (x, \bar{\partial}^s_u y, y, 0)$, and $l : (e, f, g, h) \mapsto (e + \partial^u_o h, g + \partial^u_s h)$. We have $l \circ k = \mathrm{Id}$ and $k \circ l = \mathrm{Id} + \check{e} \circ K + K \circ \check{e}$ for chain homotopy $K : (e, f, g, h) \mapsto (0, h, 0, 0)$. Moreover, $j_* = \bar{j}_* \circ k_*$ and $\bar{i}_* = k_* \circ i_*$.

□

4.2. **Weizenböck, 4d-3d expression, and energies.** We follow [KM], also c.f. [Morgan].

Let $X = (X, g_X)$ be a compact oriented Riemannian 4-manifold with $\partial X = Y$, where the metric is cylindrical metric near Y (as $[-\epsilon, 0] \times Y$). A spin^c structure $\mathfrak{s}_X = (S_X, \rho_X)$ induces $s = (S, \rho)$ along the boundary Y as follows. Denote $n \in \Gamma(Y)$ the outward unit normal vector field, then $\rho_X(n) : S^+|_Y \xrightarrow{\cong} S^-|_Y$, and define $S := S^+|_Y$. Let $v \in TY$, then $\rho(v)$ is $S^+|_Y \xrightarrow{\rho_X(v)} S^-|_Y \xrightarrow{\rho(n)^{-1}} S^+|_Y$.

Let A (or ∇_A) be a spin^c connection in temporal gauge, so restricts to Y to a spin^c connection on Y.

We have Weitzenböck formula $D^-_A D^+_A \Phi = \nabla^*_A \nabla_A \Phi + \frac{1}{2}\rho_X(F^+_{A^t})\Phi + \frac{1}{4}s\Phi$, where s is the scalar curvature for Levi-Civita connection ∇ on X recalled below:

For curvature $F(X,Y)Z := \nabla_X \nabla_Y \tilde{Z} - \nabla_Y \nabla_X \tilde{Z} - \nabla_{[\tilde{X},\tilde{Y}]|}\tilde{Z}$, where $\tilde{\cdot}$ denotes any extension to a vector field and $\cdot|$ denotes restriction to a point (well-defined independent of choices), we have $g_X(F(X,Y)Z, W)$ anti-symmetric in X and Y, anti-symmetric in Z and W, and symmetric in (X, Y) and (Z, W). Define Ricci curvature $\mathrm{Ric}(X, Y) := \sum_i g_X(F(e_i, X)Y, e_i)$ for any orthonormal basis e_i (this order of summing reproduces Gaussian curvature in 2d), and define $s := \mathrm{trRic}$.

Note that adjoint operator is defined using $C^\infty_c(X \backslash \partial X)$ (smooth function of compact support away from boundary) in L^2 metric, so having boundry or not does not affect the formula. As $D_A = \begin{pmatrix} 0 & D^-_A \\ D^+_A & 0 \end{pmatrix})$ is self-adjoint, D^-_A is adjoint to D^+_A. ρ_X maps imaginary valued self-adjoint 2-form to Hermitian (self-adjoint). So all four operators in front of Φ are self-adjoint.

Adjoint expression involving terms on Y (omitting $d$vol):

$\int_Z \langle \Phi, D_A^- D_A^+ \Phi \rangle = \int_X |D_A^+ \Phi|^2 - \int_Y \langle \rho_X(n)\Phi, D_A^+\Phi\rangle$, and
$\int_Z \langle \Phi, \nabla_A^* \nabla_A \Phi \rangle = \int_X |\nabla_A \Phi|^2 - \int_Y \langle \Phi, (\nabla_A)_n \Phi\rangle$.
One can derive from the definition that

$$D_B(\Phi|_Y) = (\rho_X(n)^{-1} D_A^+\Phi - (\nabla_A)_n \Phi)|_Y + \frac{H}{2}\Phi|_Y,$$

where H-term can be dropped if using cylindrical metric near Y.

Take $\langle \Phi, \cdot \rangle$ to the Weitzenböck formula and integrate and using the above expression about adjoint (involving boundary terms), we have:
$\|\mathcal{F}(A,\Phi)\|_{L^2}^2 := \int_X (|\frac{1}{2}\rho_X(F_{A^t}^+) - (\Phi\Phi^*)_0|^2 + |D_A^+\Phi|^2) = \mathcal{E}^{\text{an}} - \mathcal{E}^{\text{top}}$, where
$\mathcal{E}^{\text{an}} := \frac{1}{4}\int_X |F_{A^t}|^2 + \int_X |\nabla_A \Phi|^2 + \frac{1}{4}\int_X (|\Phi|^2 + \frac{s}{2})^2 - \int_X \frac{s^2}{16}$, and
$\mathcal{E}^{\text{top}} := \frac{1}{4}\int_X F_{A^t} \wedge F_{A^t} - \int_Y \langle \Phi|_Y, D_B(\Phi|_Y)\rangle + \int_Y \frac{H}{2}|\Phi|^2$.
In the cylindrical situation $[t_1,t_2]\times Y$ (which we currently have near Y) (denoting $\gamma = (A,\Phi)$ in 4d as $\gamma(t)$ in 3d), $\mathcal{E}^{\text{an}} = \int_{t_1}^{t_2} (|\dot\gamma|^2 + |\nabla \mathcal{L}(\gamma)|^2)dt$ (see [KM] (4.20) for a gauge invariant expression), and $\mathcal{E}^{\text{top}} = 2(\mathcal{L}(t_1) - \mathcal{L}(t_2))$.

For SW equation solution (A,Φ), $\mathcal{E}^{\text{an}} = \mathcal{E}^{\text{top}}$.

Exercise 4.2. Show $\int |F_{A^t}|^2 - \int F_{A^t}\wedge F_{A^t} = 2\int_X |F_{A^t}^+|^2$, and use this and the above to show for $s \geq 0$, we have $\Phi = 0$ for SW solution.

4.3. **Compactness theorem.**

Theorem 4.3. *Let us be in the above setting.*

(1) For any constant C, only finitely many $\mathfrak{s}_X$'s admit solution (A,Φ) to SW with $\mathcal{E}^{top}(A,\Phi) \leq C$.
(2) Let (A_n,Φ_n) be a sequence of smooth SW solution with $\mathcal{E}^{top}$-bound C. Then exist smooth gauge transformations $u_n : X \to S^1$ such that
  - *(a) a subsequence of $u_n(A_n,\Phi_n) \overset{\text{weakly in } L_1^2}{\rightarrow} (A,\Phi)$ for some $(A,\Phi) \in L_1^2$ (explained below);*
  - *(b) if the same subsequence (denoted with same index) satisfies*

$$\mathit{limsup}\, \mathcal{E}^{top}(A_n,\Phi_n) = \mathcal{E}^{top}(A,\Phi),$$

    then convergence $u_n(A_n,\Phi_n)$ to (A,Φ) in L_1^2 is strong; and
  - *(c) the same subsequence (without need to satisfying hypothesis in (b)) converges in C^∞ on every $X' \subset\subset X \backslash \partial X$.*

Here L_k^p is Sobolev space, completion of smooth functions/sections in $\|f\|_{L_k^p} := (\sum_{0\leq i\leq k}\int_X |\nabla^i f|^p d\text{vol})^{1/p}$, with $1 < p < \infty$. Finite regularity but complete.

Let H be a Banach space, with dual H^*, $a_n \to a$ weakly, if for all $f \in H^*$, $f(a_n) - f(a) \to 0$. If H is Hilbert with inner product $\langle\cdot,\cdot\rangle$, $a_n \to a$ weakly if for all $f \in H$, $\langle a_n - a, f\rangle \to 0$. $a_n \to a$ (strongly) in H, if $\|a_n - a\|_H \to 0$. As an example, a orthonormal countable basis converges weakly to 0 but not strongly.

Proof. For SW solution, we have $\mathcal{E}^{\text{an}} = \mathcal{E}^{\text{top}}$. We then have $\int_X |F_{A_n^t}|^2 \leq C_1$, $\int_X |\Phi_n|^4 \leq C_2$ (can also seen from first SW equation), and $\int_X |\nabla_{A_n}\Phi_n|^2 \leq C_3$ (as we can bound s due to compactness). The first gives that $c_1(\mathfrak{s}_X)$ lies in a compact set. spin^c structure in 4d is also affine over H^2, which gives conclusion (i).

Therefore, can restrict to a fixed spin^c structure and fix a base spin^c connection A_0, we want to choose $u_n' : X \to S^1$ such that

$$d^*(A_n^t - A_0^t - 2(u_n')^{-1}du_n') = 0 \text{ in } X$$
$$\langle A_n^t - A_0^t - 2(u_n')^{-1}du_n', n\rangle = 0 \text{ at } \partial X$$

where non-subscript n is the unit outward normal.

u_n' can be of the form e^{ξ_n} for $\xi_n : X \to i\mathbb{R}$, if we can solve

$$2\Delta\xi_n = d^*(A_n^t - A_0^t) \quad \text{in } X$$
$$2\langle d\xi_n, n\rangle = \langle A_n^t - A_0^t, n\rangle \quad \text{at } \partial X.$$

This is Neumann boundary value problem. e^{ξ_n} is unique up to multiplying by constant for this trivial homotopy class case $[u_n'] = 0$.

$A_n - (u_n')^{-1}du_n' =: u_n'(A_n)$ is said to be in Coulomb-Neumann gauge if the above pair of conditions for $u_n' = e^{\xi_n}$ holds.

For non-trivial homotopy class $a \in [X, S^1]$, there exists $v : X \to S^1$ with $[v] = a$ satisfying the homogeneous equation (thus the Coulomb-Neumann gauge condition can be solved for any homotopy class)

$$d^*(v^{-1}dv) = 0 \quad \text{in } X$$
$$\langle v^{-1}dv, n\rangle = 0 \quad \text{at } \partial X.$$

We have uniqueness if asking further $i\int \beta_r \wedge (A_n^t - A_0^t - 2u_n^{-1}du_n) \in [0, 2\pi)$, where $\{\beta_r\}$ represents basis of $H^3(X;\mathbb{R})$ (this can be viewed as period condition on loops via Poincaré duality).

We need a lemma whose proof is delegated to the exercise session (see [KM] 5.1.2, 5.1.3 and the paragraph that follows):

Lemma 4.4. *For any imaginary-valued 1-form a satisfying $\langle a, n\rangle = 0$ at ∂X and $i\int \beta_r \wedge a \in [0, 2\pi)$, we have*

$$\|a\|_{L_1^2}^2 := \int_X (|\nabla a|^2 + |a|^2)dvol \leq K_1 \int_X (|d^*a|^2 + |da|^2)dvol + K_2$$

for K_i constant.

To see (2)(a), write $(\tilde{A}_n, \tilde{\Phi}_n) := (u_n(A_n), u_n\Phi_n)$. Apply Lemma 4.4 to $\tilde{A}_n^t - A_0^t$, then $|d^*a|^2$-term on RHS is 0 due to Coulomb gauge, and $\int_X |da|^2$ term is bounded due to curvature bound (and finiteness of $F_{A_0^t}$ in L^2), so we get a L_1^2 bound for $\tilde{A}_n^t - A_0^t$.

We have Sobolev embedding $L_1^p \hookrightarrow L^{p^*}$, where $\frac{1}{p^*} := \frac{1}{p} - \frac{1}{\dim X} = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$. (Useful when $\dim X > p$ so that $p^* > 0$. Note that $p^* > p > 1$.) So we have L^4 bound for $\tilde{A}_n^t - A_0^t$.

We have $\|\nabla_{\tilde{A}_n}\tilde{\Phi}_n\|_{L^2}$ bounded at the start of the proof.

Then $\nabla_{A_0}\tilde{\Phi}_n = \nabla_{\tilde{A}_n}\tilde{\Phi}_n - (\tilde{A}_n - A_0)\tilde{\Phi}_n$ is L^2 bounded as the last term has both factors L^4 bounded, thus itself L^2 bounded (by Cauchy-Schwarz inequality) and the first term is $\nabla_{\tilde{A}_n}\tilde{\Phi}_n = u_n(\nabla_{A_n}(u_n^{-1}(u_n(\Phi_n)))) = u(\nabla_{A_n}\Phi_n)$ has the same norm as $\nabla_{A_n}\Phi_n$ which is L^2 bounded at the start of the proof.

We also have $\tilde{\Phi}_n$ L^2 bounded (due to L^4 bounded and compactness of X via Cauchy-Schwarz), thus $\|\tilde{\Phi}_n\|_{L_1^2}$ is uniformly bounded.

$(L^2_1)^* \cong L^2_1$. As unit ball in $(L^2_1)^*$ is weakly compact. We have a subsequence $\tilde{\Phi}_n$ weakly converging to a limit. This completes (2)(a).

To be continued. □

5. Lecture 5

We continue to follow [KM] closely and at a few places flesh out some details.

5.1. **Recall where we were at from last time.** We have Seiberg-Witten equation for (A, Φ) over a compact Riemannian 4-manifold X with $\partial X = Y$:

$$\frac{1}{2}\rho_X(F^+_{A^t}) - (\Phi\Phi^*)_0 = 0$$
$$D^+_A \Phi = 0$$

We have two notions of energies:

$\mathcal{E}^{\mathrm{top}}(A, \Phi) = \frac{1}{4}\int_X F_{A^t} \wedge F_{A^t} - \int_Y \langle \Phi|_Y, D_B(\Phi|_Y) + \int_Y \frac{H}{2}|\Phi|^2$, where $N(V, W)n := (\nabla_V \tilde{W})^\perp$ and mean curvature $H := \mathrm{tr}_Y N$.

$\mathcal{E}^{\mathrm{an}}(A, \Phi) := \frac{1}{4}\int_X |F_{A^t}|^2 + \int_X |\nabla_A \Phi|^2 + \frac{1}{4}\int_X (|\Phi|^2 + \frac{s}{2})^2 - \int_X \frac{s^2}{16}$, where $s = \mathrm{tr}_X \mathrm{Ric}$ and $\mathrm{Ric}(V, W) = \mathrm{tr}_X g_X(R(\cdot, V)W, \cdot)$.

For SW solution (A, Φ), $\mathcal{E}^{\mathrm{top}}(A, \Phi) = \mathcal{E}^{\mathrm{an}}(A, \Phi)$.

We stated and proved the (1) and (2)(a) of the following compactness theorem (interior compactness up to gauge transformation under finite topological energy):

Theorem 5.1. *(1) Finiteness of spin^c structures admitting SW solutions under a given $\mathcal{E}^{top}$-bound.*

(2) Sequential compactness up to gauge transformation under finite $\mathcal{E}^{top}$-bound: Let (A_n, Φ_n) be a sequence of SW solution with $\mathcal{E}^{top}(A_n, \Phi_n) \le C < \infty$. There exists $u_n : X \to S^1$ such that

(a) a subsequence of $(\tilde{A}_n, \tilde{\Phi}_n) := u_n(A_n, \Phi_n) \underset{in\ L^2_1}{\overset{weakly}{\to}} (A, \Phi) \in L^2_1$;

(b) If $\mathrm{limsup}_n \mathcal{E}^{top}(A_n, \Phi_n) = \mathcal{E}^{top}(A, \Phi)$, then the same subsequence in (a) converges to (A, Φ) (strongly) in L^2_1; and

(c) the subsequence in (a) converges in $C^\infty_{loc}(X \backslash \partial X)$ (namely, in $C^\infty(X')$ for any open domain $X' \subset\subset X \backslash \partial X$).

We recalled again what Banach L^k_p and weak convergence for Banach/Hilbert space are.

5.2. **Proof of (2)(b), norm preserving plus weak convergence imply strong convergence.** We prove (2)(b).

Note that $\mathcal{E}^{\mathrm{top}}(A_n, \Phi_n) = \mathcal{E}^{\mathrm{an}}(A_n, \Phi_n) = \mathcal{E}^{\mathrm{an}}(\tilde{A}_n, \tilde{\Phi}_n)$ as

$$|\nabla_{A_n}\Phi_n| = |u_n(\nabla_{A_n}\Phi_n)| = |(u_n \circ \nabla_{A_n} \circ u_n^{-1})(u_n\Phi_n)| = |\nabla_{\tilde{A}_n}\tilde{\Phi}_n|.$$

Then the hypothesis of (2)(b) means that we have uniform L^2-bound of the following $(F^+_{\tilde{A}_n})$, $\nabla_{\tilde{A}_n}\tilde{\Phi}_n$, and $(\tilde{\Phi}_n\tilde{\Phi}^*_n)_0$. Recall that L^2 norm of the last is a constant factor of $\|\tilde{\Phi}_n\|_{L^4}$ as we have seen.

$L^2 \cong (L^2)^*$, by Banach-Alaoglu which says that the unit/bounded ball in dual space is weakly compact, we have a common subsequence of triples weakly converges in L^2 to some limit. We also have $(\tilde{A}_n, \tilde{\Phi}_n)$ converges strongly in L^2.

We want to establish the weak limit of the triples:

- L^2 weak limit of $F^+_{\tilde{A}_n}$ is F^+_A.

  Indeed, we have $\langle \tilde{A}_n - A, d^*b\rangle = \langle \tilde{A}_n - dA, b\rangle \to 0$ for any smooth b compactly supported away from ∂X. Its self-dual projection says that $\langle F^+_{\tilde{A}_n} - F^+_A, b\rangle \to 0$ for all b.
- Weak limit of $\nabla_{\tilde{A}_n}\tilde{\Phi}_n$ is $\nabla_A\Phi$.

  Let $\tilde{A}_n = A_0 + a_n$ and $A = A_0 + a$ for a base connection A_0. We have $\nabla_{\tilde{A}_n}\tilde{\Phi}_n = \nabla_{A_0}\tilde{\Phi}_n + a_n\tilde{\Phi}_n$. The first term on RHS converges weakly in L^2 to $\nabla_{A_0}\Phi$ by an argument similar to the previous item. The second term on RHS converges in L^1 to $a\Phi$ in particular weakly converges to $a\Phi$. To see the L^1 convergences, note that both factors converge in L^2 to a and Φ, and we use Cauchy-Schwarz (CS) $\int|\alpha\beta| \le (\int|\alpha|^2)^{1/2}(\int|\beta|^2)^{1/2}$. So we have $\nabla_{\tilde{A}_n}\tilde{\Phi}_n$ converges weakly in L^2 to $\nabla_{A_0}\Phi + a\Phi = \nabla_A\Phi$.
- Similarly, weak limit of $(\tilde{\Phi}_n\tilde{\Phi}_n^*)$ in L^2 is $(\Phi\Phi^*)_0$.

In particular, (A, Φ) is an SW solution.

Recall a lemma, for Hilbert space (here we look at L^2), if $x_n \to x$ weakly in L^2 and $\lim_n \|x_n\|$ exists and equals to $\|x\|_{L^2}$, then x_n converges strongly to x in L^2. Proof is one line,

$$\|x_n - x\|^2 = \langle x_n - x, x_n - x\rangle = \|x_n\|^2 + \|x\|^2 - 2\langle x_n, x\rangle \to 2\|x\|^2 - 2\langle x, x\rangle = 0.$$

We have the norm preserving statement for three terms together, we can separate them because we have $\|x\| \le \text{limsup}_n\|x_n\|$.

L^2 norm preserving in limit for $F^+_{\tilde{A}_n}$, $\nabla_{\tilde{A}_n}\tilde{\Phi}_n$ and $(\tilde{\Phi}_n\tilde{\Phi}_n^*)_0$ respectively means strong convergence in L^2 to F^+_A, $\nabla_A\Phi$ and $(\Phi\Phi^*)_0$ respectively.

First strong convergence says $\tilde{A}_n \to A$ in L^2_1 thus in L^4, recall that Sobolev embedding we went over $2^* = 4$ in this case. Third strong convergence means $\tilde{\Phi}_n \to \Phi$ in L^4. Putting both together and using CS again, we have $(A_0 - \tilde{A}_n)\tilde{\Phi}_n$ to $(A_0 - A)\Phi$ in L^2 strongly. Together with $\nabla_{\tilde{A}_n}\tilde{\Phi}_n \to \nabla_A\Phi$ strongly in L_2 before, $\nabla_{A_0}\tilde{\Phi}_n = \nabla_{\tilde{A}_n}\tilde{\Phi}_n + (A_0 - \tilde{A}_n)\tilde{\Phi}_n$ converges strongly in L^2 to $\nabla_A\Phi + (A_0 - A)\Phi = \nabla_{A_0}\Phi$. This finishes (2)(b).

5.3. **(2)(c) Two claims and abstract SW with gauge fixing.** We can prove (2)(c) if the following two claims hold.

Claim 1: L^2_1-converging sequence of smooth solutions in Coulomb gauge converges in C^∞ on every interior domain $X' \subset\subset X\backslash\partial X$.

Claim 2: On any interior domain, hypothesis in (2)(b) holds.

So basically, Claim 1 says that we can get the conclusion of (2)(c) from conclusion of 2(b); and Claim 2 says that we prove starting point of (2)(b) on any interior domain. Validity of both claims immediately gives (2)(c). (So (2)(b) was a tool in the proof.)

To prove Claim 1 using elliptic estimate, we first render SW into abstract form, so that the argument is instructive and transferable to other similar settings.

Denote $A = A_0 + a$. SW+Coulomb gauge fixing is

$$\begin{aligned} \frac{1}{2}\rho_X(F_{A_0^+}) + \rho_X(d^+a) - (\Phi\Phi^*)_0 &= 0 \\ D_{A_0}\Phi + a\Phi &= 0 \\ d^*a &= 0 \end{aligned}$$

Terms are collected into $D : \Gamma(iT^*X \oplus S^+) \to \Gamma(i\mathbb{R} \oplus i\mathfrak{su}(S^+) \oplus S^-)$,

$$(a, \Phi) \mapsto (d^*a, \rho(d^+a), D_{A_0}\Phi).$$

Write $\gamma := (a, \Phi)$.

Terms can be written as $Q(\gamma, \gamma)$ for a symmetric bilinear form

$$Q(\gamma, \hat{\gamma}) := (-\frac{1}{2}(\Phi\hat{\Phi}^* + \hat{\Phi}\Phi^*), \frac{1}{2}(a\hat{\Phi} + \hat{a}\Phi, 0)).$$

The leftover term is denoted by $-b := (\frac{1}{2}\rho_X(F_{A_0^+}), 0, 0)$. So we have the abstract expression of the SW under Coulomb gauge. $D\gamma + Q(\gamma, \gamma) = b$.

5.4. **Elliptic operator and estimate.** The key fact is that D is elliptic, which allows the following elliptic estimate (semi-Fredholm estimate):

Theorem 5.2. *(Gårding inequality) Let D be a first order elliptic operator. Let $X^{(1)} \subset\subset X$. Then there exists constant C and for any smooth γ, we have*

$$\|\gamma\|_{L^p_{k+1}(X^{(1)})} \leq C(\|D\gamma\|_{L^p_k(X)} + \|\gamma\|_{L^p(X)}).$$

A differential operator $D : \Gamma(E) \to \Gamma(F)$ of order k (over $\mathbb{R}$ coefficient) between sections of bundles E and F over the same base X, if over trivializing neighborhood $U \subset \mathbb{R}^d$, $E|_U = U \times \mathbb{R}^m$ and $F|_U = U \times \mathbb{R}^n$ (so $\Gamma(E|_U) = (C^\infty(U))^m$ and $\Gamma(E|_U) = (C^\infty(U)^n)$), D is of the form

$$(f_1, \cdots, f_m) \mapsto (\sum_{i,|\alpha|\leq k} a_{1i\alpha}\partial^\alpha f_i, \cdots, \sum_{i,|\alpha|\leq k} a_{ni\alpha}\partial^\alpha f_i),$$

where $\partial^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}}$ for multi-index $\alpha = (\alpha_1, \cdots, \alpha_d)$ and $|\alpha| = \sum_j \alpha_j$.

The symbol of $D(x, \xi) : E_x \to F_x$ for $\xi \in T_x^*X$ in the above local coordinate is $(v_1, \cdots, v_m) \mapsto (\sum_{i,|\alpha|=k} a_{1i\alpha}\xi^\alpha v_i, \cdots, \sum_{i,|\alpha|=k} a_{ni\alpha}\xi^\alpha v_i)$, where $\xi^\alpha := \xi_1^{\alpha_1} \cdots \xi_d^{\alpha_d}$.

For coordinate-free way, $D(x, \xi)(v)$ is defined by choosing $f \in C^\infty(X)$ with $f(x) = 0$ and $d_x f = \xi$, and $e \in \Gamma(E)$ with $e(x) = v$, then $D(x, \xi)(v) := D(\frac{f^k}{k!}e)(x)$.

We will also write, for $\eta \in \Omega^1(X)$ and $V \in \Gamma(E)$,

$$\sigma(D, \eta)V := (x \mapsto D(x, \eta_x)V_x) \in \Gamma(F).$$

As an example, for $d : \Gamma(\wedge^p(T^*X)) \to \Gamma(\wedge^{p+1}(T^*X))$, $d(x, \xi)(\eta_x) = \xi \wedge \eta_x$.

A differential order is elliptic, if for any $\xi \in T_x^*X \backslash \{0\}$, $D(x, \xi) : E_x \to F_x$ is invertible.

Exercise 5.3. $D_{A_0} : \Gamma(S^+) \to \Gamma(S^-)$ is elliptic. For $\Gamma(E_1) \xrightarrow{D_1} \Gamma(E_2) \xrightarrow{D_2} \Gamma(E_3)$ where D_1 and D_2 are first order operators, such that for $\xi \in T_x^*X$ and for any $x \in X$,

$$(E_1)_x \xrightarrow{D_1(x,\xi)} (E_2)_x \xrightarrow{D_2(x,\xi)} (E_3)_x$$

is exact, then $\Gamma(E_2) \xrightarrow{D_1^* + D_2} \Gamma(E_1) \oplus \Gamma(E_3)$ is elliptic. Show $d^* + d^+$ is elliptic. Thus D in the SW with Coulomb gauge fixing is Elliptic.

5.5. **Proof of Claim 1.** For any interior domain $X' \subset\subset X\backslash\partial X$, choose cut off β with $\beta|_{X'} = 1$ and compactly supported in $X\backslash\partial X$.

In hypothesis, we have $\gamma_n \to \gamma$ in L^2_1. So for any $\epsilon > 0$, there exists i_0 such that $\|\gamma_i - \gamma_{i_0}\|_{L^2_1} \leq \epsilon$ for all $i \geq i_0$.

From the abstract expression, we have

$$0 = D(\gamma_i - \gamma_j) + (Q(\gamma_i, \gamma_i) - Q(\gamma_j, \gamma_j)) = D(\gamma_i - \gamma_j) + Q(\gamma_i - \gamma_j, \gamma_i + \gamma_j).$$

Then (‡) $\|\beta(\gamma_i - \gamma_j)\|_{L^p_{k+1}(X)} \leq C(\|D(\beta(\gamma_i - \gamma_j))\|_{L^p_k(X)} + \|\beta(\gamma_i - \gamma_j)\|_{L^p(X)})$ by Gårding.

First term on RHS inside the norm is $\beta D(\gamma_i - \gamma_j) + \sigma(D, d\beta)(\gamma_i - \gamma_j)$, see the notation for the second term in the previous subsection, which in L^p_k in particular is bounded multiple of $\|\gamma_i - \gamma_j\|_{L^p_k}$ (so is $\|\beta(\gamma_i - \gamma_j\|_{L^p})$.

We have

$$\begin{aligned} -\beta D(\gamma_i - \gamma_j) &= \beta Q(\gamma_i - \gamma_j, \gamma_i + \gamma_j) \\ &= Q(\beta(\gamma_i - \gamma_j), \gamma_i + \gamma_j - 2\gamma_{i_0}) + Q(\beta(\gamma_i - \gamma_j), 2\gamma_{i_0}). \end{aligned}$$

Recall Q here involves no differentiation and is just an algebraic bilinear form and can be regarded as (the projection with constant weight of) product of the factors.

Now we specialize to $L^p_{k+1} = L^3_1$ ($p = 3, k = 0$), we use $L^3_1 \times L^2_1 \to L^3, (a, b) \mapsto ab$ is bounded/continuous. First Q term in $L^2 \leq C\|\beta(\gamma_i - \gamma_j)\|_{L^3_1}\|\gamma_i + \gamma_j - 2\gamma_{i_0}\|_{L^2_1}$ whose second factor can be as small as we like ($\leq \epsilon$) and this term can be moved to the LHS of (‡) at the expense of increasing C by a factor. Thus we get

$$\|\beta(\gamma_i - \gamma_j)\|_{L^3_1} \leq C\|\gamma_i - \gamma_j\|_{L^3}.$$

We increase the regularity by 1 (here X' is arbitrary).

Specialize to $L^p_{k+1} = L^2_2$ and use $L^2_2 \times L^3_1 \to L^2_1$, we can L^2_2 bound in terms of L^2_1 bound.

Specialize to L^2_3 and use $L^2_3 \times L^2_2 \to L^2_2$, we can L^2_2 bound in terms of L^2_2 bound.

Specialize to L^{k+1}_2, for $k \geq 3$, we have the Banach algebra $L^2_k \times L^2_k \to L^2_k$, we get L^2_{k+1} bound in terms of L^2_k bound.

The above argument of getting increasingly better regularity is called elliptic bootstrapping.

Sobolve embedding $L^p_k \subset C^m$ for any $0 \leq m < k - \frac{\dim X}{p} \overset{\text{our case}}{=} k - \frac{4}{2} = k - 2$. So we have $L^2_{k+3} \subset C^k$. This finishes Claim 1.

Exercise 5.4. Using $L^2_k \subset C^{k-3}$ for $k \geq 3$ to show Banach algebra property $L^2_k \times L^2_k \to L^2_k$ for $k \geq 3$.

5.6. **Proof of Claim 2.** Only show the cylindrical case near ∂X (namely, metrically $[-\epsilon, 0] \times Y$) which is sufficient for what follows. Denote $X_s := X\backslash(s, 0] \times Y$ with $s \in [-s, 0]$. Define $f_n(s) := \mathcal{S}^{\text{an}}_{X_s}(A_n, \Phi_n) : [-\epsilon, 0] \to \mathbb{R}$ by integrating over X_s only.

(f_n) has uniform bound from above and below with $f'_n \geq 0$ and with uniformly bounded integral. Thus, we must have $\mu(\{f'_n \leq M\}) \geq \delta > 0$ independent of n where μ is the Lebesgue measure. (Otherwise, for any i, there exists $n_i \to \infty$, $\delta_i \to 0$ such that $\mu(\{f'_{n_i} \leq i\}) < \delta_i$, so integral $\geq i(\epsilon - \delta_i) \to \infty$, which contradicts to the uniform boundedness of integral.)

We need a lemma: Let $\{S_\alpha\}_{\alpha \in A}$ where $S_\alpha \subset [a, b]$ and A is an infinite index set. If $\mu(S_\alpha) \geq \delta > 0$. Then there exists infinite $B \subset A$ such that $\bigcap_{\alpha \in B} S_\alpha \neq \emptyset$. (Exercise or see [KM] 5.1.6.)

Apply this lemma to $S_n := \{f_n' \le M\} \subset [-\epsilon, 0]$. There exists $s_0 \in [-\epsilon, 0]$, $f_n'(s_0) \le M$. LHS is $-\frac{d}{ds}\mathcal{L}(\gamma_n|_{\{s\}\times Y}(s_0) = \|\nabla_{\gamma_n(s_0)}\mathcal{L}\|^2_{L^2}$.

We need a 3d analogue of Lemma 1.4 in the last Lecture notes, which says: $(B_n, \Psi_n) := \gamma_n(s_0)$ on Y with $\|\nabla_{(B_n,\Psi_n)}\mathcal{L}\|^2_{L^2} \le M$. Then there exists v_n such that $v_n(B_n, \Psi_n)$ converges in $L^2_{1/2}$ norm to a L^2_1-limit. Here $L^2_{1/2}$-norm is defined using Laplacian, which can be taken as a black box or a reading assignment on a small chapter on pseudo-differential operator on e.g. Wells' Differential analysis on complex manifolds (or most books/notes on index theorem). For our purpose, it means $\int b_n \wedge db_n$ and $\int \langle D_{B_n}\tilde{\Phi}_n|_{\{s_0\}\times Y}, \tilde{\Phi}_n|_{\{s_0\}\times Y}\rangle$ are controlled (which is like 1/2 derivative in L^2).

Let $S := S^+|_{\{s_0\}\times Y}$ the spin bundle.

If $c_1(S)$ is torsion, then $\mathcal{L}$ is gauge-invariant and continuous in $L^2_{1/2}$ norm, which is the starting point of (2)(a).

If $c_1(S)$ not torsion, then $\mathcal{L}$ is a constant multiple of two $L^2_{1/2}$-terms above $+\frac{1}{4}\int b_n \wedge F_{B_0^+}$, as $b_n \in L^2_{1/2} \overset{\text{compact}}{\subset} L^2$, we have starting point of (2)(a) again. This finishes Claim 2, thus (2)(c) and compactness theorem.

We do not have bubbling phenomenon in the interior (which makes this theory drastically simpler, this is also why we spent some time on this part explaining some heavy lifting by analysis to go beyond just story telling), but the theorem does not discuss about what happens near the boundary Y, where (possibly several levels of) SW solutions on invariant cylinder break off. We will take a quick look at this after explaining how to deal with singular SW solution $(A, 0)$ which has stabilizer group S^1 in the configuration space quotiented by gauge group.

6. Lecture 6

In $4d$, the configuration space $\mathcal{C}(X, \mathfrak{s}) = \mathcal{A} \times \Gamma(S^+) \ni (A, \phi)$. $\mathcal{A}$ is the space of spin^c connections which is an affine space over $\Gamma(iT^*X) = i\Omega^1(X)$, where we have suppressed $\cdot \otimes \mathrm{Id}_{S_X}$. The gauge group $\mathcal{G}_X = \{u : X \to S^1\}$ acts with the quotient $\mathcal{B}(X, \mathfrak{s}_X) := \mathcal{C}(X, \mathfrak{s}_X)/\mathcal{G}_X$.

(A, ϕ) is called irreducible if $\phi \neq 0$. The irreducible configurations are $\mathcal{C}^*(X, \mathfrak{s}) = \mathcal{A} \times (\Gamma(S^+)\backslash\{0\})$. $\mathcal{G}_X$ acts on $\mathcal{C}^* = \mathcal{C}^*(X, \mathfrak{s}_X)$.

We have $S^1 \overset{\text{constant functions}}{\hookrightarrow} \mathcal{G}_X \overset{(-u^{-1}du, u\cdot)}{\hookrightarrow} \mathcal{A} \times (\Gamma(S^+)\backslash\{0\}) \to \mathcal{B}^*(X, \mathfrak{s}_X)$, the latter two-arrow diagram is a principle bundle (with the middle arrow as inclusion of the fiber). This induces

$$S^1 \hookrightarrow P \to \mathcal{B}^*(X, \mathfrak{s}_X)$$

(namely, $P := \mathcal{A} \times (\Gamma(S^+)\backslash\{0\})/(\mathcal{G}_X/S^1))$, and this is an S^1 bundle over a manifold (the action being free on the irreducibles). More on this can be found in the next lecture. This part is to motivate why we are interested in S^1-action and the way of resolving singularity of this action in our setting.

6.1. **Toy example.**

Ultimately, we want to deal with $(\Gamma(S^+), \langle\cdot,\cdot\rangle_{L^2})$ in $4d$ and $(\Gamma(S), \langle\cdot,\cdot\rangle_{L^2})$ in $3d$ in infinite dimensions. But we consider the toy model first: $(\mathbb{C}^n, \langle\ ,\ \rangle)$, where the latter is the standard inner product.

Let L be a Hermitian matrix on $\mathbb{C}^n$ (which plays the role of D_B in infinite dimension later).

Define function $\Lambda(z) := \frac{\langle z, Lz\rangle}{\|z\|^2}$ on $\mathbb{C}^n\backslash\{0\}$, and it is $\mathbb{C}^* := \mathbb{C}\backslash\{0\}$-invariant; and real-valued, as $\langle z, Lz\rangle = \langle L^*z, z\rangle = \langle Lz, z\rangle = \overline{\langle z, LZ\rangle}$.

So $\Lambda : \mathbb{C}^n\backslash\{0\} \to \mathbb{R}$ descends to $\mathbb{CP}^{n-1} = (\mathbb{C}^n\backslash\{0\})/\mathbb{C}^* \to \mathbb{R}$, mapping from the complex projective space. For $\mathbb{CP}^{n-1}$, we have another sphere model $\mathbb{CP}^{n-1} = S^{2n-1}/S^1$ where $S^{n-1} = \{\|\cdot\| = 1\}$.

Consider function $f(z) = \frac{1}{2}\langle z, Lz\rangle$ on $\mathbb{C}^n$.

The negative gradient flow equation for f is linear: $\frac{dz}{dt} = -Lz$ for $z : \mathbb{R} \to \mathbb{C}^n$.

Claim: The negative gradient flow for $\frac{1}{2}\Lambda$ on $\mathbb{CP}^{n-1}$ is $z : \mathbb{R} \to \mathbb{C}^n\backslash\{0\}$ satisfying $\frac{dz}{dt} = -Lz$ under the projection $\pi : \mathbb{C}^n\backslash\{0\} \to \mathbb{CP}^{n-1}$.

To see this, switch to S^{2n-1} viewpoint where $\frac{1}{2}\Lambda = f$. For $w \in S^{2n-1}$, ∇f in $\mathbb{C}^n$ has normal component along w (recall $\|w\| = 1$), which is

$$\langle \frac{w}{|w|}, \nabla f\rangle \frac{w}{|w|} = \langle w, \nabla f\rangle w = \langle w, Lw\rangle w = \Lambda(w)w.$$

The tangent component is $\nabla f - \Lambda(w)w = Lw - \Lambda(w)w$ which is the gradient of $f|_{S^{2n-1}}$. The image of z satisfying $\frac{dz}{dt} = -Lz$ on S^{2n-1} is $\frac{dw}{dt} = -Lw + \Lambda(w)w$.

On S^{2n-1}, the critical point w is where $\nabla_w f = Lw$ is parallel with w, i.e. $Lw = \mu w$, from which we know $\mu = \langle w, Lw\rangle = \Lambda(w)$. The critical point w is precisely the eigenvector of L and its eigenvalue is $\Lambda(w)$.

For $w \in S^{2n-1}$ a critical point of ∇f (iff w is an eigenvector of L), $f = \frac{1}{2}\langle z, Lz\rangle$:

$$\begin{aligned}
&(\mathrm{Hessian}_w f)(v)\\
=&\nabla_w(\nabla f)(v)\\
=&\nabla(L - \langle z, Lz\rangle z)|_{z=w}(v) \quad \text{where } v \in T_w S^{2n-1} \text{ with } \langle v, w\rangle = 0\\
=&Lv - \langle v, Lw\rangle w - \langle Lw, v\rangle w - \langle w, Lw\rangle v \quad \text{recall } Lw = \lambda_w w \text{ with eigenvector } \lambda_w\\
=&Lv - \lambda_w v\\
=&(L - \lambda_w)v.
\end{aligned}$$

Let us **assume** that the critical points of f are isolated, and the eigenspace for each eigenvalue is 1-dimensional. Order them $w_1, \cdots, w_n$ with corresponding eigenvectors $\lambda_1 < \cdots < \lambda_n$. The index $i(w_i) = \mathrm{dim}K^- = \mathrm{dim}T_{w_i}U_{w_i} = 2(i-1)$ and the unstable manifold U_{w_i} is the subspace in $\mathbb{CP}^{n-1}$ generated by $[w_1], \cdots, [w_{i-1}]$.

6.2. **Manifold situation with S^1 action.** Let P be a compact manifold with Riemannian metric (or a tame manifold with bounded geometry like $\mathbb{C}^n$). S^1 acts on the Riemannian manifold P by isometries.

$Q := P^{S^1}$ = fixed point set of S^1, which we assume is a manifold. Let S^1 acts freely $P\backslash Q$, and assume (actually a consequence of next paragraph) Q is of even codimension in P.

Let $N := N_Q P \to Q$ be the normal bundle with S^1 action, which then gives a complex vector bundle structure. Let $\mu : \mathbb{R}/2\pi\mathbb{Z} \times P \to P$ denote the S^1 action, then $re^{i\theta} \cdot v := r\mu(\theta)v$.

We want to define P^σ the (real oriented) blowup along Q with the blow-down map $\pi : P^\sigma \to P$ as follows:

For ϵ small, the disk bundle $N^\epsilon \overset{\exp}{\hookrightarrow} P$ diffeomorphic onto the image. Away from the zero section, $N^\epsilon\backslash\{(q,0)\}_{q\in Q} = (0,\epsilon) \times S(N) \overset{\Theta}{\hookrightarrow} P\backslash Q$ with $\Theta(r, v) := \exp(rv)$.

We have $P^\sigma := ([0,\epsilon) \times S(N)) \cup_\Theta P\backslash Q$, and we have projection $\pi : P^\sigma \to P$ which comes from gluing Θ and Id in two open parts.

π is diffeomorphism over $P\backslash Q$, and over $q \in Q$, the fiber of π is $S(N_q)$.

Exercise 6.1. $h : \mathbb{R}^m \to \mathbb{R}^n$ a smooth embedding with $h(0) = 0$. Then there exists smooth $h^\sigma : (\mathbb{R}^m)^\sigma \to (\mathbb{R}^n)^\sigma$ such that

$$\begin{tikzcd} (\mathbb{R}^m)^\sigma \arrow[r, "h^\sigma"] \arrow[d, "\pi"] & (\mathbb{R}^n)^\sigma \arrow[d, "\pi"] \\ \mathbb{R}^m \arrow[r, "h"] & \mathbb{R}^n \end{tikzcd}$$

commutes.

Construct h^σ, and show that h^σ is a smooth embedding.

(Hint: on $(\mathbb{R}^m)^\sigma = [0, \infty) \times S^{m-1}$, $h(rv) = r\tilde{h}(r, v)$ for a (unique) smooth and everywhere nonzero $\tilde{h}$, define $h^\sigma : (r, v) \mapsto (r\|\tilde{h}(r, v)\|, \tilde{h}(r, v)/\|\tilde{h}(r, v)\|)$, and show $\|\tilde{h}\|$ is smooth.)

S^1 action on P lifts to an S^1 action on P^σ, which is free. $P^\sigma\backslash Q = P\backslash Q$. S^1 acts on $\partial P^\sigma = S(N)$ freely.

Define $B^\sigma = P^\sigma/S^1 \xrightarrow{\pi} B$. Over Q, we have $\partial B^\sigma \xrightarrow{\pi} Q$ with fiber over q being $S(N_q)/S^1 = \mathbb{P}(N_q)$.

6.3. **Morse function on the blow-up.** $\tilde{f} : P \to \mathbb{R}$ invariant under S^1 has gradient $\tilde{V} = \nabla\tilde{f}$. $\tilde{V}$ defined on $P\backslash Q = P^\sigma\backslash\partial P^\sigma$ extends to smooth $\tilde{V}^\sigma$ on P^σ and we have $\tilde{V}^\sigma|_{\partial P^\sigma} \subset T\partial P^\sigma$. (Use the above exercise applied to the flow of $\tilde{V}$). $\tilde{V}^\sigma$ not a gradient, but as $\nabla\tilde{V}^\sigma$ at the zeros of $\tilde{V}^\sigma$ being symmetric with real eigenvalues, we can still define $K^\pm$ as before.

Example: $\tilde{f}(p) = \frac{1}{2}\langle p, Lp\rangle$, $p \in P = \mathbb{C}^n$, and $P^\sigma = [0, \infty) \times S^{2n-1} \ni (s, \phi)$.

The negative "gradient" equation $\dot{\phi} = -L\phi + \Lambda(\phi)\phi$, $\dot{s} = -\Lambda(\phi)s$.

Earlier, we have looked at $\mathbb{CP}^{n-1}$, while here we have $\mathbb{CP}^{n-1} \times [0, \infty)$.

$\mathrm{Hessian}_w = (L-\lambda_w, \lambda_w)$, where w is critical point/eigenvector of L. So if $\lambda_w < 0$, index $= i_{\mathbb{CP}^{n-1}}(w) + 1$.

6.4. **4d SW.** We have $\mathcal{C}^\sigma(X, \mathfrak{s}_X) := \mathcal{A}(X, \mathfrak{s}_X) \times \mathbb{R}^{\geq 0} \times \mathbb{S}(\Gamma(S^+))$ the blow-up of $\mathcal{C}(X, \mathfrak{s}_X) = \mathcal{A}(X, \mathfrak{s}_X) \times \Gamma(S^+)$ along reducible configurations $\{(A, 0)\}$, where $\mathbb{S}(\Gamma(S^+)) := \{\|\cdot\|_{L^2} = 1\}$.

$\mathcal{C}^\sigma(X, \mathfrak{s}_X) \to \mathcal{C}(X, \mathfrak{s})$, $(A, s, \phi) \mapsto (A, s\phi)$. The fiber over $(A, 0)$ is $\{(A, 0, \phi)\} \cong \mathbb{S}(\Gamma(S^+))$. Seiberg-Witten map $\mathcal{F} : \mathcal{C}(X, \mathfrak{s}_X) \to \Gamma(i\mathfrak{su}(S^+) \oplus S^-) =: \mathcal{V}$ as a section of the trivial bundle $\underline{\mathcal{V}} := \mathcal{C}(X, \mathfrak{s}) \times \mathcal{V}$.

The blowup section $\mathcal{F}^\sigma : \mathcal{C}^\sigma(X, \mathfrak{s}) \to \pi^*\underline{\mathcal{V}}$ is defined as

$$\mathcal{F}^\sigma : (A, s, \phi) \mapsto (\frac{1}{2}\rho_X(F^+_{A^t}) - s^2(\phi\phi^*)_0, D^+_A\phi).$$

This is not a pullback of $\mathcal{F}$. (The pullback section is not Fredholm.)

If $s \neq 0$, $\mathcal{F}^\sigma(A, s, \phi) = 0$ iff $\mathcal{F}(A, s\phi) = 0$. If $s = 0$, $\mathcal{F}^\sigma(A, 0, \phi) = 0$ iff $\mathcal{F}(A, 0) = 0$ and $D^+_A\phi = 0$. $\mathcal{G}_X$ acts on $\mathcal{F}^\sigma$ equivariantly.

6.5. **The restriction map and the blow-up flow equation.** Let $X^1 \subset\subset X$ be an open domain. $r : \mathcal{C}^\sigma(X, \mathfrak{s}_X) \dashrightarrow \mathcal{C}^\sigma(X', \mathfrak{s}_X|_{X'})$. The domain of this map is $\mathrm{dom}(r) := \{\phi|_{X'} \not\equiv 0\}$. Let $\gamma^\sigma = (A, s, \phi) \in \mathrm{dom}(r)$,

$$(A, s, \phi) \mapsto (A, s\|\phi\|_{L^2(X')}, \frac{\phi}{\|\phi\|_{L^2(X')}}).$$

The unique continuation (whose detail is covered in the exercise session) ensures $(\mathcal{F}^\sigma)^{-1}(0) \subset \mathrm{dom}(r)$. (If a SW solution restricts to X' and falls out of $\mathrm{dom}(r)$, then it is identically 0 which contradicts to our starting point.)

Exercise 6.2. In a temporal gauge, a solution $\mathcal{F}^\sigma(\gamma^\sigma) = 0$ on $X = I \times Y$ can be written as

$$\begin{cases} \frac{1}{2}\frac{d}{dt}B^t & = \frac{1}{2} * F_{B^t} - r^2\rho^{-1}(\psi\psi^*)_0 \\ \frac{d}{dt}r & = -\Lambda(B, r, \psi_r) \\ \frac{d}{dt}\psi & = -(D_B\psi - \Lambda(B, r, \psi)\psi)), \end{cases}$$

where $\Lambda(B, r, \psi) := \langle \psi, D_B\psi\rangle_{L^2(Y)}$. Here, $D_B\psi$ plays the role of Lz in the toy example.

7. Lecture 7

We will do the following in this lecture:

- Put the configuration space (and its blowup along irreducibles) and the bundle over it (where the section of SW expression lives), and the gauge group acting in this setting into Banach space setting (so that the finite dimensional intuition largely carries over and to ultimately apply Sard-Smale theorem).
- Using a slice to locally parametrize a quotient manifold using a submanifold through a point.
- Morse theory (i) needs ∇f non-degenerate at critical points (can be viewed as requiring $y \mapsto \nabla_y f$ as a transverse section of $TB \to B$), and (ii) needs $M(a, b)$ to be a manifold of correct dimension, by requiring
$$\big(t \mapsto x(t)\big) \mapsto \big(t \mapsto \dot{x}(t) + \nabla_{x(t)} f\big)$$
to be a transverse section of $\bigcup_{\gamma \in \mathrm{Map}(\mathbb{R}, B)} \Gamma(\gamma^* TB) \to \mathrm{Map}(\mathbb{R}, B)$ in a suitable function space setting built upon point (i) (this can be generalized more readily instead of the formulation of stable and unstable submanifolds intersecting transversely). Analogously in the SW infinite dimensional picture, need (i) $\nabla\mathcal{L}$ non-degenerate, and (ii) need 4d SW solution space on a cylinder up to gauge transformation (flow lines of $-\nabla\mathcal{L}$ on 3d up to 3d gauge transformation) to a manifold with expected dimension. Each will be dealt with as a perturbation of a Fredholm defining section into a transverse section.

  We achieve by perturbing $\nabla\mathcal{L}$ into $\nabla\not{\mathcal{L}}$ by adding the gradient of a "generic" cylinder function, while keeping compactness.

7.1. **Functional space setting.** Use M to denote either X (possibly with ∂X) in $4d$, or Y in $3d$ setting, compact (or bounded geometry such as $\mathbb{R} \times Y$) Riemannian with a spin^c structure $\mathfrak{s}$. Use W to mean the (positive) spin bundle S^+ for X, or S for Y.

The configure space $\mathcal{C}(M, \mathfrak{s}) = (A_0, 0) + \Gamma(iT^*M \oplus W) = \big(A_0 + \Gamma(iT^*M)\big) \times \Gamma(W)$, for example $\Gamma(iT^*X \oplus S^+)$ in $4d$, and $\Gamma(iT^*Y \oplus S)$ in $3d$.

$\mathcal{C}_k(M, \mathfrak{s}) = \big(A_0 + L^2_k(iT^*M)\big) \times L^2_k(W)$. Banach/Hilbert manifold, a manifold based on local models of open sets in Banach/Hilbert space (which is a function space with finite regularity).

The blow-up with regularity

$$\mathcal{C}^\sigma_k(M, \mathfrak{s}) = \{(A, s, \phi) \in \big(A_0 + L^2_k(iT^*M)\big) \times \mathbb{R} \times L^2_k(W) \mid s \geq 0, \|\phi\|_{L^2} = 1\}.$$

The gauge group $\mathcal{G}_{k+1}(M) = \{u \in L^2_{k+1}(M; \mathbb{C}) \mid |u(p)| = 1\}$, here we ask $2(k+1) > \dim M$, so u is continuous and the condition makes sense by Sobolev embedding $L^p_k(M) \hookrightarrow C^r$ if $k - \frac{n}{p} > r$ (thus the function space without the condition is a Banach algebra).

$\mathcal{C}^\sigma_k(M, \mathfrak{s})$ Hilbert manifold with boundary with a Hilbert Lie group $\mathcal{G}_{k+1}$ acting smoothly and freely (always having $2(k+1) > \dim M$ in place).

The tangent space at $\gamma = (A_0, s_0, \phi_0)$ to $\mathcal{C}^\sigma_k := \mathcal{C}^\sigma_k(M, \mathfrak{s})$ is

$$\mathcal{T}_k|_\gamma := T_\gamma \mathcal{C}^\sigma_k := \{(a, s, \phi) \in L^2_k(iT^*M) \times \mathbb{R} \times L^2_k(W) \mid \mathrm{Re}\langle \phi_0, \phi\rangle_{L^2} = 0\}.$$

They fit into the tangent bundle $\mathcal{T}_k$, and one can also complete the fibers using weaker L^2_l norms and denote it as $\mathcal{T}_l$ for $l \leq k$, replacing k by l in the definition.

7.2. **Quotient.** We can form quotient $B_k(M, \mathfrak{s}) = \mathcal{C}_k(M, \mathfrak{s})/\mathcal{G}_{k+1}$, and the blow-up version $B^\sigma_k := \mathcal{C}^\sigma_k/\mathcal{G}_{k+1}$, where drop the dependence of manifold and spin^c structure for brevity. B^σ_k is a Hilbert manifold with boundary being a quotient of a free and smooth group action with closed orbit (the image of $\mathbb{d}_\gamma$ below is closed), and Hausdorff.

The group action $\mathrm{Act} : \mathcal{G}_{k+1} \times C^\sigma_k \to C^\sigma_k, (g, \gamma) \mapsto g\gamma$. The differential of this at the identity $g = e$ and a general configuration γ, is denoted by

$$\mathbb{d}_\gamma := d_{(e,\gamma)}\mathrm{Act} : T_e\mathcal{G}_{k+1} \to T_\gamma \mathcal{C}^\sigma_k.$$

We locally parametrize the quotient structure using a slice: if we choose any $\mathcal{S} \underset{\text{submanifold}}{\overset{\text{locally closed}}{\subset}} \mathcal{C}$ containing a given $\gamma \in \mathcal{G}\gamma$ such that $T_\gamma\mathcal{C} = \mathrm{im}\mathbb{d}_\gamma \oplus T_\gamma S$, then

$$\bar{\iota} : \mathcal{S} \to \mathcal{C}/\mathcal{G}$$

obtained as the composition $\mathcal{S} \overset{\text{inclusion } \iota}{\subset} \mathcal{C} \overset{\text{quotient}}{\subset} \mathcal{C}/\mathcal{G}$ is a diffeomorphism from an open neighborhood of γ onto the image, which is an open neighborhood of $\mathcal{G}\gamma$ in $\mathcal{C}/\mathcal{G}$ (by inverse function theorem).

7.3. **Construct a slice for the blow-up configuration space.** First consider the irreducible configuration space, then extend it to the blow-up.

$\gamma := (A_0, \Phi_0) \in \mathcal{C}^*_k(M, \mathfrak{s}) \subset \mathcal{C}_k(M, \mathfrak{s})$ with $\Phi_0 \neq 0$.

$\mathbb{d}_\gamma : L^2_{j+1}(i\mathbb{R}) \to \mathcal{T}_j|_\gamma, \xi \mapsto (-d\xi, \xi\Phi_0)$, for $j \leq k$.

Let $\mathcal{J}_j|_\gamma := \mathbb{d}_\gamma(L^2_{j+1}(i\mathbb{R}))$.

$\mathcal{K}_j|_\gamma$ denotes its L^2 orthogonal in $\mathcal{T}_j|_\gamma$, explicitly

$$\{(a, \phi) \mid -d^*a + i\mathrm{Re}\langle i\Phi_0, \phi\rangle = 0,\ \langle a|_{\partial M}, n\rangle = 0\},$$

where n is the outwards normal to ∂M. The first condition is $\mathbb{d}^*_\gamma(a, \phi) = 0$.

$\mathcal{J}_j = \bigcup_{\gamma \in \mathcal{C}^*} \mathcal{J}_j|_\gamma$ and similarly $\mathcal{K}_j$ are closed subbundles of $\mathcal{T}_j|_{\mathcal{C}^*}$ and they are orthogonal and complementary.

$\mathcal{J}_j$ extends (to the boundary of $\mathcal{C}^\sigma_k$) to $\mathcal{J}^\sigma_j$ over $\mathcal{C}^\sigma$ naturally.

For $\gamma = (A_0, s_0, \phi_0) \in \mathcal{C}^\sigma_k$, we define

$$\mathcal{K}^\sigma_j|_\gamma := \{(a, s, \phi) \mid -d^*a + is_0^2\mathrm{Re}\langle i\phi_0, \phi\rangle = 0, \langle a|_{\partial M}, n\rangle = 0, \mathrm{Re}\langle i\phi_0, \phi\rangle_{L^2} = 0\}.$$

They fit together $\mathcal{K}^\sigma_j$ subbundle of $\mathcal{T}^\sigma_j$ that is complementary to $\mathcal{J}^\sigma_j$ in $\mathcal{T}^\sigma_j$.

Want to find a closed submanifold $S^\sigma_{k,\gamma} \subset \mathcal{C}^\sigma_k$ based at $\gamma = (A_0, s_0, \phi_0)$ s.t. $T_\gamma S^\sigma_{k,\gamma} = \mathcal{K}^\sigma_j|_\gamma$. We define:

$$S^\sigma_{k,\gamma} := \{(A_0 + a, s, \phi) \mid -d^*a + iss_0\mathrm{Re}\langle i\phi_0, \phi\rangle = 0, \langle a|_{\partial M}, n\rangle = 0, \mathrm{Re}\langle i\phi_0, \phi\rangle_{L^2(M)} = 0\}.$$

$S^\sigma_{k,\gamma}$ has a well-defined limit as s_0 in $\gamma = (A_0, s_0, \phi_0)$ goes to 0, which defines $S^\sigma_{k,\gamma}$ for γ on the boundary.

Note that the construction of $\mathcal{S}^\sigma_{k,\gamma}$ is motivated from the proper transform of the slice $\mathcal{S}_{k,\gamma} := \{(A,\Phi) \mid -d^*a + i\mathrm{Re}\langle i\Phi_0, \Phi\rangle = 0, \langle a_{\partial M}, n\rangle = 0\}$ through any point $\gamma = (A_0, \Phi_0) \in \mathcal{C}_k$.

$\bar{\iota} :=$ quotient $\circ\, \iota : S^\sigma_{k,\gamma} \to B^\sigma_k$ with ι denoting the inclusion, called Coulomb-Neumann chart/slice (because as we shall see that to bring a general configuration into this slice based at a reducible configuration $(A_0, 0)$, one solves the same equation as in the Coulomb-Neumann gauge fixing previously in the argument of compactness).

7.4. **SW map as a section from the blow-up space with finite regularity.** We have trivial bundle

$$\mathcal{V}_{k-1} := \mathcal{C}_k \times L^2_{k-1}(i\mathfrak{su}(S^+) \oplus S^-) \to \mathcal{C}_k$$

and the blow-down map $\mathcal{C}^\sigma_k \xrightarrow{\pi} \mathcal{C}_k$. Define $\mathcal{V}^\sigma_{k-1} := \pi^*\mathcal{V}_{k-1}$ for the compact manifold.

$F^\sigma(A, s, \phi) = (\frac{1}{2}\rho_X(F^+_{A^t}) - s^2(\phi\phi^*)_0, D^+_A\phi)$ is a section of $\mathcal{V}^\sigma_{k-1} \to \mathcal{C}^\sigma_k$ with $\mathcal{G}_{k+1}$ acting equivariantly and smoothly.

In $3d$, $\nabla\mathcal{L}$ smooth section $\mathcal{T}_{k-1} \to \mathcal{C}_k$, and $(\nabla\mathcal{L})^\sigma$ smooth section for $\mathcal{T}^\sigma_{k-1} \to \mathcal{C}^\sigma_k$.

7.5. **As a preliminary, global slice: bring a general point in $\mathcal{C}_k$ to the slice $\mathcal{S}_{k,\gamma_0}$ at a reducible $\gamma_0 = (A_0, 0)$.** Recall that the defining condition for the slice $\mathcal{S}_{k,\gamma_0}$ is

$$\{-d^*a = 0, \langle a|_{\partial M}, n\rangle = 0\}.$$

To find a gauge transformation of the form $u = e^\xi$ to put $(A,\phi) = (A_0 + a, \phi)$ into the slice, one needs to solve (substituting into the above defining condition) $\Delta\xi = d^*a$, $\langle d\xi|_{\partial M}, n\rangle = \langle a|_{\partial M}, n\rangle$. ξ is unique if asking $\int_M \xi = 0$.

Define $\mathcal{G}^\perp_{k+1} = \{e^\xi \mid \int_M \xi = 0\}$.

We have a diffeomorphism

$$\mathcal{G}^\perp_{k+1} \times \mathcal{S}_{k,\gamma_0} \to \mathcal{C}_k, \quad (e^\xi, (a,\phi)) \mapsto (A_0 + (a - d\xi), e^\xi\phi),$$

restricted from the group action map Act.

B_k is quotient of $\mathcal{C}_k/\mathcal{G}_{k+1} := \iota(\mathcal{S}_{k,\gamma_0})/(\mathcal{G}_{k+1}/\mathcal{G}^\perp_{k+1})$. Here $\mathcal{G}^h := \mathcal{G}_{k+1}/\mathcal{G}^\perp_{k+1}$ can be realized as the extension $S^1 \to \mathcal{G}^h \to H^1(M;\mathbb{Z})$, where the second map is taking the associated homotopy class, and $H^1(M;\mathbb{Z})$ is the components of gauge group. (The notation for $\mathcal{G}^h$ comes from an alternative realization as harmonic maps $u : M \to S^1$ with Neumann boundary condition $\Delta u = 0, \langle\nabla u, n\rangle = 0$.)

Exercise 7.1. From the above, show we have homotopy equivalences

$$B^\sigma_k \cong \iota(\mathcal{S}_{k,\gamma_{\gamma_0}} \cap \mathcal{C}^*_k)/\mathcal{G}^h \times (L^2_k(S)\backslash\{0\})/S^1 \cong H^1(M; i\mathbb{R})/2\pi H^1(M; i\mathbb{Z}) \times \mathbb{CP}^\infty.$$

7.6. **Tame perturbation.** In $3d$, we will take perturbation of the following form $f : \mathcal{C}(Y) \to \mathbb{R}$ invariant under $\mathcal{G}$. $\not{\mathcal{L}}{:=}\mathcal{L} + f$ perturbed CSD functional. The perturbation to the equation is $\mathfrak{q} = \nabla f$.

When $\mathfrak{q}$ having less regularity, we call it a **formal gradient** of some f if for all smooth $\gamma : [0,1] \to \mathcal{C}(Y)$, we have $(f\circ\gamma)(1) - (f\circ\gamma)(0) = \int_0^1 \langle\dot\gamma, \mathfrak{q}\rangle_{L^2} dt$.

$\mathfrak{q} = (\mathfrak{q}^0, \mathfrak{q}^1) \in L^2(iT^*Y) \oplus L^2(S)$. $\nabla\not{\mathcal{L}} = \nabla\mathcal{L} + \mathfrak{q}$.

Lifted to the blow-up, $(\nabla\not{\mathcal{L}})^\sigma = (\nabla\mathcal{L})^\sigma + \mathfrak{q}^\sigma$. We will write out the LHS (which also then defines $\mathfrak{q}^\sigma$ in terms of $\mathfrak{q}$):

$$(\nabla\not{\mathcal{L}})^\sigma := \begin{pmatrix} \frac{1}{2} * F_{B^t} + r^2\rho^{-1}(\psi\psi^*)_0 + \mathfrak{q}^0(B,r,\psi) \\ \Lambda_\mathfrak{q}(B,r,\psi)r \\ D_B\psi + \tilde{\mathfrak{q}}^1(B,r,\psi) - \Lambda_\mathfrak{q}(B,r,\psi)r \end{pmatrix},$$

where $\tilde{\mathfrak{q}}^1(B,r,\psi)=\int_0^1\left(d_{(B,sr\psi)}\mathfrak{q}^1\right)(0,\psi)ds$, and

$$\Lambda_{\mathfrak{q}}(B,r,\psi)=\mathrm{Re}\langle\psi,D_B\psi+\tilde{\mathfrak{q}}^1(B,r,\psi)\rangle_{L^2}.$$

Exercise 7.2. Using the expression of $\nabla\mathcal{L}$ and the above to write down $\mathfrak{q}^\sigma$ in terms of $\mathfrak{q}=(\mathfrak{q}^0,\mathfrak{q}^1)$ explicitly.

(B,r,ψ) is a crtical point of $(\nabla\cancel{\mathcal{L}})^\sigma$ iff:
when $r\neq 0$, $(B,r\psi)$ is a critical point of $\nabla\cancel{\mathcal{L}}$; when $r=0$, $(B,0)$ is a critical point of $\nabla\cancel{\mathcal{L}}$ and ψ is an eignevector of $\phi\mapsto D_B\phi+\left(d_{(B,0)}\mathfrak{q}^1\right)(0,\phi)$.

We need tame perturbation (the following definition is a bit technical and one can proceed to cylinder functions right after it, and we hope to add a sketch of proof for the last assertion of this lecture which might make the definition more palatable):

Definition 7.3. (τ-model) Let $Z=[t_1,t_2]\times Y$. As the $4d$ blow-up model (with regularity) viewed as a path in $3d$ blow-up model, one needs to divide a factor to rescale the last variable so that it satisfies $\|\ \|_{L^2(Y)}=1$ (we need to invoke unique continuation theorem which implies that we never divide by 0), the rescaling factor needed to multiplied to the second variable, which now is a non-negative function $s(t)$ on $t\in[t_1,t_2]$. This is the τ-model and we use superscript τ to signify it, and we ask

$$s\in L^2_k([t_1,t_2])\cap\{s(t)\geq 0\}.$$

$\mathcal{C}^\tau_k$ is not a Hilbert manifold with boundary but a closed submanifold of an obvious Hilbert manifold (where the middle factor is $L^2_k([t_1,t_2])$).

Exercise 7.4. Work out the tangent bundle, and slice in τ-model.

Definition 7.5. (tame) Let $Z=[t_1,t_2]\times Y$. We want to use gauge invariant norm $L^2_{k,A}$, then we no longer have a trivial bundle $\mathcal{V}_k$ as a normed vector bundle. $L^2_{-k}:=(L^2_k)^*$.

Given $\gamma\in\mathcal{C}^\tau(Z)$, we view it as a path $\check{\gamma}(t)$ in $\mathcal{C}^\sigma_k(Y)$, then we have $\mathfrak{q}^\sigma(\check{\gamma}(t))\in L^2_k(Y;iT^*Y)\oplus\mathbb{R}\times L^2_k(Y;S)$, using Clifford multiplication to identify iT^*Y with $i\mathfrak{su}(S^+)$ and S with S^-. Then this becomes an element in $\mathcal{V}^\tau_k$, denoted by $\hat{\mathfrak{q}}^\tau(\gamma)$. One also has the non-blow-up version $\hat{\mathfrak{q}}$ of the above.

For an integer $k\geq 2$, a perturbation $\mathfrak{q}:\mathcal{C}(Y)\to\mathcal{T}_0$ is k-tame if it is the formal gradient of continuous $\mathcal{G}_Y$-invariant function on $\mathcal{C}(Y)$, such that

- $\gamma\mapsto\hat{\mathfrak{q}}(\gamma)$ is a smooth section of $\mathcal{V}_k(Z)\to\mathcal{C}_k(Z)$,
- for every integer $j\in[1,k]$, $\hat{\mathfrak{q}}$ is a continuous section of $\mathcal{V}_j(Z)\to\mathcal{C}_j(Z)$,
- for every integer $j\in[-k,k]$, $d\hat{\mathfrak{q}}$ which is a smooth section of

$$\mathrm{Hom}(T\mathcal{C}_k(Z),\mathcal{V}_k(Z))\to\mathcal{C}_k(Z),$$

  extends to a smooth section $D\mathfrak{q}$ of $\mathrm{Hom}(T\mathcal{C}_j(Z),\mathcal{V}_j(Z))\to\mathcal{C}_k(Z)$,
- there exists constant m_2, $\|\mathfrak{q}(B,\psi)\|_{L^2}\leq m_2(\|\psi\|_{L^2}+1)$ for $(B,\Psi)\in\mathcal{C}_k(Y)$,
- for any smooth connection A_0, there exists a real function μ_1 such that $\|\hat{\mathfrak{q}}(A,\Phi)\|_{L^2_{1,A}}\leq\mu_1(\|A-A_0,\Phi\|_{L^1_{1,A_0}})$,
- $\mathfrak{q}:\mathcal{C}_1(Y)\to\mathcal{T}_0$ in Y is C^1.

If $\mathfrak{q}$ is tame for all $k\geq 2$, then $\mathfrak{q}$ is called **tame**.

This technical definition will become handy in the proof of achieving transversality while keeping compactness. One cannot help but notice some similarity with an ingredient in the definition of sc-smoothness in polyfold theory.

Readers can skip the above definition on the first reading and just proceed to the following construction of a class of functions called cylinder functions that will give tame perturbations.

7.7. **Cylinder functions.** We break up the construction into several bite-size pieces:

7.7.1. *Choose a simpler quotient representative.* We have $\mathcal{G}_{k+1} = \mathcal{G}_{k+1}^{\perp} \times \mathcal{G}^h$ from the above.

In $3d$ setting, at $\gamma_0 = (B_0, 0)$, we have difeomorphism $\mathcal{G}_{k+1}^{\perp} \times \mathcal{S}_{k,\gamma_0} \to \mathcal{C}_k(Y)$ which is the group action, where $\mathcal{S}_{k,\gamma_0} = \{d^*b = 0\}$ as above.

We want $\mathcal{G}_{k+1}$-invariant function. Due to the above splitting and identification, only need to construct functions on $\mathcal{S}_{k,\gamma_0}$ that are invariant under $\mathcal{G}^h$.

7.7.2. *Torus-valued function on configuration pairing with 1-forms.* Given a coclosed 1-form $c \in \Gamma(iT^*Y) = \Omega^1(i\mathbb{R})$ (namely, $d^*c = 0$), define

$$r_c : \mathcal{C}(Y) \to \mathbb{R}, \ \ (B_0 + b, \psi) \mapsto \int_Y b \wedge *\bar{c} = \langle b, c\rangle_Y.$$

$u \in \mathcal{G}_{k+1}$, $-u^{-1}du$ represents $h \in 2\pi i H^1(Y;\mathbb{Z})$, $r_c \circ u - r_c = (h \cup [*\bar{c}])[Y]$. If $c = d^*c'$ coexact, r_c is invariant under $\mathcal{G}_{k+1}$.

Denote $\mathbb{T} := H^1(Y; i\mathbb{R})/2\pi i H^1(Y;\mathbb{Z})$, we can choose i-valued harmonic 1-forms modulo those with $2\pi i$ periods as preferred representatives. Choose integral basis $w_1, \cdots, w_t \in H^1(Y, i\mathbb{R})$, so $\mathbb{T} \cong \mathbb{R}^t/2\pi\mathbb{Z}^t$.

$\mathcal{C}(Y) \to \mathbb{T}$, $(B_0 + b, \psi) \mapsto [b_{\text{harm}}]$.

We have $(B, \psi) \mapsto (r_{w_1}(B,\psi), \cdots, r_{w_t}(B,\psi)) \bmod 2\pi\mathbb{Z}^t$.

7.7.3. *$\mathbb{C}$-valued equivariant function from pairing of the spinor direction.* We choose a splitting $S^1 \to \mathcal{G}^h \underset{v}{\rightleftarrows} H^1(Y;\mathbb{Z})$, e.g. using harmonic gauge transformations such that $u(x_0) = 1$ for some base point x_0.

Define $\mathcal{G}_{k+1}^0(Y) = \mathcal{G}^{h,0}(Y) \times \mathcal{G}_{k+1}^{\perp}(Y)$, where $\mathcal{G}^{h,0}(Y) = \mathrm{im}(v)$.

$\mathcal{G}_{k+1}^0(Y)$ acts freely on $\mathcal{C}_k(Y)$ with quotient B_k^0. The S^1 action on B_k^0 induced by the $\mathcal{G}_{k+1} = S^1 \times \mathcal{G}_{k+1}^0$ action on $\mathcal{C}_k$ is $e^{i\theta} \in S^1$ acting as $(B,\psi) \mapsto (B, e^{i\theta}\psi)$. We have $B_k := \mathcal{C}_k/\mathcal{G}_{k+1} = B_k^0/S^1$.

$H^1(Y; i\mathbb{R}) \times S$ (with S spin bundle) acted by $\mathcal{G}^{h,0} = \mathrm{im}(v)$, with the quotient $\mathbb{T} \times S =: \mathbb{S}$ which fibers over $\mathbb{T} \times Y$.

For a smooth section Υ of $\mathbb{S} \to \mathbb{T} \times Y$, we can always choose a lift $\tilde{\Upsilon}$ which is a section of $H^1(Y; i\mathbb{R}) \times S \to H^1(Y; i\mathbb{R}) \times Y$. Denoting $\tilde{\Upsilon}_b(y) := \tilde{\Upsilon}(b, y)$, we have $\tilde{\Upsilon}_{\alpha+\kappa}(y) = \big(v(\kappa)\big)(y)\tilde{\Upsilon}_\alpha(y)$, where v is the splitting above.

Recall quickly on Hodge theory. In particular, $\Delta = d^*d$ has Green function $G : L_{k-1}^2(Y) \to L_{k+1}^2(Y)$, $\Delta \circ G = \mathrm{Id}$. Given Υ of $\mathbb{S}$, we define a $\mathcal{G}^0$-equivariant map $\Upsilon^\dagger : \mathcal{C}(Y) \to \Gamma(S), (B_0 + b, \psi) \mapsto e^{-Gd^*b}\tilde{\Upsilon}_{b_{\text{harm}}}$.

Define $q_\Upsilon : (B,\psi) \mapsto \int_Y \langle \psi, \Upsilon^\dagger(B,\psi)\rangle = \langle \psi, \tilde{\Upsilon}^\dagger\rangle_Y$. q_Υ is $\mathcal{G}^0$-invariant, and also S^1 equivariant.

7.7.4. *Invariant function from finitely many directions picked out.* Choose coclosed $c_1, \cdots, c_{n+t}$, where the first n coexact, and last t being basis w_i above. Choose m sections $\Upsilon_1, \cdots, \Upsilon_m$ of $\mathbb{S}$.

Define function $p : \mathcal{C}(Y) \to \mathbb{R}^n \times \mathbb{T} \times \mathbb{C}^m$,

$$(B,\psi) \mapsto (r_{c_1}(B,\psi), \cdots, r_{c_{n+t}}(B,\psi), q_{\Upsilon_1}(B,\psi), \cdots, q_{\Upsilon_m}(B,\psi)).$$

We have an S^1-action on $\mathbb{R} \times \mathbb{T} \times \mathbb{C}^m$ acting on the last factor $\mathbb{C}^m$. Choose an S^1-invariant compactly supported $g : \mathbb{R}^n \times \mathbb{T} \times \mathbb{C}^m \to \mathbb{R}$.

7.7.5. *Cylinder function and being large enough.* Define $f := g \circ p$ and $\mathfrak{q} := \nabla f$. A function of this form is called a **cylinder function**.

Exercise 7.6. Show that space of cylinder functions is 'large enough': for any

$$[B, \psi] \in B_k^*(Y) := \mathcal{C}_k^*/\mathcal{G}_{k+1},$$

any non-zero tangent vector $v \in T_{[B,\psi]}B_k^*(Y)$. There exists a cylinder function $f : \mathcal{C}_k(Y) \to \mathbb{R}$ with the quotiented restriction $\bar{f} := (f|_{\mathcal{C}_k^*})/\mathcal{G}_{k+1} : B_k^*(Y) \to \mathbb{R}$ such that $(d_{[B,\Psi]}\bar{f})(v) \neq 0$.

Make a countable collection of such choices of cylinder functions. The space $\mathcal{P}$ of countable sequence of constant weights used for linear combination of cylinder functions is the Banach space of perturbations.

7.8. **Transversality in** $3d$**.** Let $\mathcal{P}$ be the Banach space of weights for linear combinations for cylinder functions.

There exists a residual (complement of countable intersection of open dense sets, especially non-empty) subset $\mathcal{P}_{\text{res}}$ of $\mathcal{P}$, such that for any $\mathfrak{q} \in \mathcal{P}_{\text{res}}$, zeros of $(\nabla \not{\mathcal{L}})^\sigma = (\nabla \mathcal{L})^\sigma + \mathfrak{q}^\sigma$, which is a section of $\mathcal{T}_{k-1}^\sigma \to \mathcal{C}_k^\sigma(Y)$, is non-degenerate (This means the following: We have for $j \leq k$, $\mathcal{T}_j^\sigma \cong \mathcal{J}_j^\sigma \oplus \mathcal{K}_j^\sigma$ as before, $(\nabla \not{\mathcal{L}})^\sigma$ is transverse to the subbundle $\mathcal{J}_{k-1}^\sigma$, recalling it being invariant under group action).

This is proved using Sard-Smale theorem.

8. Lecture 8: Transversality in 3d making all zeros of perturbed SW map non-degenerate

8.1. **Perturbation of SW map / the gradient of CSD.** No exercise session (so that the final lecture will have one).

$\mathcal{L}$ CSD functional. SW map $\nabla \mathcal{L}$.

Perturb it by $q = \nabla f = (q^0, q^1) \in L^2(iT^*Y) \oplus L^2(S)$.

$\tilde{q}^1(B, r, \psi) = \int_0^1 (d_{B,sr\psi} q^1)(0, \psi) ds$ for $(B, r, \psi) \in \mathcal{C}^\sigma$.

$\Lambda_q(B, r, \psi) = \text{Re}\langle \psi, D_B\psi + \tilde{q}^1(B, r, \psi)\rangle_{L^2}$.

Let $\not{\mathcal{L}} := \mathcal{L} + f$, thus $\nabla \not{\mathcal{L}} = \nabla \mathcal{L} + q = \begin{pmatrix} \frac{1}{2} * F_{B^t} + r^2\rho^{-1}(\psi\psi^*)_0 + q^0(B, r, \psi) \\ \Lambda_q(B, r, \psi)r \\ D_B\psi + \tilde{q}^1(B, r, \psi) - \Lambda_q(B, r, \psi)r \end{pmatrix}$.

8.2. **Splitting the tangent bundle complementary to the group action.** Last time $\mathcal{C}_k^\sigma$ has tangent bundle $T_k^\sigma = \mathcal{J}_k^\sigma \oplus \mathcal{K}_k^\sigma$, where the first factor is tangent to the gauge group orbit. We complete this in lower regularity to have $T_j^\sigma = \mathcal{J}_j^\sigma \oplus \mathcal{K}_j^\sigma$ for $j \leq k$ and most relevant case is $j = k - 1$.

A zero (perturbed SW solution) $\mathfrak{a} \in \mathcal{C}_k^\sigma(Y)$ of $(\nabla \not{\mathcal{L}})^\sigma$ is non-degenerate if $(\nabla \not{\mathcal{L}})^\sigma \in \Gamma(T_{k-1}^\sigma)$ is transverse to $\mathcal{J}_{k-1}^\sigma$ (natural from quotient space viewpoint).

Want to show that: for $\mathcal{P}$ Banach space of tame perturbation, there exists a $\mathcal{P}^{\text{res}} \subset \mathcal{P}$ (complemenet of countable intersection of open dense subsets, in particular, non-empty), such that $q \in \mathcal{P}^{\text{res}}$, we have $(\nabla \not{\mathcal{L}})^\sigma = (\nabla \mathcal{L})^\sigma + q^\sigma$ has only non-degenerate zeros.

8.3. **Characterization of non-degeneracy.** We want to abstractizing the Hessian $\nabla\big((\nabla\pounds)^\sigma\big)$:

Definition 8.1 (k-ASAFOE)**.** An operator L acting on sections of a vector bundle $E \to Y$ is called k-almost self-adjoint first order elliptic (k-ASAFOE) if $L = L_0 + h$, where

- L_0 is SAFOE (self-adjoint first order elliptic) operator with smooth coefficients.
- $h : C^\infty(E) \to L^2(E)$ an operator extends to a bounded operator $L^2_j(E) \to L^2_j(E)$, for all $|j| \le k$ (here, $L^2_{-m} = (L^2_m)^*$ with respect to L^2 inner product), but h is not necessarily self-adjoint.

It is called ASAFOE, if k-ASAFOE for all k.

Properties:

- L is k-ASAFOE, then regularizing: $u \in L^2_{-k}$, $Lu = v \in L^2_j$ with $|j| \le k$, then $u \in L^2_{j+1}$.
- L is k-ASAFOE, then $L : L^2_j(E) \to L^2_{j-1}(E)$ is Fredholm of index 0 (due to self-adjoint) for $-k \le j \le k$.
- The previous item implies that $L : L^2_j \to L^2_{j-1}$ invertible iff injective. Moreover, invertible for one j implies invertible for all $|j| \le k$.
- So λ is an eigenvalue iff $(L-\lambda) : L^2_j \to L^2_{j-1}$ not invertible (independent of j).
- $L : L^2_j \to L^2_{j-1}$ with $L = L_0 + h$ k-ASAFOE.

$$\text{Then: } \begin{cases} \text{If } h \text{ symmetric, then eigenvalues are real,} \\ \quad \text{there exists complete orthonormal eigenvectors in } L^2_{k+1}, \text{ dense in } L^2. \\ \text{If } h \text{ non-symmetric, then imaginary parts of eigenvalues of} \\ \quad \text{complexification of } L \otimes 1_{\mathcal{C}} \text{ bounded by } L^2\text{-operator norm of } \frac{h-h^*}{2}. \end{cases}$$

Remark 8.2. h symmetric, eigenvalues are unbounded in both directions.

Denote the tangent bundle for $\mathcal{A}_k$ acted by $\mathcal{G}_{k+1}$ by $T^{\mathrm{red}}_j = \mathcal{J}^{\mathrm{red}}_j \oplus \mathcal{K}^{\mathrm{red}}_j$ for $j \le k$. Fibers are exact and coclosed 1-forms in $i\mathbb{R}$.

$\mathrm{pr}_{T^{\mathrm{red}}_{k-1}} \circ (\nabla\pounds|_{\mathcal{A}_k \times \{0\}})$ defines $(\nabla\pounds)^{\mathrm{red}} : \mathcal{A}_k \to T^{\mathrm{red}}_{k-1}$.

For $B \in \mathcal{A}_k$, $D_{q,B} : L^2_k(S) \to L^2_{k-1}(S)$, $\phi \mapsto D_B\phi + (d_{(B,0)}q^1)(0,\phi)$ is k-ASAFOE, S^1-equivariant, then it is complex linear operator.

Definition 8.3 (characterization)**.** A zero $\mathfrak{a} = (B, r, \psi) \in \mathcal{C}^\sigma_k$ is a non-degenerate

$$\text{zero of } (\nabla\pounds)^\sigma \text{ iff} \begin{cases} r \neq 0, (B, r\psi) \text{ is non-degenerate zero of } \nabla\pounds, \\ r = 0, \psi \text{ eigenvector of } D_{q,B} \text{ with simple eigenvalue } \lambda \neq 0 \\ \quad (\text{for } D_{q,B} \text{ as complex operator}), \text{ B non-degenerate zero of } (\nabla\pounds)^{\mathrm{red}}. \end{cases}$$

Remark 8.4. First, recall that $\mathrm{Act}^\sigma : \mathcal{G}_{k+1} \times \mathcal{C}^\sigma_k \to \mathcal{C}^\sigma_k$, $\mathbb{d}^\sigma_{\mathfrak{a}} := d_{(\mathrm{id},\mathfrak{a})}\mathrm{Act}^\sigma : T_{\mathrm{id}}\mathcal{G}_{k+1} \to T_{\mathfrak{a}}\mathcal{C}^\sigma_k$.

$(\nabla\pounds)^\sigma$ non-degenerate at a reducible $\mathfrak{a} = (B, 0, \psi)$ is equivalent to surjectivity of

$$\mathbb{d}^\sigma_{\mathfrak{a}} \oplus d_{(B,0,\psi)}(\nabla\pounds)^\sigma = \begin{pmatrix} -d & d_B(\nabla\pounds)^{\mathrm{red}} & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ \psi & 0 & 0 & D_{q,B} - \lambda, \end{pmatrix}$$

where the last 3×3 matrix is $d_{\mathfrak{a}}(\nabla\pounds)^\sigma$.

8.4. h **non-symmetric in our cases, eigenvalues are real.** On irreducible $\mathcal{C}^*_k(Y)$, $T_j|_{\mathcal{C}^*_k(Y)} = \mathcal{J}_i \oplus \mathcal{K}_j$. The slice $\mathcal{S}_{k,\alpha} = \alpha + \mathcal{K}_{k,\alpha}$ through $\alpha = (B_0, \Psi_0)$, where $\{(b,\phi) \mid -d^*b + i\mathrm{Re}\langle i\Psi_0, \phi\rangle = 0\}$.

$\mathrm{Hess}_{q,\alpha} = \mathrm{pr}_{\mathcal{K}_{k-1,\alpha}} \circ d_\alpha(\nabla \text{₤})|_{\mathcal{K}_{k,\alpha}}$ is $\mathcal{G}_{k+1}$-equivariant.

Being symmetric implies that there exists a complete orthonormal basis $\{e_n\}$ in $\mathcal{K}_{0,\alpha}$ which are smooth with real eigenvalues λ_n. The span $\{e_n\}$ is dense in all $\mathcal{K}_{j,\alpha}$. The operator is Fredholm with index 0.

To show this, consider the extended Hessian

$$\begin{aligned}
&\left(\hat{\mathrm{Hess}}_{q,\alpha} : T_{k,\alpha} \oplus L^2_k(i\mathbb{R}) \to T_{k-1,\alpha} \oplus L^2_{k-1}(i\mathbb{R})\right) \\
=&\begin{pmatrix} d_\alpha \nabla \text{₤} & \mathbb{b}_\alpha \\ \mathbb{d}^*_\alpha & 0 \end{pmatrix} \\
=&\begin{pmatrix} D_{B_0} & 0 & 0 \\ 0 & *d & -d \\ 0 & d^* & 0 \end{pmatrix} + h \text{ with 3 coordinates being } L^2_k(S) \oplus L^2_k(iT^*Y) \oplus L^2_k(i\mathbb{R}) \\
=&\begin{pmatrix} 0 & x & \mathbb{d}_\alpha \\ x & \mathrm{Hess}_{q,\alpha} & 0 \\ \mathbb{d}_\alpha & 0 & 0 \end{pmatrix} \text{ with 3 coordinates being } \mathcal{J}_k \oplus \mathcal{K}_k \oplus L^2_k(i\mathbb{R}),
\end{aligned}$$

where $x = \mathrm{pr}_{\mathcal{J}_{k-1,\alpha}} \circ d_\alpha \nabla \text{₤}|_{\mathcal{K}_{k,\alpha}}$, is k-ASAFOE. $x = 0$ at zeros of ∇₤.

It has $\mathrm{Hess}_{q,\alpha}$ as a block, so it has a complete orthonormal basis.

For blown-up, Hessian no longer the second derivative of a function. We have $\mathrm{Hess}^\sigma_{q,\mathfrak{a}} : \mathcal{K}^\sigma_{k,\mathfrak{a}} \to \mathcal{K}^\sigma_{k-1,\mathfrak{a}}$ similarly. $d_\mathfrak{a}(\nabla \text{₤})^\sigma = \begin{pmatrix} 0 & x \\ y & \mathrm{Hess}^\sigma_{q,\mathfrak{a}}. \end{pmatrix}$

$x = 0$, $y = 0$ at zero of $(\nabla \text{₤})^\sigma$.

Non-degenerate iff $\mathrm{Hess}^\sigma_{q,\mathfrak{a}}$ is surjective.

Definition 8.5. Extended $\hat{\mathrm{Hess}}^\sigma_{q,\mathfrak{a}} : T^\sigma_{k,\mathfrak{a}} \oplus L^2_k(i\mathbb{R}) \to T^\sigma_{k-1,\mathfrak{a}} \oplus L^2_{k-1}(i\mathbb{R})$ as a block matrix below:

$\mathbb{d}^\sigma_\mathfrak{a}$ is defined similar as before.

$\mathbb{d}^{\sigma,\dagger}_\mathfrak{a} : T^\sigma_{k,\mathfrak{a}} \to L^2_{k-1}(i\mathbb{R})$ s.t. $\ker \mathbb{b}^{\sigma,\dagger}_\mathfrak{a} = \mathcal{K}^\sigma_{k,\mathfrak{a}}$.

Let $\mathfrak{a} = (B_0, s_0, \psi_0)$, $\mathbb{d}^{\sigma,\dagger}_\mathfrak{a} : (b, s, \psi) \mapsto -d^*b + is_0^2 \mathrm{Re}\langle i\psi_0, \psi\rangle + i|\psi_0|^2 \mathrm{Re}(\frac{\int_Y \langle i\psi_0, \psi}{\int_Y 1})$.

For $\mathfrak{a}$ zero, we have $\hat{\mathrm{Hess}}^\sigma_{q,\mathfrak{a}} = \begin{pmatrix} d_\mathfrak{a}(\nabla \text{₤})^\sigma & \mathbb{d}^\sigma_\mathfrak{a} \\ \mathbb{d}^{\sigma,\dagger}_\mathfrak{a} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \mathbb{d}^\sigma_\mathfrak{a} \\ 0 & \mathrm{Hess}^\sigma_{q,\mathfrak{a}} & 0 \\ \mathbb{d}^{\sigma,\dagger}_\mathfrak{a} & 0 & 0 \end{pmatrix}$.

This is not a perturbation of an elliptic operator on Y. ψ is orthogonal to ψ_0, r is not section unconstrainted. One way to remedy this is to define $\Psi := \psi + r\psi_0$. $(b, \Psi, c) \in L^2_j(iT^*Y \oplus S \oplus i\mathbb{R})$ in this coordinate above is $L_0 + h_\mathfrak{a}$, where $L_0 = \begin{pmatrix} *d & 0 & -d \\ 0 & D_{B_0} & 0 \\ -d^* & 0 & 0 \end{pmatrix}$ is k-ASAFOE. Not symmetric but spectrum is real (see [KM] Lemma 12.4.3).

8.5. **Transversality.**

Lemma 8.6. *$\mathcal{E}$, $\mathcal{F}$ and $\mathcal{P}$ separable (countable dense subset) Banach manifolds. $S \subset \mathcal{F}$ closed submanifold. $F : \mathcal{E} \times \mathcal{P} \to \mathcal{F}$, $F_p(e) = F(e,p)$. Suppose F is* *transverse to S and for all $(e,p) \in F^{-1}(S)$,*

$$T_e\mathcal{E} \overset{d_eF_p}{\to} T_f\mathcal{F} \overset{\pi}{\to} T_fF/F_fS,$$

where $f := F(e,p)$, is Fredholm. Then there exists a residual $\mathcal{P}^{res} \subset \mathcal{P}$ such that for any $p \in \mathcal{P}^{res}$, $F_p : \mathcal{E} \to \mathcal{F}$ is transverse to S.

8.5.1. *Irreducible case.* For irreducible case, suffice to look at $\nabla \not{\mathcal{L}}$ on $\mathcal{C}^*_k(Y)$. Consider $\mathfrak{g} : \mathcal{C}^*_k \times \mathcal{P} \to \mathcal{K}_{k-1}$, $(\alpha, q) \mapsto (\nabla \not{\mathcal{L}})(\alpha) = (\nabla \mathcal{L})(\alpha) + q(\alpha)$.

Claim: $\mathfrak{g}$ transverse to the zero section of $\mathcal{K}_{k-1}$. Namely, the surjectivity of

$$\big((b,\psi), \delta q\big) \mapsto \mathrm{Hess}_{q,\alpha}(b,\psi) + \delta q(\alpha)$$

at zero $\alpha = (b,\psi)$ and tangent δq.

Indeed, cokernerl of $\mathrm{Hess}_{q,\alpha}$ is finite dimensional, and L^2-orthogonal to the range $= \ker \mathrm{Hess}_{q,\alpha}$.

Suffice to show for $v \in \ker(\mathrm{Hess}_{q,\alpha})\backslash\{0\}$. There exists $\delta q \in \mathcal{P}$ s.t.

$$\langle \delta q(\alpha), v\rangle_{L^2} = \nabla(\delta f) = d(\delta f)(v) \neq 0.$$

True by tame perturbation via cylinder function being large enough and approxminated by dense $\mathcal{P}$. So $\mathfrak{g}^{-1}(0)$ Banach manifold.

$\mathcal{Z} = \mathfrak{g}^{-1}(0)/\mathcal{G}_{k+1} \subset B^*_k(Y) \times \mathcal{P}$, and we have $\mathcal{Z} \to \mathcal{P}$, restricted from pr_2, smooth Fredholm of index 0. The set of regular values is a residual set.

8.5.2. *Reducible case.* $\mathfrak{g}^{\mathrm{red}} : \mathcal{A}_k \times \mathcal{P} \to \mathcal{K}^{\mathrm{red}}_{k-1}$, $(B,q) \mapsto (\nabla \not{\mathcal{L}})^{\mathrm{red}}(B) = (\nabla \mathcal{L})^{\mathrm{red}}(B) + q(B,0)$.

Same proof shows $\mathfrak{g}^{\mathrm{red}}$ transverse to the zero section of $\mathcal{K}^{\mathrm{red}}_{k-1}$. To achieve other conditions in the characterization, perturb in direction normal to the irreducibles.

$\mathcal{P}^\perp \subset \mathcal{P}$ that consists of q vanishing at reducible locus.

Let $\mathrm{Op}^{\mathrm{sa}}$ be the space of self-adjoint Fredholm map $L^2_k(S) \to L^2_{k-1}(S)$ of form $D_{B_0} + h$.

B_0 spin^c connection, h self-adjoint operator extendable to bounded $L^2_j \to L^2_j$, $j \leq k$.

$\mathrm{Op}^{\mathrm{sa}}$ Banach, stratified by kernel dimension. Let $L \in \mathrm{Op}^{\mathrm{sa}}$ with $\ker L = V$, the tangent space to L in its stratum is $\ker(\mathrm{Op}^{\mathrm{sa}} \to \mathrm{Op}^{\mathrm{sa}}(V))$, $N \mapsto \mathrm{pr}|_V \circ N|_V$.

In $\mathrm{Op}^{\mathrm{sa}}$, the set of operators whose spectrum is not simple is countable union of the images of F_n, Fredholm operator of negative index. To define it, first denote $\mathrm{Op}^{\mathrm{sa}}_n \subset \mathrm{Op}^{\mathrm{sa}}$ the space of operators having 0 as eigenvalue of multiplicity exactly n. Then $F_n : \mathrm{Op}^{\mathrm{sa}}_n \times \mathbb{R} \to \mathrm{Op}^{\mathrm{sa}}$, $(L,\lambda) \mapsto L + \lambda$. F_n is local embedding, the normal bundle at $L + \lambda$ is isomorphic to the space of traceless self-adjoint $\ker L \to \ker L$.

D_{q,B_0} defined earlier.

Define $M : \mathcal{A}_k \times \mathcal{P}^\perp \to \mathrm{Op}^{\mathrm{sa}}$, $(B, q^\perp) \mapsto D_{q^\perp, B}$.

Claim: M is transverse to the stratification of $\mathrm{Op}^{\mathrm{sa}}$ and transverse to F_n for all n.

Proof. Let $q^\perp = \nabla f^\perp$. $V := \ker D_{q^\perp,B}$. Regard V as the subspace of normal bundle of $\mathcal{A}_k$ in $\mathcal{C}_k(Y)$.

Facts: Let K compact subset of finite dimensional C^1 submanifold $N \subset B^0_k(T) = \mathcal{C}_k(Y)/\mathcal{G}^0_{k+1}(Y)$, the latter of which has been introduced in the last lecture.

K, N invariant under S^1.

There exist a collection of coclosed forms c_ν and sections Υ of $\mathbb{S}$ and a neighborhood U of K in N s.t. $p : B_k^0(Y) \to \mathbb{R}^n \times \mathbb{T} \times \mathbb{C}^m$ gives an embedding of U.

Choose p s.t. dp embeds S^1-invariant $V = \ker D_{q^\perp,B}$ in $\mathbb{C}^m \subset T_{p(B,0)}(\mathbb{R}^n \times \mathbb{T} \times \mathbb{C}^m)$ from choices made to construct $\mathcal{P}$.

Choose S^1-invariant function δg on $\mathbb{R}^n \times \mathbb{T} \times \mathbb{C}^m$ s.t. $\delta f = \delta g \circ p$.

Hess of $\delta f|_V$ is any S^1-equivariant complex linear self-adjoin $V \to V$. Take δf approximated from $\mathcal{P}$.

Similar for showing transverse to F_n □

$(\mathfrak{g}^{\mathrm{red}})^{-1}(0) \subset \mathcal{A}_k \times \mathcal{P}$.

$(\mathfrak{g}^{\mathrm{red}})^{-1}(0)/\mathcal{G}_{k+1} \subset \mathcal{A}_k/\mathcal{G}_{k+1} \times \mathcal{P}$, we have a map from the former to $\mathcal{P}$ restricted from the pr_2 of the latter. This is Fredholm of index 0.

$\mathcal{P}^\perp \times (\mathfrak{g}^{\mathrm{red}})^{-1}(0)/\mathcal{G}_{k+1} \overset{\mathrm{Id}\times\mathrm{pr}_2}{\to} \mathcal{P}^\perp \times \mathcal{P}$.

Same argument shows $\mathcal{W} \subset \mathcal{P}^\perp \times (\mathcal{A}_k/\mathcal{G}_{k+1})$ consists of $(q^\perp, [B])$ where $D_{q^\perp,B}$ either is non-simple or consists 0 as eigenvalue is a countable union of Banach submanifolds $\mathcal{W}_n$, n of finite positive codimension.

Indeed, at each $x \in \mathcal{W}$, there exists complement to $T_x\mathcal{W}$ contained in $TP^\perp$ direction.

Take product with $\mathcal{P}$, $\mathcal{W} \times \mathcal{P} \subset \mathcal{P}^\perp \times (\mathcal{A}_k/\mathcal{G}_{k+1}) \times \mathcal{P}$.

Similar statement implies that $\mathcal{P}^\perp \times (\mathfrak{g}^{\mathrm{red}})^{-1}(0)/\mathcal{G}_{k+1}$ and $\mathcal{W} \times \mathcal{P}$ are transverse, so the intersection is a locally finite union of Banach submanifolds $U \subset \mathcal{P}^\perp \times (\mathfrak{g}^{\mathrm{red}})^{-1}(0)/\mathcal{G}_{k+1}$ of finite positive codimension.

Projection to $\mathcal{P}^\perp \times \mathcal{P}$ for each component of U is Fredholm of negative index. Sard-Smale gives regular values. $\mathcal{P}^\perp \times \mathcal{P} \overset{\text{addition}}{\to} \mathcal{P}$ maps a residual set to a residual set.

9. Lecture 9

This lecture is about moduli spaces of trajectories and regularity. Exercise session fills some details skipped during the lecture and gives the proof for regularity.

We focus on shifting the functional analytic setting from $I \times Y$ with compact I to $\mathbb{R} \times Y =: Z$.

We saw in the lecture notes that $\mathcal{C}^\tau(I \times Y)$ the τ model (instead of σ model) adapted to the flow picture.

σ model: A typical point in the blow-up configuration space is (A, s, ψ) with constant $s \geq 0$ and $\|\psi\|_{L^2(Z)} = 1$ in 4d.

We restrict spinor $\check{\psi}(t) := \psi(t, \cdot)$ on each slice $Y \cong \{t\} \times Y$ (unique continuation property implies that if $\psi \neq 0$, then $\check{\psi}(t) \neq 0$ for all t.)

To make each slice a σ model (with last entry being of unit length), we need to divide the spinor by its norm and multiply this norm to the middle entry, $(\check{A}(t), s\|\check{\psi}(t)\|_{L^2(Y)}, \frac{\check{\psi}(t)}{\|\check{\psi}(t)\|_{L^2(Y)}})$, making it into a non-negative function $\mathbb{R} \to \mathbb{R}^{\geq 0}$. This is τ model.

$\mathcal{C}_k^\tau(Y)$ with the middle function asked to be in L_k^2. This is no longer a Hilbert/Banach manifold with boundary, but a closed subspace of a Hilbert manifold $\tilde{\mathcal{C}}_k^\tau(Y)$ where the middle function in the latter is $\mathbb{R} \to \mathbb{R}$ (no constraint).

For I compact, we have the correspondence between a point in $\mathcal{C}^\tau(I\times Y)/\mathcal{G}(I\times Y)$ in 4d and a smooth path in $\mathcal{C}^\sigma(Y)/\mathcal{G}(Y)$ in 3d.

But for $I=\mathbb{R}$, need care: $\tilde{\mathcal{C}}_{k,loc} :=$

$\{(A,s,\phi)\in A_0+L^2_{k,loc}(iT^*Z)\times L^2_{k,loc}(\mathbb{R},\mathbb{R})\times L^2_{k,loc}(S^+)\mid \|\check{\psi}(t)\|_{L^2(Y;S)}=1\}$, here $Y\cong\{t\}\times Y$ and $S\cong S^+|_{\{t\}\times Y}$. The subscript loc means L^2_k norm is bounded for function/section restricted to compact $I\times Y$.

We also consider the un-tilded version $\mathcal{C}^\tau_{k,loc}$ where $s(t)\geq 0$.

The gauge group $\mathcal{G}_{k+1,loc}=\{u\in L^2_{k+1,loc}(Z,\mathbb{C})\mid |u(\cdot)|=1\}$, with quotients $B^\tau_{k,loc}\subset\tilde{B}^\tau_{k,loc}$.

$q\in\mathcal{P}^{res}$ residual part such that zeros of $(\nabla\pounds)^\sigma$ non-degenerate.

4d SW map is a section $\mathcal{F}^\tau_q:\tilde{\mathcal{C}}^\tau_{k,loc}(\mathbb{R}\times Y)\to\mathcal{V}^\tau_{k-1,loc}(\mathbb{R}\times Y)$, the fiber of the latter at (A_0,s_0,ϕ_0) is $\{(a,s,\phi)\in$ $L^2_{k-1,loc}(i\mathfrak{su}(S^+))\oplus L^2_{k-1,loc}(\mathbb{R},\mathbb{R})\oplus L^2_{k-1,loc}(S^-)\mid \mathrm{Re}\langle\check{\phi}_0(t),\check{\phi}(t)\rangle_{L^2(Y)}=0 \text{ for all } t\}$. This bundle is not a locally trivial vector bundle, but ok if we choose some projection.

If $\mathfrak{b}$ is zero of $(\nabla\pounds)^\sigma$, $\mathfrak{b}$ corresponds to a translation-invariant $\gamma_\mathfrak{b}\in\mathcal{C}^\tau_{k,loc}(Z)$ s.t. $\mathcal{F}^\tau_q(\gamma_\mathfrak{b})=0$. So $\check{\gamma}_\mathfrak{b}(\cdot)$ is constant.

Definition 9.1. $[\gamma]\in\tilde{B}^\tau_{k,loc}(Z)$ is asymptotic to $[\mathfrak{b}]$ as $t\to\pm\infty$, if $[\tau^*_t\gamma]\to[\gamma_\mathfrak{b}]$ in $\tilde{B}^\tau_{k,loc}$, where $\tau^*_t\gamma:=\gamma(\cdot+t)$. Written as $\underrightarrow{\lim}\gamma=[\mathfrak{b}]$ when $t\to+\infty$, and $\underleftarrow{\lim}[\gamma]=[\mathfrak{b}]$ when $t\to-\infty$.

Definition 9.2. A moduli of trajectories is $M([\mathfrak{a}],[\mathfrak{b}])=\{[\gamma]\in B^\tau_{k,loc}(Z)\mid\mathcal{F}^\tau_q(\gamma)=0,\ \underleftarrow{\lim}\gamma=[\mathfrak{a}],\ \underrightarrow{\lim}\gamma=[\mathfrak{b}]\}$. It is independent of k due to elliptic regularity before. We also have $\tilde{M}([\mathfrak{a}][\mathfrak{b}])$, where we have $[\gamma]\in\tilde{B}^\tau_{k,loc}(Z)$ and with the same other constraints.

$[\gamma]\in M([\mathfrak{a}],[\mathfrak{b}])$ corresponds to $[\check{\gamma}(\cdot)]$ in $B^\sigma_k(Y)$ connecting from $[\mathfrak{a}]$ to $[\mathfrak{b}]$. It determines a relative homotopy class $z\in\pi_1(B^\sigma_k(Y);[\mathfrak{a}],[\mathfrak{b}])$ which is an affine space over $H^1(Y;\mathbb{Z})$, the components of gauge group, via action. So it decomposes into components, $M([\mathfrak{a}],[\mathfrak{b}])=\bigsqcup_z M_z([\mathfrak{a}],[\mathfrak{b}])$. This is most natural way to describe the moduli space of trajectories.

But we need more direction version for transversality:

Choose lifts $\mathfrak{a}$ and $\mathfrak{b}$ in $\mathcal{C}^\sigma_k(Y)$ of zeros $[\mathfrak{a}]$ and $[\mathfrak{b}]$ of $(\nabla\pounds)^\sigma$. Choose smooth $\gamma_0=(A_0,s_0,\phi_0)\in\mathcal{C}^\tau_{k,loc}(\mathbb{R}\times Y)$, which is $\gamma_\mathfrak{a}$ near $-\infty$ and $\gamma_\mathfrak{b}$ near $+\infty$ and $[\check{\gamma}_0]\in z$.

Define $\tilde{\mathcal{C}}^\tau_k(\mathfrak{a},\mathfrak{b})=\{\gamma\in\mathcal{C}^\tau_{k,loc}(Z)\mid\gamma-\gamma_0\in L^2_k(iT^*Z)\times L^2_k(\mathbb{R},\mathbb{R})\times L^2_{k,A_0}(S^+)\}$, here L^2_k is global, and L^2_{k,A_0} means (higher) covariant derivative are defined using the connection A_0 from γ_0.

Sitting inside, we have $\mathcal{C}^\tau_k(\mathfrak{a},\mathfrak{b})$ where we have the middle variable in $L^2_k(\mathbb{R},[0,\infty))$.

The gauge group is defined as $\mathcal{G}_{k+1}:=\{u\in\mathcal{G}_{k+1,loc}\mid u(\mathcal{C}^\tau_k(\mathfrak{a},\mathfrak{b}))\subset\mathcal{C}^\tau_k(\mathfrak{a},\mathfrak{b})\}$. A fact is $\mathcal{G}_{k+1}(Z)=\{u\in\mathcal{G}_{k+1,loc}\mid 1-u\in L^2_{k+1}(Z;\mathbb{C})\}$.

Define $B^\tau_{k,z}([\mathfrak{a}],[\mathfrak{b}])=\mathcal{C}^\tau_k(\mathfrak{a},\mathfrak{b})/\mathcal{G}_{k+1}(Z)$. $B^\tau_k([\mathfrak{a}],[\mathfrak{b}]):=\bigsqcup_z B^\tau_{k,z}([\mathfrak{a}],[\mathfrak{b}])$. The tilde version are defined the same way. They are Hausdorff.

Theorem 9.3. *Let $[\gamma]\in M_z([\mathfrak{a}],[\mathfrak{b}])$. Choose any lift $\gamma\in\mathcal{C}^\tau_{k,loc}(Z)$, choose lifts $\mathfrak{a},\mathfrak{b}$ and γ_0 such that $[\check{\gamma}_0]\in z$. Then there exists $u\in\mathcal{G}_{k+1,loc}$ s.t. $u(\gamma)\in\mathcal{C}^\tau_k(\mathfrak{a},\mathfrak{b})$ (namely, there exists a gauge representative that lies in the direct description). Any two such u and u', we have $u'u^{-1}\in\mathcal{G}_{k+1}(Z)$. So*

$$\gamma\mapsto[(u(\gamma))]=[(u'u^{-1})(u(\gamma))]=[u'(\gamma)]\in B^\tau_{k,\tau}([\mathfrak{a}],[\mathfrak{b}])$$

is well-defined independent of choice u. Then actually this map descends to an injective map from $M_z([\mathfrak{a}],[\mathfrak{b}])$, and this map has the image

$$\{[\gamma] \in B^{\tau}_{k,\tau}([\mathfrak{a},\mathfrak{b}]) \mid \mathcal{F}^{\tau}_q(\gamma) = 0\},$$

and this bijection is homeomorphism. Similarly, we have the statement for the tilde version.

9.0.1. *Local structure of moduli space of trajectories.* We have just realized $M_z([\mathfrak{a}],[\mathfrak{b}])$ as the zero set of $\mathcal{F}^{\tau}_q$. Now we want to show $\mathcal{F}^{\tau}_q$ to be locally non-linear Fredholm between Banach manifolds.

$L_0 + h$ k-ASAFOE on sections of $E \to Y$ as in the last lecture (L_0 SAFOE, $h : \mathcal{C}^{\infty}(E) \to L^2(E)$ extends to bounded $h : L^2_j(E) \to L^2_j(E)$ for all j with $|j| \leq k$). We pull $E \to Y$ back to $E \to Z = \mathbb{R} \times Y$.

Consider the translation-invariant $D = \frac{d}{dt} + L_0 + h$ is bounded $L^2_{j+1}(E) \to L^2_j(E)$ for $|j| \leq k$.

Spectrum for an operator on a real Hilbert space means spectrum of its complexification.

Definition 9.4. Let L_0+h be a k-ASAFOE operator. It is hyperbolic, if spectrum is disjoint from the imaginary axis in $\mathbb{C}$.

Proposition 9.5. *$L_0 + h$ hyperbolic, then $D = \frac{d}{dt} + L_0 + h : L^2_{j+1}(E) \to L^2_j(E)$ is invertible (thus Fredholm).*

Now consider time-dependent h. $D := \frac{d}{dt} + L_0 + h : L^2_1 \to L^2$ (independent of k so we consider lowest k). Family $L_0 + h_t$, $t \in [0,1]$, which is a continuous path in $\{$bounded operator $L^2 \to L^2\}$ with $L_0 + h_0$ and $L_0 + h_1$ hyperbolic.

The spectral flow $\mathrm{sf}(L_0 + h_t)$ = "net number of eigenvalues whose real parts go from negative to positive". We make this precise in the exercise session as a genericity statement.

Proposition 9.6. *L_0 SAFOE on sections of $E \to Y$, h_t bounded $L^2(E) \to L^2(E)$ continuous in t in operator norm with $h_{\pm\infty} = h_{\pm}$, and $L_0 + h_{\pm}$ hyperbolic. Then $Q = \frac{d}{dt} + L_0 + h_t : L^2(\mathbb{R} \times Y; E) \to L^2(\mathbb{R} \times Y; E)$ is Fredholm with index $= \mathrm{sf}(L_0 + h_t)$.*

9.0.2. *Slice.* T^{τ}_j denotes the L^2_j fiber-completion of the tangent bundle of $\tilde{\mathcal{C}}^{\tau}_k(\mathfrak{a},\mathfrak{b})$ (the latter of which has the constraint $\mathrm{Re}\langle \phi_0|_t, \phi|_t \rangle_{L^2(Y;S)} = 0$ in its definition), where ϕ_0 is from the base and ϕ is from the vector.

Write the derivative of the gauge group action as before as $\mathbb{d}^{\tau}_{\gamma}\xi = (-d\xi, 0, \xi\phi_0)$ with $\gamma = (A_0, s_0, \phi_0)$.

Define $\mathcal{S}^{\tau}_{k,\gamma} := \{(A = A_0 + a, s, \phi) \in \tilde{\mathcal{C}}^{\tau}_k(\mathfrak{a},\mathfrak{b}) \mid \mathrm{Coul}^{\tau}_{\gamma}(A_0 + a, s, \phi) = 0\}$, where $\mathrm{Coul}^{\tau}_{\gamma} : \tilde{\mathcal{C}}^{\tau}_k(\mathfrak{a},\mathfrak{b}) \to L^2_{k-1}(i\mathbb{R})$,

$$(A_0 + a, s, \phi) \mapsto -d^* a + iss_0 \mathrm{Re}\langle i\phi_0, \phi\rangle + i|\phi_0|^2 \mathrm{Re}(\frac{\int_Y \langle i\phi_0, \phi\rangle}{\int_Y 1}).$$

The point of this map is its linearization $d_{\gamma}\mathrm{Coul}^{\tau}_{\gamma}$ extends to $\mathbb{d}^{\tau,\dagger} : T^{\tau}_j \to L^2_{j-1}(i\mathbb{R})$ has the following property: $\mathcal{K}^{\tau}_{j,k} := \ker \mathbb{d}^{\tau,\dagger}_{\gamma}$ and $\mathcal{J}^{\tau}_{j,\gamma} = \mathrm{im}\mathbb{d}^{\tau}_{\gamma}$ are complementary closed subspaces spanning $T^{\tau}_{j,\gamma}$ which vary smoothly over the base.

Want to show that restricting to the slice denoted by $\cdot|$, the equation has Fredholm linearization.

$\mathcal{F}^\tau_q|$ a section of $\mathcal{V}^\tau_{k-1}| \to \mathcal{S}^\tau_{k,\gamma}$, where a tame perturbation q chosen s.t. zeros of $(\nabla \text{\pounds})^\sigma$ are non-degenerate.

$\mathcal{V}^\tau_{k-1}$ is not a trivial vector bundle along the path, and we need a projection Π^τ_γ to define linearization (just like differentiating on a sphere), where

$$\Pi^\tau_\gamma : L^2_j(i\mathfrak{su}(S^+)) \oplus L^2_j(\mathbb{R},\mathbb{R}) \oplus L^2_{j,A_0}(S^-) \to \mathcal{V}^\tau_{j,\gamma},$$

$(\eta, r, \psi) \mapsto (\eta, r, \Pi^\perp_{\phi_0(t)}\psi)$ where $\Pi^\perp_{\phi_0(t)}\psi = \psi - \mathrm{Re}\langle \check{\phi}_0(t), \psi(t)\rangle_{L^2(Y;S)}\phi_0$.

$d\mathcal{F}^\tau_q$ is defined by taking derivative in the ambient Banach space, then projecting.

$a = b + cdt$ where b is in temporal gauge, and we denote

$$(a, r, \phi) = ((b, r, \psi), c) = (V, c).$$

$d_{\gamma_0}\mathcal{F}^\tau_q : (V, c) \mapsto \frac{d}{dt}V + d(\nabla\text{\pounds})^\sigma(V) + \mathbb{d}^\sigma_{\gamma_0(t)}c$. Here $\frac{d}{dt}V := \left(\frac{db}{dt}, \frac{dr}{dt}, \Pi^\perp_{\phi_0(t)}(\frac{d\phi}{dt})\right)$. (We used iT^*Y with $i\mathfrak{su}(S^+)$).

We impose the Coulomb gauge fixing condition $0 = \mathbb{d}^{\tau,\dagger}_0(V, c) = \frac{dc}{dt} + \mathbb{d}^{\sigma,\dagger}_{\gamma_0(t)}(V)$. We also assume γ_0 is in temporal gauge for convenience.

SW + gauge fixing, $Q_{\gamma_0} = d_{\gamma_0}\mathcal{F}^\tau_q \oplus \mathbb{d}^{\tau,\dagger}_{\gamma_0}$. In path notation, $(V, c) \mapsto \frac{d}{dt}(V, c) + L_{\gamma_0(t)}(V, c)$.

Here, if denoting $\mathfrak{c} := \gamma_0(t)$, $L_\mathfrak{c} = \begin{pmatrix} d_\mathfrak{c}(\nabla\text{\pounds})^\sigma & \mathbb{d}^\sigma_\mathfrak{c} \\ \mathbb{d}^{\sigma,\dagger}_\mathfrak{c} & 0 \end{pmatrix} = \widehat{\mathrm{Hess}}^\sigma_{q,\mathfrak{c}}$ before.

Theorem 9.7. *Q_{γ_0} is Fredholm for $1 \le j \le k$ with index independent of j and satisfying the semi-Fredholm estimate/Gårding inequality $\|u\|_{L^2_j} \le C_1\|Q_{\gamma_0}u\|_{L^2_{j-1}} + C_2\|u\|_{L^2_{j-1}}$.*

$d(\mathcal{F}^\tau_q|_{\text{slice}}) : \mathcal{K}^\tau_{j,\gamma} \to \mathcal{V}^\tau_{j-1,\gamma}$ Fredholm with index same as that of Q_{γ_0}, also called relative grading

$$\mathrm{gr}_z([\mathfrak{a}], [\mathfrak{b}]) = \mathrm{gr}(\mathfrak{a}, \mathfrak{b}) = \mathrm{sf}(\widehat{\mathrm{Hess}}^\sigma_{q,\hat{\gamma}_0(t)}) = \mathrm{sf}(\begin{pmatrix} 0 & \mathbb{d}^\sigma_{\gamma(t)} \\ \mathbb{d}^{\sigma,\dagger}_{\gamma_0(t)} & 0 \end{pmatrix} \oplus \mathrm{Hess}^\sigma_{q,\gamma_0(t)}) = \mathrm{sf}(\mathrm{Hess}^\sigma_{q,\gamma_0(t)}).$$

9.0.3. *Regularity.* $M_z([\mathfrak{a}], [\mathfrak{b}]) \subset \tilde{M}_z([\mathfrak{a}], [\mathfrak{b}]) \subset \tilde{B}^\tau_{k,z}([\mathfrak{a}], [\mathfrak{b}])$, where the first one has $s(t) \ge 0$.

A neighborhood of $[\gamma]$ in $M_z([\mathfrak{a}], [\mathfrak{b}])$ is the zero set of $\mathcal{F}^\tau_q|_{U_\gamma} : \mathcal{U}_\gamma \to \mathcal{V}^\tau_{k-1}$. If $d_\gamma\mathcal{F}^\tau_q| : \mathcal{K}^\tau_{k,\gamma} \to \mathcal{V}^\tau_{k-1,\gamma}$ is surjective, then $M_z([\mathfrak{a}], [\mathfrak{b}])$ is a manifold near $[\gamma]$ of dimension $\dim\ker d_\gamma\mathcal{F}^\tau_q| = \mathrm{ind}Q_\gamma = \mathrm{gr}_z([\mathfrak{a}], [\mathfrak{b}])$.

Unique continuation means a SW solution $\gamma = (A, s, \phi)$ has either $s \equiv 0$ or $s : \mathbb{R} \to \mathbb{R}\backslash\{0\}$.

In the second case if $s > 0$, $[\gamma] \in M_z([\mathfrak{a}], [\mathfrak{b}])$, so

$$M_z([\mathfrak{a}], [\mathfrak{b}]) = \tilde{M}_z([\mathfrak{a}], [\mathfrak{b}])/(i : [A, s, \phi] \mapsto [A, -s, \phi]).$$

In the flow form of $4d$ SW equation, we have appearance of $\Lambda_q(\mathfrak{a})$, which plays the eigenvalue role in the finite dimensional case. If $\mathfrak{a}$ is a reducible zero, $\mathfrak{a}$ is called boundary-stable if $\Lambda_q(\mathfrak{a}) > 0$, and boundary-unstable if $\Lambda_q(\mathfrak{a}) < 0$.

Lemma 9.8. *If $M_z([\mathfrak{a}], [\mathfrak{b}])$ conatins an irreducible trajectory, then $\mathfrak{a}$ is either irreducible or boundary-unstable, and $\mathfrak{b}$ is either irreducible or boundary-stable.*

Proof. $\frac{ds}{dt} = -\Lambda_q(\check{\gamma}(t))s$ and $s > 0$. We have $\Lambda_q(\check{\gamma}(t)) \to \Lambda_q(\mathfrak{a})$ and to $\Lambda_q(\mathfrak{b})$.

If $\mathfrak{a}$ reducible, $s \to 0$ at $-\infty$, then $\Lambda_q(\mathfrak{a}) < 0$.

If $\mathfrak{b}$ reducible, similarly, we have $\Lambda_q(\mathfrak{b}) > 0$. ∎

If γ reducible, $Q_\gamma = Q^\partial_\gamma \oplus Q^\nu_\gamma$.

$Q^\partial_\gamma = (d_\gamma \mathcal{F}^\tau_q)^\partial \oplus \mathbb{d}^{\tau,\dagger}_\gamma$ whose first factor is invariant under involution i, and $Q^\nu_\gamma : L^2_k(i\mathbb{R}) \to L^2_{k-1}(i\mathbb{R}), s \mapsto \frac{ds}{dt} + \Lambda(\check{\gamma})s$.

After calculation, one can see $(\dim\ker Q^\nu_\gamma, \mathrm{dimcok} Q^\nu_\gamma)$ is $(1,0)$ if $\mathfrak{a}$ is ∂-unstable and $\mathfrak{b}$ is boundary-stable; $(0,1)$ if $\mathfrak{a}$ is boundary-stable and $\mathfrak{b}$ is boundary-unstable (this case is said to be boundary-obstructed, this is still ok due to be constant dimension of cokernel); $(0,0)$ if both $\mathfrak{a}$ and $\mathfrak{b}$ are boundary-stable, or both $\mathfrak{a}$ and $\mathfrak{b}$ are boundary-unstable.

The definition of regular is stated.

The regularity theorem says that there exists a residual $q \in \mathcal{P}^{\mathrm{res}}$, (i) all zeros of $(\nabla\pounds)^\sigma$ non-degenerate; (ii) $M_z([\mathfrak{a}],[\mathfrak{b}])$ is regular.

The proof is provided in the exercise session.

10. Lecture 10

We will discuss gluing and neighborhood of stratum of broken trajectories.

10.0.1. *Compactness of moduli spaces of broken trajectories.* Let $[\mathfrak{a}]$ and $\mathfrak{b}$ be (non-degenerate) zeros of $(\nabla\pounds)^\sigma$.

$M_z([\mathfrak{a}],[\mathfrak{b}])$ is called non-trivial if $[\mathfrak{a}] \neq \mathfrak{b}$ or z is non-trivial.

For any $[\gamma]$ in a non-trivial moduli space, $[\tau_s\gamma]$ will be a different element, where the shift $\tau_s : \gamma \mapsto [\gamma(\cdot + s)]$ descends to a map between the gauge equivalence classes. Let $[\check{\gamma}]$ denote its equivalence class under shift (not to be confused with $\check{\gamma}(t)$ with t variable regarded as a path in $3d$), called an unparametrized trajectory.

$\check{M}_z([\mathfrak{a}],[\mathfrak{b}])$ denotes the moduli space of unparametrized non-trivial trajectories.

Definition 10.1. An unparametrized broken trajectory joining $[\alpha]$ to $[\mathfrak{b}]$ is a tuple $([\check{\gamma}_1],[\check{\gamma}_2],\cdots,[\check{\gamma}_n])$, where

- $n \geq 0$,
- $[\check{\gamma}] \in \check{M}_{z_i}([\mathfrak{a}_{i-1}],[\mathfrak{a}_i])$,
- $[\mathfrak{a}_i]$ for $0 \leq i \leq n$ are zeros of $(\nabla\pounds)^\sigma$, with $\mathfrak{a}_0 = \mathfrak{a}$ and $\mathfrak{a}_n = \mathfrak{b}$, and $z = z_1 * \cdots z_n$ the concatenated homotopy path.

We denote this as $[\check{\gamma}^+]$ (as the mathbb font in the lecture is not native for Greeks in TeX) and call it n-broken (note that we have $n-1$ broken points). If $n = 0$, then $[\check{\gamma}^+] = [\alpha_0]$ by convention.

The moduli space of such broken trajectories is denoted by $\check{M}^+_z([\mathfrak{a}],[\mathfrak{b}])$. $\check{\gamma}_i$ is the representative of the i-th component $[\check{\gamma}_i]$.

We now define the topology for $\check{M}^+_z([\alpha],[\beta])$.

Fix a point $[\check{\gamma}^+] \in \check{M}^+_z([\mathfrak{a}],[\mathfrak{b}])$, we define a neighborhood of it as follows:

Choose $[\gamma_i]$ lifting $[\check{\gamma}_i]$ for all i, and let $U_i \subset B^\tau_{k,loc}(\mathbb{R}\times Y)$ an open neighborhood of $[\gamma_i]$. Let $T \in \mathbb{R}^+$ nonnegative reals.

Consider $[\check{\delta}^+] = ([\delta_1],\ldots,[\delta_m])$, where $m \leq n$ (possibly less components), we need to assign how n components get allocated to those (possibly less) m components via a surjective and order-preserving allocation map $j : \underline{n} := \{1,\cdots,n\} \to \underline{m}$, and if several adjacent components indexed by a subset $i_1, i_k+1,\cdots,i_1+k$ of $\underline{n}$ mapped to same component indexed $j \in \underline{m}$, one should picture that each of those different shifted $[\tau_{s_{i_1+i}}\delta_j]$ is in the neighborhood U_{i_1+i} of $[\gamma_{i_1+i}]$, so $[\delta_j]$ is in the neighborhood of a trajectory 'glued' from $[\gamma_{i_1+i}],\cdots,[\gamma_{i_1+k}]$. Let us package the component allocation and shift allocation map by $(j,s) : \underline{n} \to \underline{m}\times\mathbb{R}$, where

$1 \le i_1 < i_2 \le n$ implies either $j(i_i) \le j(i_2)$ or '$j(i_1) = j(i_2)$ and $s(i_1) + T \le s(i_2)$ (adjacent components have relative shift of at least T away'.

To summarize the above, define

$\Omega := \Omega(U_1, \cdots, U_n, T) := \{[\check{\delta}^+] \in \check{M}_z^+([\mathfrak{a}], [\mathfrak{b}]) \mid [\check{\delta}^+]$ is m-broken for some $m \le n$,

$\exists$ allocation $(j, s) : \underline{n} \to \underline{m} \times \mathbb{R},$ s.t $[\tau_{s(i)}\delta_{j(i)}] \in U_i]\}$.

Ω is considered as an open neighborhood for $[\check{\gamma}^+]$. Those Ω for different $[\check{\gamma}^+]$, U_i's and T define a basis which gives the topology for $\check{M}_z^+([\mathfrak{a}], [\mathfrak{b}])$.

A misleadingly simple to state, but carrying a lot of heavy-lifting in the proof where the most technical ingredient we have covered before is the following:

Theorem 10.2. $\check{M}_z^+([\mathfrak{a}], [\mathfrak{b}])$ *is compact.*

10.0.2. *Stratified spaces.* We give a primitive version of stratified spaces for our purpose (only counting points in codimension 1 strata).

Definition 10.3. N^d is a d-(dim) stratified space if we have filtered inclusion $\emptyset = N^{-1} \subset N^0 \subset \cdots \subset N^{d-1} \subset N^d$, s.t. $N^e \backslash N^{e-1}$ is either empty or e-dim manifold, and in the later case, it is called e-dim stratum (which can consist of several connected components).

Proposition 10.4. *If $M_z([\mathfrak{a}], [\mathfrak{b}])$ is non-empty and dim d (we can always make it regular so this notion of dim makes sense). Then $\check{M}_z^+([\mathfrak{a}], [\mathfrak{b}])$ is a $(d-1)$-stratified space. If $M_z([\mathfrak{a}], [\mathfrak{b}])$ contains an irreducible (namely, an unbroken $[\gamma]$ with $\gamma = (A, s, \phi)$ with $s > 0$), then $(d-1)$-dim/top stratum of $\check{M}_z^+([\mathfrak{a}], [\mathfrak{b}])$ is $\{$irreducibles$\}$.*

We now want to know more about $(d-2)$-dim stratum (aka codim-1 stratum).

Consider (‡): $\check{M}_{z_1}([\mathfrak{a}_0], \mathfrak{a}_1) \times \cdots \times \check{M}_{z_l}([a_{l-1}], [a_l]) \subset \check{M}_z^+([\mathfrak{a}_0], \mathfrak{a}_l)$.

Let the relative grading of each be $\mathrm{gr}_{z_i}([\mathfrak{a}_{i-1}], [\mathfrak{a}_i]) = d_i - \epsilon_i$, where d_i is its dimension and e_i is 1 if it is boundaryt-obstructed (namely $[\mathfrak{a}_{i-1}]$ is boundary-stable and $[\gamma_i]$ is boundary-unstable) and 0 otherwise. We call $(d_1 - \epsilon_1, \cdots, d_l - \epsilon_l)$ the grading vector, and $(\epsilon_1, \cdots, \epsilon_l)$ the obstruction vector. If we reserve d_i for dimension, then we can read obstruction vector from the grading vector (vice versa).

The $(d-1)$-dim stratum, aka top stratum, is the irreducible part of $\check{M}_z([\mathfrak{a}_0], [\mathfrak{a}_l])$. (Note that $[\mathfrak{a}_l]$ is just a notation as the limit agreeding with a given element under consideration, this by no means implies that the elements are l-broken.)

The $(d-1)$-dim stratum, aka codimension-1 stratum, is the union of

- top stratum of (‡) with grading vector (d_1, d_2) (thus obstruction vector $(0, 0)$),
- top stratum of (‡) with grading vector $(d_1, d_2 - 1, d_3)$ (thus obstruction vector $(0, 1, 0)$), and
- (only if $M_z([\mathfrak{a}_0], [\mathfrak{a}_l])$ contains both reducibles and irreducibles)

$$\check{M}_z([\mathfrak{a}_0], [\mathfrak{a}_l]) \cap \{\text{reducibles}\}.$$

10.0.3. *Moduli space on finite cylinders.* We will capture the behavior of moduli space on infinity cylinder using the finite cylinders where action is most concentrated.

Now consider $Z = I \times Y$ with I compact.

$\mathcal{C}_k^\tau(Z) \subset \tilde{\mathcal{C}}_k^\tau(Z)$ with quotient $B_k^\tau(Z) \subset \tilde{B}_k^\tau(Z)$.

Since we do not have boundary condition imposed on, we do not have a Fredholm problem. But still $M(Z) = \{[\gamma] \in B_k^\tau(Z) \mid \mathcal{F}_q^\tau(\gamma) = 0\} \subset B_k^\tau(Z)$ as a Hilbert submanifold. We also have the tilde version. $M(Z)$ is a Hilbert submanifold with boundary and identified as $\tilde{M}(Z)/i$ where i is the involution changing signs of s variable.

10.0.4. *Spectral boundary condition on $\partial Z = \bar{Y} \sqcup Y$.* Let $R_Y : \tilde{M}(Z) \to B^{\sigma}_{k-1/2}(Y)$, $R_{\bar{Y}} : \tilde{M}(Z) \to B^{\sigma}_{k-1/2}(\bar{Y})$ be restricting to the boundary, the regularity $k-1/2$ is due to Trace theorem, as taking trace costs $1/2$ derivative with $p=2$ here and is onto. (We came across fractional Sobolev space before and can safely suppress this technicality now).

$[\gamma] \in \tilde{M}(Z)$, $\mathfrak{a} := \gamma|_Y$, $\bar{\mathfrak{a}} := \gamma|_{\bar{Y}}$.

$T_{[\alpha]} B^{\sigma}_{k-1/2}(Y) \cong K^{\sigma}_{k-1/2,\mathfrak{a}}$ (transverse to the gauge orbit), so

$$(dR_Y, dR_{\bar{Y}}) : T_{[\gamma]}\tilde{M}(Z) \to K^{\sigma}_{k-1/2,\mathfrak{a}}(Y) \times K^{\sigma}_{k-1/2,\bar{\mathfrak{a}}}(\bar{Y}).$$

Using $\mathrm{Hess}^{\sigma}_{q,\mathfrak{a}}$ operator, we have spectral decomposition $K^{\sigma}_{k-1/2,\mathfrak{a}} = K^{+}_{\mathfrak{a}} \oplus K^{-}_{\mathfrak{a}}$ (ignoring the $(\cdot)$ signifying the boundary), $K^{-}_{\bar{\mathfrak{a}}}(\bar{Y}) \cong K^{+}_{\mathfrak{a}}(Y)$.

Let $\Pi : K^{\sigma}_{k-1/2,(\bar{\mathfrak{a}},\mathfrak{a})}(\bar{Y} \sqcup Y) \to K^{-}_{(\bar{\mathfrak{a}},\mathfrak{a})}(\bar{Y} \sqcup Y)$, where the LHS side is defined as $K^{\sigma}_{k-1/2,\bar{\mathfrak{a}}}(\bar{Y}) \oplus K^{\sigma}_{k-1/2,\mathfrak{a}}(Y)$ and the RHS is defined as $K^{+}_{\bar{\mathfrak{a}}}(\bar{Y}) \oplus K^{-}_{\mathfrak{a}}(Y)$ with the kernel the complement of the range in the above decomposition.

Theorem 10.5. $\Pi \circ (dR_{\bar{Y}}, dR_Y)$ *and* $(1-\Pi) \circ (dR_{\bar{Y}}, R_Y)$ *are Fredholm and compact.*

Specifying Lagrangian boundary condition, it becomes Fredholm problem again.

10.1. **Gluing in finite dimension.** L invertible (self-adjoint) SA linear operator $\mathbb{R}^n \to \mathbb{R}^n$. $\dot{\gamma}(t) = -L\gamma(t)$, $M(T) := \{\text{solution } \gamma : [-T,T] \to \mathbb{R}^n\}$.

We have restriction $r : M(T) \to \mathbb{R}^n \times \mathbb{R}^n$, $\gamma \mapsto (\gamma(-T), \gamma(T))$, each factor in the image determines γ. So $\mathrm{im} r = \mathbb{R}^n$.

Want to parametrize $\mathrm{im} r$ so that it converges nicely with respect to $T \to \infty$.

Write $\mathbb{R}^n \times \mathbb{R}^n = (H^+ \oplus H^-) \times (H^+ \oplus H^-)$ spectral decomposition.

$\mathrm{im}(r) = \{(u_+ + e^{2TL}u_-, e^{-2TL}u_+ + u_-) \mid (u_+, u_-) \in H^+ \times H^-\}$ to $H^+ \times H^- = M(\infty)$ decays exponentially as $T \to \infty$.

Here $M(\infty)$ can be thought of as $\{\text{solution } \gamma : R^{geq} \sqcup \mathbb{R}^{\leq} \to \mathbb{R}^n\}$ with restriction $r : M(\infty) \to \mathbb{R}^n \times \mathbb{R}^n$, $r(M(\infty)) = H^+ \times H^-$.

To summarize this above abstractly whose statement can be easily generalized to the SW setting:

For for $T > 0$, there exist parametrizations

$$u(T,\cdot) : \mathbb{R}^n \to M(T) \text{ and } u(\infty,\cdot) : \mathbb{R}^n \to M(\infty)$$

s.t. $\mu_T := r \circ (u(T,\cdot)) : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ converges $\mu_\infty := r \circ (u(\infty,\cdot))$.

Let $W = \bigcup_{T \in (0,\infty]} \mu_T(\mathbb{R}^n)$ has singularity at 0 (all 'slices' $\mu_T(\mathbb{R}^n)$ intersect at 0).

So consider

$$W^0 := \bigcup_{T \in (0,\infty]} \mu_T(\mathbb{R}^n \backslash \{0\}).$$

Unique continuation means $\mu_{T_1}(\mathbb{R}^n \backslash \{0\}) \cap \mu_{T_2}(\mathbb{R}^n \backslash \{0\}) = \emptyset$ if $T_1 \neq T_2$. So W^0 is the injective image of $(0,\infty] \times (\mathbb{R}^n \backslash \{0\})$, and it is a C^0 manifold with boundary $\mu_\infty(\mathbb{R}^{\backslash}\{0\})$, but not a smooth manifold.

10.2. **Non-linear version of the above in Morse theory.** Let B be a compact Riemannian manifold with Morse $f : B \to \mathbb{R}$. Let K_1, K_2 be closed submanifolds lying in the level sets, $f|_{K_1} = 1$, and $f_{K_2} = -1$ and there exists a unique $a \in f^{-1}([-1,1])$, the flow is linear, consider $\dot\gamma = -\nabla f \circ \gamma = -L\gamma$.

$M_S(K_1, K_2) := \{\gamma : [-S, S] \to B \mid \gamma(-S) \in K_1, \gamma(S) \in K_2, \dot\gamma = -\nabla f \circ \gamma\}$, $M(K_1, K_2) = \bigcup_{S>0} M_S(K_1, K_2)$ has a compactification at infinity by attaching $M_\infty(K_1, K_2) = M(K_1, a) \times M(a, K_2)$, where the first factor is solution $\gamma : \mathbb{R}^{\geq 0} \to B$ with $\gamma(0) \in K_1$ and $\gamma(\infty) = a$ and similarly for the second factor.

If $S_a \pitchfork K_1$ and $U_a \pitchfork K_2$ (transversely intersecting), then the compactification is a C^0 manifold with boundary in the neighborhood of $M_\infty(K_1, K_2)$.

10.2.1. *Abstract statement of gluing theorem.* Let $E_0 \to Y$ and let E denote the pullback of E_0 over $\mathbb{R} \times Y$. Let $Du = \frac{du}{dt} + Lu$ with $L : L^2_k(E_0) \to L^2_{k-1}(E_0)$.

We have $D : L^2_k(Z; E) \to L^2_{k-1}(Z; E)$ where $Z = Z^T$ and Z^∞.

Let $L^2_{k,\delta}(Z^\infty; E) = \{s \mid e^{\delta|t|}s \in L^2_k(Z^\infty; E)\}$ and we have $D : L^2_{k,\delta}(Z^\infty; E) \to L^2_{k-1,\delta}(Z^\infty; E)$.

Suppose $\bar\Pi : L^2_{k-1/2}(\bar Y \sqcup Y; E_0) \to H$ where H is a Hilbert space and $\Pi := \bar\Pi \circ R$ with R the boundary restriction. Write $\mathcal{E}^T := L^2_k(Z^T; E)$ and $\mathcal{F}^\infty = L^2_{k-1}(Z^T; E)$, similarly we have $\mathcal{E}^\infty$, $\mathcal{E}^\infty_\delta$, $\mathcal{F}^\infty$, $\mathcal{F}^\infty_\delta$.

Suppose $(D, \Pi) : \mathcal{E}^\infty \to \mathcal{F}^\infty \oplus H$ invertible (then so is the weighted version for δ close enough to 0).

(†) Let C_0 be a constant dominating the norm of the inverse of (D, Π).

Suppose we have smooth $\alpha : L^2_k([-1,1] \times Y; E) \to L^2_{k-1}([-1,1] \times Y; E)$ of the following form: There exists a continuous $\alpha_0 : C^\infty(E_0) \to L^2(E_0)$ and α is the extension from the induced map $C^\infty(E) \to L^2_{loc}(E)$ defined by $\gamma \mapsto (\alpha_0 \circ \check\gamma(t))\hat{}$, where $\hat{}$ denotes the inverse of $\check{}$. Suppose additionally, $\alpha : L^2_k([-1,1] \times Y; E) \to L^2_{k-1}([-1,1] \times Y; E)$ satisfies $\alpha(0) = 0$ and $d_0\alpha = 0$. This then implies that α is smooth as a map $\mathcal{E}^T \to \mathcal{F}^T$, $\mathcal{E}^\infty \to \mathcal{F}^\infty$, $\mathcal{F}^\infty_\delta \to \mathcal{F}^\infty_\delta$, and for all $\epsilon > 0$, there exists $\eta > 0$ s.t. $\|u\|, \|u'\| \leq \eta$ implies that $\|\alpha(u) - \alpha(u')\| \leq \epsilon\|u - u'\|$.

Suppose η_1 chosen from the above for $\epsilon := \frac{1}{2C_0}$ with C_0 chosen above.

$F^T = D + \alpha : \mathcal{E}^T \to \mathcal{F}^T$ with $M(T) = (F^T)^{-1}(0)$, and $F^\infty = D + \alpha : \mathcal{E}^\infty \to \mathcal{F}^\infty$ with $M(\infty) = (F^\infty)^{-1}(0)$. $M(T) \subset \mathcal{E}^T$ and $M(\infty) \subset \mathcal{E}^\infty$ Hilbert submanifolds.

Then there exists $\eta > 0$ and smooth $u(T, \cdot) : B_\eta(H) \to M(T)$ and $u(\infty, \cdot) : B_\eta(H) \to M(\infty)$ each diffeomorphism onto the image, with $\Pi \circ (u(T, \cdot)) = \mathrm{Id} = \Pi \circ (u(\infty, \cdot))$, and for $T \in [T_0, \infty]$, $\mu_T := r \circ (u(T, \cdot))$ smooth embedding from $B_\eta(H)$ and $[T_0, \infty) \times B_\eta(H) \ni (T, h) \mapsto \mu_T(h)$, $\mu_T \to \mu_\infty$ in C^∞_{loc}, and there exists $\eta' > 0$ (independent of T), s.t. $\mathrm{im}(u(T, \cdot)) \supset \{u \in M(T) \mid \|u\| \leq \eta'\}$.

10.2.2. *Applying to the SW setting.* Let $\mathfrak{a} \in \tilde{\mathcal{C}}^\sigma_k(Y)$ be a non-degenerate (by q) zero of $(\nabla \text{£})^\sigma$. Let $\gamma_\mathfrak{a}$ be the associated translational-invariant solution in 4d.

For each $T > 0$, think $\gamma_\mathfrak{a}$ lives on $Z^T = [-T, T] \times Y$. Let

$$Z^\infty = (\mathbb{R}^{\leq} \times Y) \sqcup (\mathbb{R}^{\geq} \times Y).$$

We can define $\tilde{\mathcal{C}}^\tau_{k,loc}(Z^\infty)$ etc.

$\tilde M(Z^\infty, [\mathfrak{a}]) \overset{\text{Hilbert submanifold}}{\subset} \tilde B^\tau_{k,loc}(Z^\infty)$, where we have the limit to be $[\mathfrak{a}]$ at the ends of two half cylinders.

We have r restricting to the boundary and we have spectral decomposition.

Let $\mathcal{S}^\tau_{k,\mathfrak{a}}(Z^T) = \{(A = A_0 + a, s, \phi) \in \tilde{\mathcal{C}}^\tau_k(Z) \mid \langle a|_{\partial Z}, n\rangle = 0, \mathrm{Coul}^\tau_{\gamma_\mathfrak{a}}(A, s, \phi) = 0\}$ with n be normal to the boundary.

$r : \tilde{\mathcal{C}}^\tau_k(Z^T) \to \tilde{\mathcal{C}}^\sigma_{k-1/2}(\bar{Y} \sqcup Y) \times L^2_{k-1/2}(i\mathbb{R})$ where the last coordinate is $\langle a|_{\partial Z}, n\rangle$.

$\mathcal{T}^\sigma_{k-1/2,\mathfrak{a}} \cong \mathcal{J}^\sigma_{k-1/2,\mathfrak{a}}(Y) \oplus \mathcal{K}^\sigma_{k-1/2,\mathfrak{a}}(Y)$.

$\hat{\mathrm{H}}\mathrm{ess}$ ASAFOE hyperbolic ($\mathfrak{a}$ non-degenerate) gives a spectral decomposition $K^+ \oplus K^-$.

Let $H^-_Y = \{0\} \oplus K^- \oplus L^2_{k-1/2}(i\mathbb{R})$ and $H^-_{\bar{Y}} = \{0\} \oplus K^+ \oplus L^2_{k-1/2}(i\mathbb{R})$.

Let $H := H^-_{\bar{Y}} \oplus H^-_Y$ and $\Pi^-_Y : \mathcal{T}^\sigma \oplus L^2_{k-1/2}(i\mathbb{R}) \to H^-_Y$ and $\Pi^-_{\bar{Y}}$, and define $\Pi := \Pi^-_{\bar{Y}} \oplus \Pi^-_Y$.

Apply abstract theorem to

$$\begin{cases} \mathcal{F}^\tau_q(\gamma) = 0 \\ \mathrm{Coul}^\tau_\mathfrak{a}(\gamma) = 0 \\ (\Pi \circ i^{-1} \circ r)(\gamma) = h, \end{cases}$$

where i is the identification of $\mathcal{T}^\tau$ to a subspace in $\tilde{\mathcal{C}}^\tau$, and verify the hypothesis of the abstract theorem.

11. Lecture 11

Lecture 11 is about gluing and the accompanying exercise session is about orientation of moduli spaces. (Lecture 12 finishes up the construction of SWF, and lectures 13-15 cover some calculations and applications.)

11.0.1. *Concatenation of trajectories on adjacent finite cylinders being a trajectory on the union.* Let I be compact. $I = I_1 \sqcup I_2$ with $I_1 \cap I_2 = \{0\}$.

We have restriction map to sub-cylinders

$$\rho : \tilde{M}(I \times Y) \to \tilde{M}(I_1 \times Y) \times \tilde{M}(I_2 \times Y)$$

and restriction to the common boundary

$$R_i : \tilde{M}(I_1 \times Y) \times \tilde{M}(I_2 \times Y) \to B^\sigma_{k-1/2}(\{0\} \times Y).$$

We have $\mathrm{im}\rho \subset \mathrm{Fib}(R_1, R_2) := \{m = (m_1, m_2)] \mid R_1(m) = R_2(m)$.

ρ, being a restriction to a Hilbert submanifold from a smooth map on the configuration space, is smooth.

- $\mathrm{im}\rho = \mathrm{Fib}(R_1, R_2)$, and ρ is a homeomorphism onto this image.
- (R_1, R_2) is transverse to the diagonal in the image, so $\mathrm{Fib}(R_1, R_2)$ is a smooth Hilbert submanifold.
- Then ρ is a diffeomorphism.

The above is true for $\tilde{M}(\mathbb{R} \times Y)$ is fixing $[\mathfrak{a}]$ and $[\mathfrak{b}]$.

If $\tilde{M}([\mathfrak{a}], [\mathfrak{b}])$ is not boundary-obstructed, then (R_1, R_2) is transverse to the diagonal iff $\tilde{M}([\mathfrak{a}], [\mathfrak{b}])$ is regular.

The version for multiple segments also works with R_1 being restriction to the positive boundaries and R_2 being restriction to the negative boundaries.

11.0.2. *Boundary-obstructed case.* For the boundary-obstructed case ($\mathfrak{a}$ and $\mathfrak{b}$ are reducible, and $\mathfrak{a}$ boundary-stable and $\mathfrak{b}$ boundary-unstable.

We have codimension 1 inclusion $\partial B^\sigma_{k-1/2}(Y) \subset \tilde{B}^\sigma_{k-1/2}(Y)$ with the former defined by $\{s = 0\}$. (Note that we are in 3d case and middle variable is a constant.)

We have a map $\pi^\partial : \tilde{B}^\sigma_{k-1/2}(Y) \to \partial B^\sigma_{k-1/2}(Y)$, $[(B, s, \psi)] \mapsto [(B, 0, \psi)]$.

Let us deal with the general case, which is not much different.

$$\tilde{M}([\mathfrak{a}], [\mathfrak{b}]) \cong \tilde{M}(\mathbb{R} \times Y; [\mathfrak{a}], [\mathfrak{b}]) = \bigsqcup_{i=0}^{n} \tilde{M}(i),$$

where $\tilde{M}(0)$ and $\tilde{M}(n)$ are half-infinite cylinders with limit point to $\mathfrak{a}$ and $\mathfrak{b}$ respectively.

Let $1 \le i_0 \le n$. Let

$$R'_+, R'_- : \prod_{i=0}^{n} \tilde{M}(i) \to \tilde{B}' := \partial B^\sigma_{k-1/2}(\{t_0\} \times Y) \times \prod_{i \neq 0} \tilde{B}^\sigma_{k-1/2}(\{t_i\} \times Y)$$

be the restriction (R_+, R_-) to the positive and negative boundaries followed by π^∂ at the $\{t_0\} \times Y$-boundary.

ρ, restricting to sub-cylinders, is a homeomorphism from $M([\mathfrak{a}], [\mathfrak{b}])$ to $\mathrm{Fib}(R'_+, R'_-)$.

If regular, then (R'_+, R'_-) is transverse to the diagonal and ρ is a diffeomorphism; vice versa.

In this boundary-obstructed case, $\mathrm{Fib}(R_+, R_-) \subsetneq \mathrm{Fib}(R'_+, R'_-)$.

Let us specialize to a simple example for notational simplicity. Case $n = 1$ and $t_0 = 0$ and $I_0 = \mathbb{R}^{leq 0}$ and $I_1 = \mathbb{R}^{\geq 0}$.

Let $E\tilde{M}([\mathfrak{a}], [\mathfrak{b}]) := \mathrm{Fib}(R'_+, R'_-)$ has no translation action.

We have $\tilde{M}([\mathfrak{a}], [\mathfrak{b}]) \hookrightarrow E\tilde{M}([\mathfrak{a}], [\mathfrak{b}]) \xrightarrow{\delta} \mathbb{R}$, where $\delta := s([\gamma_+]|_{\{0\}\times Y}) - s([\gamma_-]|_{\{0\}} \times Y)$.

If $E\tilde{M}$ is regular at m (the fiber product is transverse to the diagonal), then near m, it is a smooth manifold of dimension $d + 1$ with $d = \mathrm{gr}_z([\mathfrak{a}], [\mathfrak{b}])$.

11.0.3. *Centered trajectory via localizing.* $\gamma \in \mathcal{C}^\sigma_{k-1/2}(Y)$, $e(\gamma) = \|(\nabla \cancel{L})^\sigma(\gamma)\|_{L^2(Y)}$ smooth in γ and gauge-invariant.

- Any pair of zeros of $\mathfrak{a}$ and $\mathfrak{b}$ of $(\nabla \cancel{L})^\sigma$ (if $[\mathfrak{a}] = [\mathfrak{b}]$, z is non-trivial). There exists $\epsilon > 0$, for any component $[\gamma_i]$ of broken $[\check{\gamma}^+] \in \check{M}^+_z([\mathfrak{a}], [\mathfrak{b}])$, there exists t such that $e(\gamma_i(t)) > \epsilon$.
- For such $\epsilon > 0$, let $\beta(t)$ be a cut-off function $\begin{cases} 0, & \text{if } t \le \epsilon. \\ > 0, & \text{if } t > \epsilon. \end{cases}$ Then, if $[\gamma] \in M_z([\mathfrak{a}], [\mathfrak{b}])$, then $(\beta \circ e \circ \gamma)(t) \ge 0$, not identically zero and supported in a compact interval.

Remark 11.1. $[\check{\gamma}] \in \check{M}_z([\mathfrak{a}], [\mathfrak{b}])$, there exists a unique parametrization $[\gamma] \in M_z([\mathfrak{a}], [\mathfrak{b}])$, the center of distribution $c(\gamma) := c(\beta \circ e \circ \gamma) := \frac{\int t(\beta \circ e \circ e)(t)dt}{\int (\beta \circ e \circ \gamma)(t)dt}$ of $(\beta \circ e \circ \gamma)(t)$ is at 0.

Definition 11.2. For finite $I = [t_1, t_2]$ of length > 2, $[\gamma] \in M(I \times Y)$ is centered, if

(i) $e(\gamma(t)) < \frac{\epsilon}{2}$ for all $t \in [t_1, t_1 + 1] \cup [t_2 - 1, t_2]$.
(ii) $e(\gamma(t)) > \epsilon$ for some $t \in [t_1, t_2]$.

(iii) $c(\gamma)$ is at the center of I, $\frac{1}{2}(t_1+t_2)$. Let $M^{\mathrm{cen}}(I\times Y)\subset M(I\times Y)$ denote the collection of centered ones.

A local centering interval of $[\gamma]\in M_z([\mathfrak{a}],[\mathfrak{b}])$ is I s.t. $[\gamma|_I]$ is centered.

We can have multiple intervals in this definition that are disjoint and ordered with increasing centers.

We can have this for broken trajectories.

$\{[\gamma]\in M(I\times Y)\mid \text{(i) and (ii)}\}$ is an open subset of $M(I\times Y)$ because conditions are open conditions. Inside it, M^{cen} is a closed smooth submanifold.

11.0.4. *Description of neighborhood of strata in spaces of unparametrized broken trajectories.* Consider $\check{M}_{z_1}([\mathfrak{a}_0],[\mathfrak{a}_1]\times\cdots\times\check{M}_{z_n}([\mathfrak{a}_{n-1}],[\mathfrak{a}_n])\subset\check{M}^+_z([\mathfrak{a}],[\mathfrak{b}])$, where $[\mathfrak{a}_0]=[\mathfrak{a}]$, $[\mathfrak{a}_\mathfrak{n}]=[\mathfrak{b}]$ and $z=z_1*\cdots*z_n$.

Let $K_m\subset M([\mathfrak{a}_{m-1}],[\mathfrak{a}_m])$ compact, which induces $\check{K}_m\subset\check{M}$ and denote $\check{\mathbb{K}}=\prod_m\check{K}_m$. Choose ϵ and β as above which gives $M^{\mathrm{cen}}(I\times Y)$.

Proposition 11.3. *We can find L_0 depending on $\check{\mathbb{K}}$ s.t. for any $L\geq L_0$, there exists a neighborhood $\check{W}\supset\check{\mathbb{K}}$ in $\check{M}^+_z([\mathfrak{a}_0],[\mathfrak{a}_n])$ s.t. for all $[\check{\gamma}^+]=([\check{\gamma}_1],\cdots,[\check{\gamma}_l])\in\check{W}$ admits a unique complete collection of local centering intervals $\{I_{i,m}\}$ of length $2L$, of n members.*

Here, a complete collection of local centering intervals is for fixed i, $\{I_{i,m}\}_{m=1}^{n_i}$ adjoint, with increasing centers of lengths $2L$ each, each such interval is local centering, and $(\beta\circ e\circ\gamma)(t)$ is supported in $\bigcup_{m=1}^{n_i}I_{i,m}$.

The proposition gives an identification
$$\mu:\{1,\cdots,n\}\to\{(i,m)\mid 1\leq i\leq l,1\leq m\leq n_i\}.$$
Then, we define $S_j=\begin{cases}C_{\mu(j+1)}-C_{\mu(j)} & \text{if } \mu(j+1),\mu(j) \text{ lie in the same component}\\ \infty \text{ otherwise.}\end{cases}$

So we have $\mathbb{S}=(S_1,\cdots,S_{n-1}):\check{W}\to(0,\infty]^{n-1}$. Local choices of $\mathbb{S}$ agree, fit together into $\mathbb{S}$ defined in $\check{W}$ neighborhood of an entire statum.

Definition 11.4. Let Q be a topological space and $q_0\in Q$. $\Pi:S\to Q$ continuous. $S_0\subset\pi^{-1}(q_0)$. π is a topological submersion along S_0, if for all $s_0\in S_0$, there exist U neighborhood of s_0 in S, Q' neighborhood of q_0 in Q and a homemomorphism $(U\cap S_0)\times Q'\to U$ which identifies the second projection with Π. (S_0 is necessarily is open $\pi^{-1}(q_0)$.)

Denote $\check{M}_i:=\check{M}_{z_i}([\mathfrak{a}_{i-1}],[\mathfrak{a}_i])$.

Theorem 11.5. *If none of the factors $\check{M}_i$ in the stratum of the moduli space is boundary-obstructed. Then there exists neighborhood $\check{W}$ of $\prod_{i=1}^n\check{M}_i$ in $\check{M}^+_z([\mathfrak{a}_0],[\mathfrak{a}_n])$ and $\mathbb{S}:\check{W}\to(0,\infty]^{n-1}$ s.t. $\mathbb{S}^{-1}((\infty,\cdots,\infty))=\prod_i\check{M}_i$ and $\mathbb{S}$ is a topological submersion along $\prod_i\check{M}_i$.*

If for all $i\in O\subset\subset\{1,\cdots,n\}$, $\check{M}_i$ is boundary-obstructed; $i\in O^c$, not boundary-obstructed.

Theorem 11.6. *There exists $\check{W}$ around $\prod_i\check{M}_i$ in $\check{M}^+([\mathfrak{a}_0],[\mathfrak{a}_n])$ s.t.*

(1) $j:\check{W}\subset E\check{W}$ topological embedding with $\mathbb{S}:E\check{W}\to(0,\infty]^{n-1}$ from which $\check{W}\to(0,\infty]^{n-1}$ is restricted via j.
(2) $\mathbb{S}:E\check{W}\to(0,\infty]^{n-1}$ topological submersion alogn fiber at $\infty:=(\infty,\cdots,\infty)$.

(3) $j(\check{W}) \subset E\check{W}$ *is the zero set of a continuous map* $\delta : E\check{W} \to \mathbb{R}^O$ *with* $\delta|_{\textit{fiber at }\infty} = 0$ *(so the fiber at* ∞ *of* $E\check{W}$ *= the fiber at* ∞ *of* $\check{W} = \prod_i \check{M}_i$*).*

(4) Let $\check{W}^0$ *and* $E\check{W}^0$ *denote the subset where non of* S_i *is* ∞*, then* $j|_{\check{W}^0}$ *is an embedding between smooth manifolds, and* $\delta|_{E\check{W}^0}$ *is transverse to zero.*

(5) For $i_0 \in O$ *and* δ_{i_0} *of* δ *associated, then for all* $z \in E\check{W}$*, we have*

$$\begin{cases} \text{if } i_0 \geq 2 \text{ and } S_{i_0-1}(z) = \infty, \text{ then } \delta_{i_0}(z) \geq 0, \\ \text{if } i_0 \leq n-1 \text{ and } S_{i_0}(z) = \infty, \text{ then } \delta_{i_0}(z) \leq 0. \end{cases}$$

To explain the last item, let us look at the simplest boundary-obstructed case:

For $n = 3$, $[\mathfrak{a}_0], \cdots, [\mathfrak{a}_3]$. Let $\check{M}_i := \check{M}_{z_i}([\mathfrak{a}_{i-1}], [\mathfrak{a}_i])$, $i = 1, 2, 3$, as above.

Suppose $[\mathfrak{a}_1]$, $[\mathfrak{a}_2]$ reducible, and $\check{M}_2$ boundary-obstructed.

Let $\check{M}_1^{\mathrm{irr}} \subset \check{M}_1$, $\check{M}_3^{\mathrm{irr}} \subset \check{M}_3$ be the irreducible parts, the top strata. $\check{M}_1^{\mathrm{irr}}$, $\check{M}_2$, $\check{M}_2^{\mathrm{irr}}$ non-empty.

Then, the theorem says that there exists $E\check{W}$ and topological submersion $\mathbb{S} = (S_1, S_2) : E\check{W} \to (0, \infty] \times (0, \infty]$ and map $\delta : E\check{W} \to \mathbb{R}$ vanishes on the fiber over (∞, ∞). The zero set of δ is identified with a neighborhood $\check{W}$ of the subset $\check{M}_1^{\mathrm{irr}} \times \check{M}_2 \times \check{M}_3^{\mathrm{irr}}$ in $\check{M}^+([\mathfrak{a}_0], [\mathfrak{a}_3])$; and $\delta(z) > 0$ if $S_1(z) = \infty$ and $S_2(z)$ finite, and $\delta(z) < 0$ if $S_2(z) = \infty$, $S_1(z)$ finite. Let $E\check{W}^0 := \{\text{finite } \mathbb{S}\}$ and $\check{W}^0$ sits in $E\check{W}^0$ as a smooth submanifold and as the transverse zero of δ.

11.0.5. *Simplistic structure of the proof.* For the unobstructed case:

Fix $L > 0$, let $T_j \in (1, \infty]$, $1 \leq j \leq n-1$, and let $\mathbb{T} := (T_1, \cdots, T_{n-1})$.

$Z^T = [-T, T] \times Y$ for $T < \infty$, $Z^\infty = Z^+ \sqcup Z^-$ two half-infinite cylinders.

$Z_{\mathbb{T}} = Z^- \sqcup Z^L \sqcup Z^{T_1} \sqcup Z^L \cdots Z^{T_n-1} \sqcup Z^L \sqcup Z^+$. Let $l := \#\{T_j = \infty\}$, and gluing boundaries give l infinite cylinders $\mathbb{R} \times Y$.

Fix $[\mathfrak{a}]$. Let d_k metric on $B_k^\tau(I \times Y)$ defined by

$$d_k([\gamma], [\gamma']) = \inf\{\|u\gamma - \gamma'\|_{L^2_{k,\mathfrak{a}}} \mid u \in \mathcal{G}_{k+1}(I \times Y)\}.$$

$M_\eta(Z^\pm; [\mathfrak{a}]) = \{d_j([\gamma], [\gamma_\mathfrak{a}]) \leq \eta\}$, $M_\eta(Z^T; [\mathfrak{a}]) = \{\text{ditto}\}$ and

$$M_\eta(Z^\infty; [\mathfrak{a}]) = \{([\gamma^+], [\gamma^-]) \mid d_k([\gamma^-], [\gamma_\mathfrak{a}])^2 + d_k([\gamma^+], [\gamma_\mathfrak{a}])^2 \leq \eta^2\}.$$

$M_{\mathbb{T}} = M(Z^-; [\mathfrak{a}_0]) \times M(Z^L) \times M(Z^{T_1}) \times \cdots \times M(Z^{T_{n-1}}) \times M(Z^L) \times M(Z^+; [\mathfrak{a}_n]$, and contained in there is:

$M^{\mathrm{cen}}_{\mathbb{T},\eta} = M_\eta(Z^-; [\mathfrak{a}_0] \times M^{\mathrm{cen}}(Z^L) \times M_\eta(Z^{T_1}; [\mathfrak{a}_1]) \times \cdots \times M_\eta(Z^{T_{n-1}}; [\mathfrak{a}_{n-1}]) \times M^{\mathrm{cen}}(Z^L) \times M_\eta(Z^+; [\mathfrak{a}_n])$.

$(R^-_{\mathbb{T}}, R^+_{\mathbb{T}})$ restricts $M_{\mathbb{T}}$ to the positive and negative boundaries.

$\mathrm{Fib}(R^-_{\mathbb{T}}, R^+_{\mathbb{T}}) \subset M_{\mathbb{T}}$ fiber product.

Write $\mathbb{M} = \bigcup_{\mathbb{T}} M_{\mathbb{T}}$, $\mathbb{M}^{\mathrm{cen}}_\eta = \bigcup_{\mathbb{T}} M^{\mathrm{cen}}_{\mathbb{T},\eta}$, and $\mathrm{Fib}(R^-, R^+) = \bigcup_{\mathbb{T}} \mathrm{Fib}(R^-_{\mathbb{T}, R^+_{\mathbb{T}}})$.

Concatenation: $\mathbb{c}_{\mathbb{T}} : \mathrm{Fib}(R^-_{\mathbb{T}}, R^+_{\mathbb{T}}) \to \check{M}^+([\mathfrak{a}_0], [\mathfrak{a}_n])$ into l-broken trajectory.

If η small, $m \in \mathrm{Fib}(R^-_{\mathbb{T}}, R^+_{\mathbb{T}}) \cap M^{\mathrm{cen}}_{\mathbb{T},\eta}$, $\mathbb{c}_{\mathbb{T}}(m)$ has a complete collection of n local centering intervals of length $2L$ from those Z^L's.

$$S_i(m) = \begin{cases} 2T_i + 2L & \text{if } T_i < \infty \\ \infty & \text{if } T_i = \infty \end{cases}.$$

Let $\mathbb{c} = \bigcup_{\mathbb{T}} \mathbb{c}_{\mathbb{T}} : \mathrm{Fib}(R^-, R^+) \to \check{M}^+([\mathfrak{a}_0], [\mathfrak{a}_n])$.

Previous lemma on the existence of unique complete collection provides a canonical right inverse for $\mathbb{c}$ on $\check{W}$.

Proposition 11.7. *Let $\breve{\mathbb{K}} \subset \prod_i \breve{M}_i$ compact. There exists $\eta_0 > 0$ s.t. $\eta < \eta_0$ and for all $L \geq L_1(\eta)$, $im(\mathbb{c}) \supset \breve{W} \supset \breve{\mathbb{K}}$ and $\mathbb{c}|_{\mathbb{c}^{-1}(\breve{W})\cap \mathbb{M}_\eta^{cen}}$ is injective.*

Obstructed case:

For $i \in O$, take $\tilde{M}([-L,0]\times Y)\times M([0,L]\times Y)$, and R'_+, R'_- restrict to $\partial B^\sigma_{k-1/2}(\{0\}\times Y)$ and repeat.

11.0.6. *Orientation of moduli spaces.* This is explained during the exercise session.

12. Lecture 12: Construction for Floer homologies and calculation for S^3

Last time in exercise session,

2 element set (of orientations)

$$\Lambda([\mathfrak{a}],q) := \bigsqcup_{[\mathfrak{a}_0]\text{ reducible}} \Lambda([\mathfrak{a}],q;[\mathfrak{a}_0],0)/\big(\lambda \sim q(\lambda,\tau^{-1}(0))\big),$$

where $[\mathfrak{a}] \in B^\sigma_k(Y)$ and q a tame perturbation, $\tau : \Lambda([\mathfrak{a}_0],0;[\mathfrak{a}'_0,0]) \to \mathbb{Z}/2$ trivialization, and q concatenation.

Remark 12.1. Analogous to orientation set of U_a in f.d. Morse theory.

$\Lambda([\mathfrak{a}_1],q_1)\Lambda([\mathfrak{a}_2],q_2) \to \Lambda([\mathfrak{a}_1],q_1;[\mathfrak{a}_2],q_2), [(\lambda_1,\lambda_2)] \mapsto q(\lambda_1,\rho(\lambda_2))$ where ρ is take the inverse of the path. This way two absolute orientation (relative orientations relative to reducibles, equivalent over all reducibles) induces a relative orientation.

Here, for 2-element sets A and B, $AB := A\times_{\mathbb{Z}/2} B$ where $\mathbb{Z}/2$ acts on both factors non-trivially.

12.1. Inducing orientation of moduli spaces from relative orientation.

Fix q, write $\Lambda([\mathfrak{a}]) := \Lambda([\mathfrak{a}],q)$, and $\Lambda[\mathfrak{a}_1],[\mathfrak{a}_2] := \Lambda([\mathfrak{a}_1],q;[\mathfrak{a}_2],q)$.

How does an element of $\Lambda([\mathfrak{a}_1],[\mathfrak{a}_2])$ determine an orientation of $M([\mathfrak{a}_1],[\mathfrak{a}_2])$?

$[\gamma] \in M([\mathfrak{a}_1],[\mathfrak{a}_2])$, $\zeta = [\check{\gamma}]$ path in $B^\sigma_k(Y)$, $P_\gamma := Q_\gamma = \frac{d}{dt} + L$. Critical points $\mathfrak{a}_i$ nondegenerate so $L(t)$ hyperbolic for $|t| \geq T$ for T large enough. $H^-(L(t))$ varies continuously for $|t|\geq T$, so $\Lambda([\mathfrak{a}_1],[\mathfrak{a}_2]) = \Lambda(\zeta(t_1),\zeta(t_2))$ for any $t_1 \leq -T$ and $t_2 \geq T$.

12.1.1. *∂-unobstructed case.* $\Lambda^{\mathrm{top}}T_{[\gamma]}M([\mathfrak{a}_1],[\mathfrak{a}_2]) \cong \det P_\gamma$.

$\gamma = \gamma_1 \cup \gamma_0 \cup \gamma_+$, where γ_- and γ_+ are for $t \leq t_1$ and $t \geq t_2$ respectively.

The earlier restriction map restricting to the fiber product is an isomorphism, which implies $\det P_\gamma = \det P_{\gamma_-} \otimes \det P_{\gamma_0} \otimes \det P_{\gamma_+}$. $P_{\gamma_\pm}$ close to constant coefficient operator, thus invertible, and orientation is canonical, and P_{γ_0} has orientation set $\Lambda(\zeta(t_1),\zeta(t_2)) = \Lambda([\mathfrak{a}_1],[\mathfrak{a}_2])$.

12.1.2. *∂-obstructed case.* $\ker P_\gamma = M([\mathfrak{a}_1],[\mathfrak{a}_2])$, and $\mathrm{cok} P_\gamma = \mathrm{cok}(\frac{d}{dt}+\lambda(t))$ which is oriented by $(0,0,-1)$ (just as already in definition of $\Lambda([\mathfrak{a}_0],q;[\mathfrak{a}'_0],q')$ done in the exercise session).

$\Lambda([\mathfrak{a}_1],[\mathfrak{a}_2]) = \det P_\gamma = \det T_{[\gamma]}M([\mathfrak{a}_1],[\mathfrak{a}_2]) \otimes \mathrm{cok}^* \cong \Lambda^{\mathrm{top}}T_{[\gamma]}M([\mathfrak{a}_1],[\mathfrak{a}_2])$.

Therefore, regularity of moduli spaces implies orientability.

12.2. Orientation for the unparametrized moduli spaces.

$\mathbb{R} \hookrightarrow M_z([\mathfrak{a}],[\mathfrak{b}]) \to \breve{M}_z([\mathfrak{a}],[\mathfrak{b}])$. $t : [\gamma] \mapsto [\tau_t^*\gamma]$.

For bundle, fiber first convention plus standard orientation for $\mathbb{R}$ implies orientation of $\breve{M}$.

12.3. **Orientation under gluing.**

12.3.1. *∂-unobstructed.* $\check{M}_{z_1}([\mathfrak{a}_0],[\mathfrak{a}_1]) \times \check{M}_{z_2}([\mathfrak{a}_1],[\mathfrak{a}_2]) \subset \check{M}^+_z([\mathfrak{a}_0],[\mathfrak{a}_2])$

Let λ_{ij} be the induced relative orientation for $\check{M}([\mathfrak{a}_i],[\mathfrak{a}_j])$.

The LHS has orientation $q(\lambda_{01},\lambda_{12})$. The RHS has an orientation induced from $\check{M}_z([\mathfrak{a}_0],[\mathfrak{a}_2])$ which has $\lambda_{0,2}$ and using outwards normal first, we have an orientation for $\partial\check{M}^+$.

Those two orientations differ by $(-1)^{\dim M_{z_1}([\mathfrak{a}_0],[\mathfrak{a}_1])}$.

12.3.2. *∂-obstructed.* $M_{ii+1} := M_{z_{i+1}}([\mathfrak{a}_i],[\mathfrak{a}_{i+1}])$. Let

$$M := \prod_{i=0}^{2} \check{M}_{ii+1} = \check{M}_{01} \times \check{M}_{12} \times \check{M}_{23}.$$

Let us assume $\check{M}_{12}$ is ∂-obstructed. We have orientation λ_{01}, λ_{12}, λ_{23}, thus an orientation $q(\lambda_{01}\lambda_{12}\lambda_{23})$ on M.

The description of neighborhood of M in $\check{M}^+([\mathfrak{a}_0],[\mathfrak{a}_3]) =: N$. $\exists\, W \overset{\text{open}}{\subset} N$ gives

$$\begin{tikzcd}
M \arrow[r, "\text{fiber}"] \arrow[d] & W \arrow[r, "\text{top. emb. } j"] \arrow[d] & EW \arrow[r, "\delta"] \arrow[d, "\text{top. subm.}"'] & \mathbb{R} \\
\{(\infty,\infty)\} \arrow[r] & (0,\infty]^2 \arrow[r] & (0,\infty]^2 &
\end{tikzcd}$$

This structure is called codim 1 δ-structure along M.

EW locally $M \times (0,\infty]^2$ (fiber first order) and get an orientation from orientation $q(\lambda_{01}\lambda_{12}\lambda_{23})$ and standard one on $(0,\infty]^2$.

$W \hookrightarrow EW \xrightarrow{\delta} \mathbb{R}$ with fiber first convention gives an orientation on W, and thus an orientation on N.

Definition 12.2. M is said to have boundary orientation if the orientation λ_{03} on the top stratum of $N := \check{M}^+([\mathfrak{a}_0],[\mathfrak{a}_3])$ and the induced orientation on W (thus N) differ by $(-1)^{\dim \check{M}([\mathfrak{a}_0],[\mathfrak{a}_3])} = (-1)^{\dim M([\mathfrak{a}_0],[\mathfrak{a}_3])+1}$.

For $M = \prod_{i=0}^2 M_{ii+1}$, the boundary orientation and $q(\lambda_{01}\lambda_{12}\lambda_{23})$ differ by $(-1)^{\dim M_{01}+1}$.

12.4. **Reducible moduli spaces.**
The reducible moduli space consists of trajectories in reducible part.

- If $[\mathfrak{a}]$ boundary-unstable and $[\mathfrak{b}]$ boundary-stable, $M^{\mathrm{red}}([\mathfrak{a}],[\mathfrak{b}]) = \partial M([\mathfrak{a}],[\mathfrak{b}])$ which has orientation from outwards normal first convention. Otherwise, $M^{\mathrm{red}} = M$ which inherits an orientation.
- Repeat early using reducible part (which behaves nicely under gluing), this gives orientation on M^{red}.
- Both orientations agree, except when both $[\mathfrak{a}]$ and $[\mathfrak{b}]$ are both boundary-unstable and they differ by $(-1)^{\dim M} = (-1)^{\dim M^{\mathrm{red}}}$.

12.5. **Evaluation at the boundary components.**
$\mathcal{U}$, an open cover for topological space B, has covering order $\le d+1$ if every $d+2$ intersection $U_{i_0} \cap \cdots \cap U_{i_{d+1}} = \emptyset$ for distinct $U_i \in \mathcal{U}$.

A metric space has covering dimension $\le d$ if every open cover has a refinement with covering order $\le d+1$.

Čech cohomology $\check{H}^n(B;\mathbb{Z}) := \varinjlim H^n_{\mathrm{Simp}}(K(\mathcal{U});\mathbb{Z})$ where $K(\mathcal{U})$ is the nerve which is a simplicial space to which one can attach simplicial cohomology.

If $B' \subset B, \mathcal{U}' := \mathcal{U}|_B := \{U \cap B\}$. $K(\mathcal{U}|_B) \overset{\text{subcomplex}}{\subset} K(\mathcal{U})$.

We have $\check{H}(B, B';\mathbb{Z}) = \varinjlim H^n_{\mathrm{Simp}}(K(\mathcal{U}), K(\mathcal{U}');\mathbb{Z})$.

If $N^d \supset N^{d-1} \supset \cdots \supset N^0$ stratified, we have covering dimension $\leq d$ (fact).

If each $M^e := N^e \backslash N^{e-1}$ is oriented, we have $\check{H}^d(N^d, N^{d-1};\mathbb{Z}) = H^d_c(M^d;\mathbb{Z})$. Here $M^d = \bigsqcup_\alpha M^d_\alpha$.

Let $I_\alpha : \check{H}^d(N^d, N^{d-1};\mathbb{Z}) \to \mathbb{Z}$, $\mu^d_\alpha \mapsto 1$, where μ^d_α is the generator for M^d_α.

LES for (N^d, N^{d-1}, N^{d-2}) gives coboundary map $\delta_* : H^{d-1}_c(M^{d-1};\mathbb{Z}) \to H^d_c(M^d;\mathbb{Z})$, namely , $\bigoplus_\beta H^{d-1}_c(M^{d-1}_\beta;\mathbb{Z}) \to \bigoplus_\alpha H^d_c(M^d_\alpha;\mathbb{Z})$, defined as $\sum \delta_{\alpha\beta}$ wrt this splitting, where $\delta_{\alpha\beta} = I_\alpha \delta_* \mu^{d-1}_\beta$ is the multiplicity of M^{d-1}_β in the boundary of M^d_α. Boundary multiplicity is $\delta_\beta = \sum_\alpha \delta_{\alpha\beta}$.

N^d d-dimensional stratified space compact with an embedding to a metric B. An open cover $\mathcal{U}$ of B is transverse to N^d if $\mathcal{U}|_{N^e}$ has covering order $\leq e+1$.

Fact: $\{N^{d_k}_k\}$ countable locally finite collection of stratified manifolds. Then every open cover of B has a refinement that is transverse to $\{N^{d_k}_k\}$.

This implies that $\check{H}^n(B;\mathbb{Z}) = \varinjlim H^n_{\mathrm{Sim}}(K(\mathcal{U};\mathbb{Z}))$.

$u \in \check{C}^d(\mathcal{U}|_{N^d};\mathbb{Z}) = C^d_{\mathrm{Sim}}(K(\mathcal{U}|_{N^d});\mathbb{Z})$ is coclosed and vanishes on $C^{d-1}_{\mathrm{Sim}}(K(\mathcal{U}|_{N^{d-1}});\mathbb{Z})$, so $[u \in \check{H}^d(N^d, N^{d-1};\mathbb{Z})]$ can integrate over $\bigsqcup_\alpha M^d_\alpha = M^d = N^d \backslash N^{d-1}$.

$\langle u, [M_\alpha]\rangle := I_\alpha[u|_{N^d}]$

Stokes theorem says for $v \in \check{C}^{d-1}(\mathcal{U};\mathbb{Z})$, $\sum_\beta \delta_{\alpha\beta}\langle v, [M^{d-1}_\beta]\rangle = \langle \delta v.[M^d_\alpha]\rangle$.

For ∂-obstructed case.

Lemma 12.3. *N^d d-dimensional compact stratified and stratum-oriented. If N^d has codim 1 δ-structure along M^{d-1}_β. Then $\delta_\beta = 1$.*

For $d = 1$, N^0 has boundary orientation, then $\#_{sign} N^0 = 0$.

12.6. **Floer homology.** Y compact connected oriented Riemannian 3-manifold (with Riem. metric g) and spin^c structure $\mathfrak{s}$. $\mathcal{P}$ tame perturbation Banach space. $q \in \mathcal{P}^{\mathrm{res}}$ chosen s.t. all zeros of $(\nabla \mathcal{L})^\sigma$ non-degenerate and $M([\mathfrak{a}],[\mathfrak{b}])$ regular. Moreover, if f $c_1(\mathfrak{s})$ not torsion, q small, $\nexists$ reducible solution/zero (doable, remark after proposition 16.4.3. in [KM]), such q is called admissible.

Define $\check{HM}_*(Y,\mathfrak{s})$, $\widehat{HM}_*(Y,\mathfrak{s})$, $\overline{HM}_*(Y,\mathfrak{s})$. M for monopole.

$\mathfrak{c} \subset B^\sigma_k(Y,\mathfrak{s})$ zeros of $(\nabla \mathcal{L})^\sigma$ split into $\mathfrak{c}^o \sqcup \mathfrak{c}^s \sqcup \mathfrak{c}^u$, irreducible, boundary-stable, boundary-unstable zeros.

$\Lambda = \{x, y\}$ 2-element (orientation) set. $\mathbb{Z}\Lambda = \mathbb{Z}\langle x, y\rangle/(x = -y) \cong \mathbb{Z}$ if preferred element in Λ is chosen.

$C^0 = \bigoplus_{[\mathfrak{a}]\in\mathfrak{c}^o} \mathbb{Z}\Lambda([\mathfrak{a}])$, C^s and C^u similarly.

Define $\check{C} = C^0 \oplus C^s$, $\hat{C} = C^0 \oplus C^u$, $\overline{C} = C^s \oplus C^u$.

A choice of $\Lambda([\mathfrak{a}],[\mathfrak{b}])$ determines an orientation of $\check{M}_z([\mathfrak{a}],[\mathfrak{b}])$, which leads to $\epsilon[\gamma] : \mathbb{Z}]\Lambda([\mathfrak{a}]) \to \mathbb{Z}\Lambda([\mathfrak{b}])$ for all $[\gamma] \in \check{M}_z([\mathfrak{a}],[\mathfrak{b}])$.

For $[\gamma] \in \check{M}^{\mathrm{red}} M_z([\mathfrak{a}],[\mathfrak{b}])$, we have similarly $\overline{\epsilon}[\gamma]$ agrees with $\epsilon[\gamma]$ except $\overline{\epsilon}[\gamma] = -\epsilon[\gamma]$ when both $[\mathfrak{a}],[\mathfrak{b}] \in \mathfrak{c}^u$.

Define $\bar{\partial} : \overline{C} \to \overline{C}$ as $\sum_{[\mathfrak{a}],[\mathfrak{b}]} \sum_{[\gamma]\in \check{M}^{\mathrm{red}}_z([\mathfrak{a}],[\mathfrak{b}),\ 0\text{-dim}} \bar{\epsilon}\gamma]$, here the sum is finite for each $\mathbb{Z}\Lambda([\mathfrak{a}])$. It can be written as $\begin{pmatrix} \bar{\partial}^s_s & \bar{\partial}^u_s \\ \bar{\partial}^s_u & \bar{\partial}^u_u \end{pmatrix}$.

Similar can define ∂^o_o, ∂^o_s, ∂^u_s, ∂^u_o,
e.g. $\partial^o_o = \sum_{[\mathfrak{a}]\in\mathfrak{c}^o \text{ supscript}} \sum_{[\mathfrak{b}]\in\mathfrak{c}^o \text{ subscript}} \sum_{[\gamma]\in \breve{M}_z([\mathfrak{a}],[\mathfrak{b}]),\ 0\text{-dim}} \epsilon[\gamma]$.

Define $\check{\partial} = \begin{pmatrix} \partial^o_o & -\partial^u_o\bar{\partial}^s_u \\ \partial^o_s & \bar{\partial}^s_s - \partial^u_s\bar{\partial}^s_u \end{pmatrix}$ on $\check{C} = C^o \oplus C^s$, this calculates Morse homology of $B^\sigma_k(Y,\mathfrak{s})$ w.r.t. gradient-like $(\nabla\pounds)^\sigma$.

Define $\hat{\partial} = \begin{pmatrix} \partial^o_o & \partial^u_o \\ -\bar{\partial}^s_u\partial^o_s & -\bar{\partial}^u_u - \bar{\partial}^s_u\partial^u_s \end{pmatrix}$ on $\hat{C} = C^o \oplus C^u$, this calculates Morse homology of $\big(B^\sigma_k(Y,\mathfrak{s}), \partial B^\sigma_k(Y,\mathfrak{s})\big)$ w.r.t. gradient-like $(\nabla\pounds)^\sigma$.

Note that we are able to upgrade mod 2 into signed coefficient after discussion of orientation.

$\bar{\partial}$, $\check{\partial}$, $\hat{\partial}$ squares to 0, by considering the 1-dimensional stratified compactified space of broken trajectory and use $\#_{\text{sign}} N^0(\breve{M}^+) = 0$ (same line of reasoning as in Morse theory with vertical boundary and see some proofs earlier in the lecture).

We have L.E.S. relating all three (see verbatim construction and proof in earlier lecture).

Definition 12.4. $HM_*(Y,\mathfrak{s}) = \mathrm{im}(j_*)$ called reduced monopole Floer homology, where $\check{HM}_*(Y,\mathfrak{s}) \xrightarrow{j_*} \widehat{HM}_*(Y,\mathfrak{s})$, $j = \begin{pmatrix} 1 & 0 \\ 0 & -\bar{\partial}^s_u \end{pmatrix}$.

Proposition 12.5. *$HM(Y,\mathfrak{s})$ is of finite rank.*

Proof. If $c_1(\mathfrak{s})$ not torsion, $\mathfrak{c} = \mathfrak{c}^o$ finite due to compactness and $\check{HM} = \widehat{HM} = HM$.

If $c_1(\mathfrak{s})$ torsion, $\bar{\partial}^s_u$, $\bar{\partial}^u_s$, ∂^u_s finitely many nonzero matrix entries, because nonzero only when $\mathrm{gr}_z([\mathfrak{a}_i],[\mathfrak{b}_j]) = d + 2i - 2j = 0$ and $i > 0$ and $j < 0$. Here $\mathfrak{a}_i$ and $\mathfrak{b}_j$ blow down $[\alpha]$ and $[\beta]$. $\mathfrak{a} = (\alpha,\lambda)$ and $\alpha = (B,0)$.

$$i = i(\mathfrak{a}) = \begin{cases} |\mathrm{Spec} D_{q,B} \cap [0,\lambda)|, \lambda > 0 \\ \frac{1}{2} - |\mathrm{Spec} D_{q,B} \cap [\lambda, 0]|, \lambda < 0 \end{cases},$$

and $j := i(\mathfrak{b}_j)$ similarly.

$\mathrm{rk} HM_*(Y,\mathfrak{s}) \le \mathrm{rk} C^0 + \mathrm{rk}\bar{\partial}^s_u$.

Suppose $H \subset HM_*(Y,\mathfrak{s})$ generated in $0 \oplus C^s$ such that $j_*|_H$ injective.

Then $j(H) \subset (0, \bar{\partial}^s_u(H))$ finite rank. □

12.7. **Grading.** Define $\mathbb{J}(\mathfrak{s})$ grading for $\check{HM}_*$ etc.

$\mathbb{J}(Y,\mathfrak{s}) := B^\sigma_k(Y, \mathfrak{s} \times \mathcal{P} \times \mathbb{Z})/\sim$, where $([\mathfrak{a}], q_1, m) \sim ([\mathfrak{b}], q_2, n])$ if $\exists$ path $\zeta = [\check{\gamma}]$ joining $[\mathfrak{a}]$ to $[\mathfrak{b}]$, and p 1-parameter perturbation from q_1 to q_2, s.t. $\mathrm{ind} D_{\gamma,p} = n - m$.

$\mathbb{Z}$ acts on $\mathbb{J}(\mathfrak{s})$, $([\mathfrak{a}, q, m]) \mapsto ([\mathfrak{a}, q, m+1])$.

$\mathrm{gr}[\mathfrak{a}] = ([\mathfrak{a}, q, 0])/\sim \in \mathbb{J}(\mathfrak{s})$.

$\mathfrak{a}$ reducible, define $\bar{\mathrm{gr}}[\mathfrak{a}] = \begin{cases} \mathrm{gr}[\mathfrak{a}], & \mathfrak{a} \in \mathfrak{c}^s, \\ \mathrm{gr}[\mathfrak{a}] - 1, & \mathfrak{a} \in \mathfrak{c}^u \end{cases}$.

$\check{C}_j = \bigoplus_{[\mathfrak{a}]\in\mathfrak{c}^o\cup\mathfrak{s}^s,\ \mathrm{gr}[\mathfrak{a}]=j} \mathbb{Z}\Lambda([\mathfrak{a}])$, etc.

$\check{\partial}$, $\hat{\partial}$, $\bar{\partial}$ differential of degree -1, $\check{\partial}(\check{C}_j) \subset \check{C}_{j-1}$.

$\mathbb{Z}$ acts transitively on $\mathbb{J}(\mathfrak{s})$ with stabilizer $\mathrm{im}([\sigma] \in H_2(Y;\mathbb{Z}) \mapsto \langle c_1(\mathfrak{s}), [\sigma]\rangle) \subset 2\mathbb{Z}$. Thus free, iff $c_1(\mathfrak{s})$ torsion.

12.8. **Calculation of Monopole Floer homology for S^3 with round metric.** The unperturbed CSD for the unique spin^c structure $\mathfrak{s}$ has 1 critical point, reducible

$[B,0]$ with $F_{B^t} = 0$ (becuase 3d Riemannian Y with scalar curative ≥ 0, the only critical $[B,\psi]$ of unperturbed CSD is reducible).

In $B_k^\sigma(S^3,\mathfrak{s})$, zeros of $(\nabla \not{L})^\sigma$ degenerate (as the spectrum D_B is not simple). Choose small $q \in \mathcal{P}$ still unique critical point, $[B,0]$ reducible, but $D_{q,B}$ now has simple nonzero eigenvalues. Label increasinglyly λ_i with λ_0 being the first positive eigenvalue. Eigenvalues are 1-1 corresponding to gauge equivalence classes of zeros of $(\nabla \not{L})^\sigma$, $[\mathfrak{a}_i] = ([B,0],\lambda_i)$.

$[\mathfrak{a}_i], i \geq 0, \in \mathfrak{c}^s$.

$[\mathfrak{a}_i], i < 0, \in \mathfrak{c}^u$.

By assigning $[\mathfrak{a}_0] \mapsto 0$, then $\mathbb{J}(S^3,\mathfrak{s}) \cong \mathbb{Z}$, having free transitive $\mathbb{Z}$-action.

If λ_i, λ_{i-1} have the same sign, then $\mathrm{gr}_z([\mathfrak{a}_i],[\mathfrak{a}_{i-1}]) = 2$ independent of z.

The remaining case: $\mathrm{gr}_z([\mathfrak{a}_0],[\mathfrak{a}_{-1}]) = 1$.

Therefore, $\mathrm{gr}[\mathfrak{a}_i]$ has $\mathbb{Z}$-grading, $= \begin{cases} 2i & i \geq 0 \\ -2i+1 & i<0 \end{cases}$.

$\Lambda[\mathfrak{a}_i] = \Lambda([\mathfrak{a}_i],[\mathfrak{a}_i])/\sim$ has preferred element.

So $\check{C}_j = \mathbb{Z}$ for even $j \geq 0$ and 0 otherwise; $\hat{C}_=\mathbb{Z}$ for odd $j<0$ and 0 otherwise; and $\overline{C}_j = \mathbb{Z}$ for even j (due to the shift in grading as detailed above).

Differentials are 0 trivially, as either domain or the target is 0. $H\check{M}_j \cong \check{C}_j$ etc, homologies equal to the chain groups.

Therefore $HM_*(S^3,\mathfrak{s}) = 0$ as $j_* = 0$.

13. Lecture 13: Non-vanishing of $\widehat{\mathrm{HM}}_*$ and Weinstein conjecture

Coupled Morse theory associates homology group to manifold Q equipped with a family of self-adjoint (s.a.) Fredholm operators of index 0 parametrized by Q. We will introduce it below.

13.1. **Family of self-adjoint operator over Q.**

Let H be a separable complex Hilbert space of ∞ dimensionality. $K : H \to H$ compact s.a. with $\ker K = 0$ (injective), and $H_1 := K(H)$ dense in H thus Hilbert, with $\|v\|_1 := \|K^{-1}v\|$.

$K = K^+ \oplus K^-$ splitting into the $\pm$ve parts with respective images $H_1^\pm$ with the closure in H being $H^\pm$. E.g. $H_1 = L_1^2(Y;E) \subset H := L^2(Y;E)$.

Definition 13.1. J compact positive s.a. operator on H with $\ker J = 0$. Let its eigenvalues enumerated as follows: $\mu_i = \lambda_i^{-1}$ with $0 < \lambda_0 \leq \lambda_1 \leq \cdots$. The spectrum of J is called mild if $\exists\, C$ s.t. $\frac{\lambda_{2N}}{\lambda_N} \leq C$.

$B(H : H_1) := \{x : H \to H \mid$ bounded linear s.t. $x(H_1) \subset H_1, x^*(H_1) \subset H_1$ with finite operator norms in $H_1\}$.

Let $U(H : H_1) := \{x \in B(H : H_1) \mid x^*x = 1\}$.

$S(H : H_1) := \{L : H \to H \mid$ Fredholm, index 0, s.a.$\} \supset \{$SAFOE diff operator$\}$ (if H is a function space).

Remark 13.2. $L \in S(H : H_1)$, $\exists$ complete orthonormal (o.n.) system $\{e_i\}_i$ for H with $e_i \in H_1$ and eigenvectors with eigenvalues λ_i.

As $\{\lambda_i\}$ has no accumulation point, we have either (i) bdd from below, (ii) bdd from above, or (iii) unbounded in both directions.

Definition 13.3. $S_*(H : H_1) := \{L \in S(H : H_1) \mid$ satisfying (iii) above, $H_1^\pm(L)$, with H-closure $H^\pm(L)$ eigenspaces of eigenvalues ≥ 0 and <0 of L, s.t. $\exists u \in U(H : H_1)$with $u(H^\pm) = H^\pm(L), u(H_1^\pm) = H_1^\pm(L)\}$.

We now define (Q, L): Q a compact Riem. maniold, $P \to Q$ principle bundle with structure group $U(H : H_1)$, with associated vector bundles $\mathcal{H}_1$ and $\mathcal{H}$ over Q.

Definition 13.4. A family of s.a. operators of type $S_*(H : H_1)$ over Q is a principle $U(H : H_1)$ bundle above with a bundle map $L : \mathcal{H}_1 \to \mathcal{H}$ between the associated vector bundles (v.b.), equivalently, a smooth section of $S_*(\mathcal{H} : \mathcal{H}_1) \to Q$ with fiber $S_*(H : H_1)$.

We have the following properties:

- (Kuiper) $U(H)$ contractible.
- If the spectrum of $|K| := K^+ - K^-$ is mild, then $U(H : H_1)$ is contractible (lengthy but ok proof, similar to proving the previous item, see [KM] 33.1.5).
- If K^+ and $-K^-$ have mild spectrum, then $S_*(H : H_1)$ has homotopy type of $U(\infty) = \lim U(n)$.
- By points 2 (we can homotopically uniquely trivialize any $U(H : H_1)$-bundle) and 3, we have that families of s.a. operator of type $S_*(H : H_1)$ over Q are classified by $[Q, U(\infty)]$ homotopy maps between them.

  This means any such family over Q is the pullback of the universal family over $U(\infty)$ via a map (called classifying map) from Q to $U(\infty)$.

Now, we give a description (a model) of the universal family of s.a. operator over $U(\infty)$. Any other over some Q is a pullback via a map above $[Q, U(\infty)]$.

We describe a criterion to tell whether two such maps are homotopic. Let $U^f(H) := \{u \in U(H) \mid u - 1 \text{ has finite rank}\}$.

Lemma 13.5. *Let $u_1, u_2 : Q \to U^f(H)$. Suppose $\exists$ continuous $\theta : Q \to U^f(H)$ s.t. $\theta u_1 = u_2 \theta$ and for all $q \in Q$, $\theta|_{\ker(u+1)} : \ker(u_1 + 1) \to \ker(u_2 + 1)$ is isomorphism, then u_1 and u_2 are homotopic.*

$U(\infty) = \lim_N U(N)$, we describe family of L over $U(N)$ and show the classifying map is (homotopic to) inclusion, which then induces the family over $U(\infty)$.

For $z \in U(N)$, let $C^\infty(S^1, z) := \{h : \mathbb{R} \to C^N \mid \text{smooth} h(t+1) = zh(t)\}$ with L^2 and L^2_1 norms (integration over $[0,1]$) and the respective completions denoted by $H(z)$ and $H_1(z)$. We denote $H := H(1)$ and $H_1 := H_1(1)$. We have a bundle over $U(N)$ with structure group $U(H : H_1)$. Let $L(z) : H_1(z) \to H(z), h \mapsto -i\frac{d}{dt}h$ of type $S_*(H : H_1)$. $L(z)$ is classified by $\psi : U(N) \to U(\infty)$, which using the above lemma is homotopic to the inclusion.

We have described the object for which we can define a homology theory, and we describe it now:

13.2. **Coupled homology $\overline{H}(Q, L)$.** Let (g, f) be a Morse-Smale pair on Q, which means g is a Riemannian metric and f is Morse function such that the moduli spaces of trajectories of ∇f defined using g are regular. Let ∇ be a connection on a principle bundle, which means 1-form valued in Lie algebra of $U(H : H_1)$ with equivariant properties (it is instructive to recall this exactly, but we will not need this below).

For each $q \in \mathrm{Crit}(f)$, a critical point, assume $L(q) : H_1(q) \to H(q)$ has simple nonzero spectrum.

Assume there is no spectral flow around loops in Q ($\iff$ classifying map $Q \to SU(\infty) \subset U(\infty)$ factoring through $SU(\infty)$), which means we can label eigenvalues of $L(q)$, $q \in \mathrm{Crit}(f)$, $\ldots \lambda_{-1}(q) < \lambda_0(0) < \lambda_1(q) < \cdots$, s.t. any path $q(t)$ joining q_1

to q_2, the spectral flow along $q(t)$ is $s(q_2) - s(q_1)$ where absolute spectral flow can be arranged as $s(q) = |\{i \mid i < 0, \text{and } \lambda_i(1) > 0\}|$.

Let $\phi_i(q)$ be a unit eigenvector for $\lambda_i(q)$.

Define $\overline{C}_n = \overline{C}_n(Q, L) = \bigoplus_{q \in \mathrm{Crit}(f)} \bigoplus_{i, \mathrm{ind} q + 2i = n} \mathbb{Z}\Lambda_q$, where Λ_q is the orientation 2-element set for the unstable manifold at q (from which we can get stable manifold orientation, and thus the moduli space orientation through intersection of two oriented submanifolds).

$\begin{cases} \frac{d}{dt}\gamma + (\nabla f)_{\gamma(t)} = 0 \\ (\gamma^* \nabla)\phi + (L(\gamma(t))\phi)dt = 0 \end{cases}$

Define $M(q_0, i; q_1, j)$ the $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$ (acting on the second variable) quotient of (γ, ϕ) of solutions of the above equation s.t.

- $\gamma(t)$ from q_0 to q_1,
- $\phi(t)$ has the leading asymptotic $c_0 e^{-\lambda_i t}\phi_i(q_0)$ as $t \to -\infty$,
- $\phi(t)$ has the leading asymptotic $c_1 e^{-\lambda_j t}\phi_j(q_1)$ as $t \to +\infty$.

We need weighted Sobolev space to make $\phi \mapsto (\gamma^*\nabla)\phi + (L(\gamma(t))\phi)dt$ a Fredholm problem with $\mathrm{ind}_{\mathbb{C}} = i - j + 1$.

We can make the moduli space regular so that $M(q_0, i; q_1, j)$ is a smooth manifold with dimension $\mathrm{ind}(q_0) - \mathrm{ind}(q_1) + 2(i - j)$. We also have $\check{M}$ and $\check{M}^+$ as before.

We define the differential as $\bar{\partial} = \sum_{\check{M},\ \dim 0} \sum_{[\gamma, [\phi]] \in \check{M}} \epsilon([\gamma, [\phi]])$.

Homotopic L's give isomorphic $\overline{H}_*(Q, L)$'s.

$\overline{H}_*(Q, L)$ is a module over $H^*(\mathbb{P}(\mathcal{H}))$

For the tautological line bundle L over $\mathbb{P}(\mathcal{H})$, $u_2 := -c_1(L)$ and we can construct Morse chain level operation $(\tilde{u}_2 \cap \cdot) : \overline{C}_k \to \overline{C}_{k-2}$ which is invertible for all k.

There is also a version with spectral flow, which we suppress, as we do not need it below.

13.3. Calculation of $\overline{HM}_*$ for 3-manifold with torsion spin^c structure.

Let Y be a 3-manifold oriented with spin^c structure $\mathfrak{s}$ s.t. $c_1(\mathfrak{s}) := c_1(S)$ torsion.

Reducible critical points of unperturbed CSD $\mathcal{L}$ in $\mathcal{B}(Y, \mathfrak{s})$
$=\mathbb{T}$ torus of gauge equivalence class of spin^c connection A s.t. $\mathrm{tr}(A)$ is flat.
$\cong H^1(Y; \mathbb{R})/H^1(Y; \mathbb{Z})$.

Let $A_*(Y)$ be the exterior algebra generated by $A_1(Y) := H^1(Y; \mathbb{Z})$. Then $A_*(Y) \cong H_k(\mathbb{T}; \mathbb{Z})$.

We have $\beta_k : A_k(Y) \to A_{k-3}(Y)$,
$\alpha_1\alpha_2 \cdots \alpha_k \mapsto \sum_{i_1 < i_2 < i_3} (-1)^{i_1+i_2+i_3} \langle \alpha_{i_1} \cup \alpha_{i_2} \cup \alpha_{i_3}, [Y] \rangle \alpha_1 \cdots \hat{\alpha}_{i_1} \cdots \hat{\alpha}_{i_2} \cdots \hat{\alpha}_{i_3} \cdots \alpha_k$.

Theorem 13.6. *Y closed connected oriented 3-manifold, $\mathfrak{s}$ has c_1 torsion. Then $\overline{HM} := \overline{HM}(Y, \mathfrak{s}; \mathbb{Q})$ has a filtration $\overline{HM} \supset \cdots \supset \mathcal{F}_s\overline{HM} \supset \mathcal{F}_{s-1}\overline{HM} \supset \cdots$, s.t. the graded pieces $\frac{\mathcal{F}_s\overline{HM}}{\mathcal{F}_{s-1}\overline{HM}} \cong \frac{\ker \beta_s}{im \beta_{s+3}} \otimes \mathbb{Q}[T^{-1}, T]$, the expression to the right of the tensor product is the polynomial ring generated by formal variables T^{-1} and T.*

Proof. We have retraction $p : B(Y, \mathfrak{s}) \to \mathbb{T}$, and f Morse on $\mathbb{T}$. Consider $f \circ p = f_1$ which is an example of a cylinder function. Define $\not{\mathcal{L}} = \mathcal{L} + f_1$.

Reducible critical points of $\not{\mathcal{L}}$is critical points $[\alpha]$ of f in $\mathbb{T} \subset B(Y, \mathfrak{s})$.

The flow of $-\nabla\not{\mathcal{L}}$ preserves $\mathbb{T}$.

In $B^\sigma(Y, \mathfrak{s})$, reducible zeros $[\alpha]$ are gauge orbits of $(\alpha, [\phi])$ where $\alpha \in [\alpha]$ is a critical point of f and $\phi \in L^2(Y; S)$ is an eigenvector of Dirac D_α.

Using perturbation in $\mathcal{P}$ that vanishes at the reducible locus, everything above remains, except D_α becomes $D_{q,\alpha}$ and $[\alpha]$ is now non-degenerate in B^σ and $M_z^{\mathrm{red}}([\alpha],[\beta])$'s are regular.

Claim: $\overline{HM}(Y,\mathfrak{s})$ is homology of $\overline{C}_*(Q,L)$ where $Q=\mathbb{T}$ and $L:=\{D_{q,\alpha}\}_{[\alpha]\in T}$.

L corresponds to a classifying map $\mathbb{T}\to SU(2)\hookrightarrow U(\infty)$. The pullback of the 3-d generator of $SU(2)=S^3$ is $\xi_3:(a_1,a_2,a_3)\mapsto\langle a_1\cup a_2\cup a_3.[Y]\rangle\in\Lambda^3H^1(Y;\mathbb{Z})^*$.

This is seen through a computation involving Chern character ch. We quickly remark $ch(V\oplus W)=ch(V)+ch(W)$, $ch(V\otimes W)=ch(V)ch(W)$, and $ch(V)=\mathrm{tr}(\exp(i\Omega/2\pi))$ where Ω is the curvature of a connection using Chern-Weil theory.

Then $\exists$ a Riemannian metric on Q, Morse f and homotopic $\tilde{L}$ s.t. $\overline{HM}(Q,\tilde{L})=C_*(Q,f)\otimes\mathbb{Z}[T^{-1},T]$, here the differential $\bar{\partial}x=\partial x+T(\tilde{\xi}\cap x)$ where ∂ is Morse differential, and $\tilde{\xi}$ is Morse chain level cap product with ξ. This squares to 0. $\square$

I gave a lecture in another lecture series about spectral sequence, its intuition generalizing SES inducing LES, and how it works and I will put that part here soon. For now, it is a way to calculate homology using doubly graded complex with differential pointing left and up and homology of it is the next page with induced longer differential, at infinite page if it stabilizes, it is said to converge/abut.

$\exists$ spectral sequence abutting to $\overline{H}_*(Q,L)$ where E^2 and E^3 terms are: $E^2_{s,2j}=E^3_{s,2j}=T^jH_s(Q)$ and differential on E^3 page is

$$d^3_{s,2j}:E^3_{s,2j}\to E^3_{s-3,2j+2},T^j[x]\mapsto T^{j+1}\xi\cap[x].$$

If the higher differentials for pages 4 or later vanish, then $\overline{H}:=\overline{H}_*(Q,L)$ has $\cdots\supset\mathcal{F}_s\overline{H}\supset\mathcal{F}_{s-1}\overline{H}\supset\cdots$ with $\mathcal{F}_s\overline{H}/\mathcal{F}_{s-1}\overline{H}\cong\ker\beta_s/\mathrm{im}\beta_{s+3}\otimes\mathbb{Z}[T^{-1},T]$.

13.4. **Non-vanishing at infinite gradings.** The previous subsection implies that $\overline{HM}_*(Y,\mathfrak{s})$ with $c_1(\mathfrak{s})$ torsion is nonzero and has infinite rank.

Proof. Suffice to show $\frac{\ker\beta_s}{\mathrm{im}\beta_{s+3}}$ has nonzero rank for at least one of s. Let $\zeta=e^{2\pi i/6}$.

$$\begin{aligned}&\sum_s\mathrm{rank}(\frac{\ker\beta_s}{\mathrm{im}\beta_{s+3}})\zeta^s\\ =&\sum_s\mathrm{rank}(A_s)\zeta^s\\ =&\text{evaluation of Poincaré polynomial of }\mathbb{T}\text{ at }\zeta\\ =&(1-\xi)^{b_1(Y)},\ \text{which is always nonzero.}\end{aligned}$$

$\square$

Thus, $\widehat{HM}_*(Y,\mathfrak{s})$ nonzero in infinitely many grading for $\mathfrak{s}$ with c_1 torsion.

Proof. $\widehat{HM}_*(Y,\mathfrak{s})$ graded by $\mathbb{J}(Y,\mathfrak{s})\cong\mathbb{Z}$ if $c_1(\mathfrak{s})$ torsion.

By definition, $\widehat{HM}_*$: non-trivial grading (grading that is not trivial) is bounded from above.

$\check{HM}_*$: non-trivial grading is bounded from below.

$\overline{HM}_*$ has nontrivial gradings that are infinite in both directions, seen above.

LES relating these 3 implies that we have isomorphism between $\overline{HM}$ and $\widehat{HM}$ when grading is sufficiently below. $\square$

13.5. **Application in proof of Weinstein conjecture.** Following Hutchings' exposition, we will give a sketch on how an application of the nonvanishing result in the previous section together with lots of new insights gives a deep result in dynamics in Taubes' proof of Weinstein conjecture. One should argue the following is loosely related to the subject matter in this lecture series, and can omit it without continuity, but partly because of the interests of the department and partly because of lots of interests of HM in recent years have come from outside like dynamics, it is helpful to be aware of reasoning using the concepts and results in the Seiberg-Witten world, even if one is to study SW per se, and we will see lots of notions we have familiarized ourselves with in past lecturees have come into play.

Let Y be a closed 3-manifold, and λ 1-form on Y such that $\lambda \wedge d\lambda$ is nowhere vanishing. $\xi := \ker \lambda$ is called a (cooriented) contact structure.

For such λ, we have $\iota_\bullet d\lambda : TY \to T^*Y, X \mapsto \iota_X d\lambda := d\lambda(X, \cdot)$ and it has 1-dimensional kernel denoted by L, and $\lambda|_L$ is a nowhere zero, so we can define a vector field $R \in \Gamma(L) \subset \Gamma(TY)$ s.t. $\iota_R d\lambda = 0$ and $\lambda(R) = 1$, so R is a nowhere zero vector field, called Reeb vector field. One can study its dynamics e.g. whether R has a closed orbit $\gamma : \mathbb{R}/T\mathbb{R} \to Y$ such that $\dot{\gamma} = R \circ \gamma$ with period $T < \infty$ as the first approximation. This does not do the justice of the following question, but it is an important question that drives lots of development in contact geometry, and earlier work by Hofer on proof for the case of $Y = S^3$ and overtwisted case introduces holomorphic curve into contact geometry and establishes the connection of existence of Reeb orbits and existence of finite energy punctured holomorphic curves, lays the foundation and inspires the creation of a powerful invariant called symplectic field theory (SFT) and its variants.

Weinstein conjecture (3d): For (Y, λ) closed contact 3-manifold. $\exists$ a closed orbit of the Reeb vector field R (associated to λ).

Taubes approached this using Seiberg-Witten theory and used in an essential way the non-vanishing result in the earlier subsection.

How do we get the initial data needed to construct SW from the current setting. First fix a Riemannian metric g on Y such that $|\lambda|_g = 1$ and $d\lambda = 2 *_g \lambda$ (both 2-forms are nonvanishing and one can define $*$ using a multiple of $\lambda \wedge d\lambda$ as volume form and adjust the proportion using g).

$\xi = \ker \lambda = K^{-1}$ (the anticanonical line bundle (for symplectization), at the following one can just regard it as a notation). Denote the associated spin^c structure as $\mathfrak{s}_\xi := (S, c)$ where $S = S_\xi = \underline{C} \oplus K^{-1}$ with the first factor denoting the trivial bundle and the Clifford multiplication acts by $c(\lambda)$ acts as i and $-i$ in each factor respectively.

Any spin^c structure $\mathfrak{s}$ is $E \otimes S = E \oplus K^{-1}E$ for some complex line bundle E.

We perturb SW equation by exact 2-form as follows

$$(\ddagger) \begin{cases} *F_A = r(\langle \rho(\cdot)\psi, \psi \rangle - i\lambda) + i\overline{w} \\ D_A \psi = 0 \end{cases}$$

Here $\overline{w}$ is a harmonic 1-form, such that $[\frac{*\overline{w}}{\pi}] = c_1(K^{-1})$ with $\pi = 3.14...$ the universal constant.

$\exists$ unique Hermitian connection A_0 and spinor $\psi_0 = (1, 0) \in \Gamma(\underline{C} \oplus K^{-1})$ s.t. $D_{A_0}\psi_0 = 0$. A trivial solution to the equation for any r.

We want to let $r \to \infty$.

After scaling, and small perturbation (to achieve regularity etc), IFT shows that we still have:

For all $\delta > 0$ if $r >> 0$ (my notation for being large enough), we have

$\exists$ unique (up to gauge transformation) (A_{triv}, ψ_{triv}) to (‡) for $\mathfrak{s}_\xi$ such that:

$$\begin{cases} 1 - |\psi_{triv}| \leq \delta \text{ on } Y \quad (\dagger) \text{ in fact, LES is of order } O(r^{-1/2}) \\ \text{grading of } (A_{triv}, \psi_{triv}) \text{in HM chain group is independent of } r. \end{cases}$$

So for δ small enough, ψ is a nowhere vanishing section.

Fix E, $\mathfrak{s} = E \otimes \mathfrak{s}_\xi$.

Let (A_n, ψ_n) be a sequence of solutions to $(\ddagger)_{r_n}$ with $r_n \to \infty$. Suppose

$$\begin{cases} (1)\ \exists \delta > 0, \text{ with } \sup_Y (1 - |\psi_n|) > \delta \text{ (thus not the trivial solution above)}, \\ (2)\ \exists C < \infty, \text{ with } i \int_Y \lambda \wedge F_{A_n} < C \text{ (energy control for the following convergence)}. \end{cases}$$

Then $\exists$ non-empty orbit set a (a formal sum of closed Reeb orbits with nonzero integer weights) s.t. $[a] = PD(c_1(E))$.

The idea is that $\psi_n = (\alpha_n, \beta) \in \Gamma(E \oplus EK^{-1})$ w.r.t the splitting. We have α_n^{-1} converges to a as a current (in the dual space of smooth compactly supported dR forms) under energy bound (2) and which would be non-empty due to (1) and uniqueness of the perturbed trivial solution, while $|\beta_n| \to 0$. Topological property is one definition of c_1. This is inspired by Taubes' earlier work of getting a holomorphic curve from SW solution (with r parameter) in closed symplectic 4-manifolds.

Fix E s.t. $\mathfrak{s}_E = E \otimes \mathfrak{s}_\xi$ has torsion c_1 (note that $c_1(\mathfrak{s}_E) = c_1(K^{-1}) + 2c_1(E)$ as $S = E \oplus K^{-1}E$).

Theorem 13.7 ([KM] in previous subsection)**.** *$\exists$ solution to $(\ddagger)_r$ for all $r \geq 1$*

Need to find a sequence of solutions satisfying conditions (1) and (2):

13.5.1. *Condition (1).* If $c_1(E) \neq 0$, $\exists c > 0$ s.t. for $r >> 0$ if (A, ψ) is solution to $(\ddagger)_r$, then $\exists$ point $p \in Y$, s.t. $1 - |\psi(p)| \geq 1 - \frac{c}{\sqrt{r}}$.

For $c_1(E) = 0$, since "trivial" solution is unique at one degree (independent of r), but $\widehat{\text{HM}}$ nonzero in infinitely many gradings.

13.5.2. *Condition (2).* In the current perturbed form $CS_{\overline{w}}(A) := -\int_Y (A - A_1) \wedge (F_A + F_{A_1} - 2i * \overline{w})$. Want to show that it is a sum of 2 functionals to control. It is gauge-invariant as c_1 is torsion.

Definition 13.8. $\mathcal{E}(A) := i \int_Y \lambda \wedge F_A$ (appeared in condition (2)).

$CSD(A, \psi) := -\frac{1}{8} \int_Y (A - A_0) \wedge (F_A + F_{A_0} - 2i\mu) + \frac{1}{2} \int_Y \langle D_A \psi, \psi \rangle d\text{vol}$ (with $d\text{vol} = \lambda \wedge d\lambda$).

($\mu = -rd\lambda - iF_{A_0} + 2 * \overline{w}$). Let the reference base point be $A_0 + 2A_1$.

From these, we have $CSD(A, \psi) = \frac{1}{2}(CS(A) - r\mathcal{E}(A)) + \frac{r}{2} \int \langle D_A \psi, \psi \rangle d\text{vol}$.

For SW solution, we have $CSD(A, \psi) = \frac{1}{2}(CS(A) - r\mathcal{E}(A))$. (So indeed we have CS written as a sum of two meaningful quantities).

We first find a piecewise smooth canonical family in r:

$\circledast$: For $r >> 0$, $\exists$ $(A(r), \psi(r))$ to $(\ddagger)_r$.

- $(A(r), \psi(r))$ is piecewise smooth in r.
- $CSD(A(r), \psi(r))$ continuous in r.
- For r. s.t. $(A(r), \psi(r))$ is smooth, $(A(r), \psi(r))$ is nondegenerate and its grading in $\widehat{HM}$ is independent of r.
- $(A(r), \psi(r))$ is not gauge equivalent to (A_{triv}, ψ_{triv}).

Proof. For any grading k, $r >> 0$, all generators defining $\widehat{\mathrm{HM}}_k(Y, \mathfrak{s})$ are irreducible, via a spectral flow argument.

This implies that the differential of Floer chain just counts for pertrubed (-ve) gradient flow of CSD without reducibles.

[KM] in the earlier s says that $\exists$ nonzero clas σ in $\widehat{H}M_*(Y, \mathfrak{s})$ and when $c_1(E) = 0$, we can assume grading of σ is not the same as (A_{triv}, ψ_{triv}). Fix σ.

(Using tame perturbation in $\mathcal{P}$, suppressed in notation,) We can assume $\widehat{\mathrm{HM}}_*$ defined for generic r.

For such r, σ is represented by $\sum_i n_i c_i$ where c_i is a critical point of CSD $\mathcal{L}$(they are distinct), and n_i's are nonzero, and the index set is finite.

Define $h(r) := \min_{\sum_i n_i^r c_i^r \in \sigma} \max_i CSD(c_i^r) =: CSD(c^r_{\text{minmax}})$, where we show the r-dependence as superscripts.

Define $(A(r), \psi(r)) := c^r_{\text{minmax}}$. Need to show $h(r)$ extends to a continuous function for all $r >> 0$, using bifurcation.

We give a baby version of this in Morse theory. (f_r, g_r) fails Morse-Smale condition for finitely many r. Let $r < r'$ be two such that M-S condition does hold.

We have continuation map $\Phi : C_*^M(f_{r'}, g_{r'}) \to C_*^M(f_r, g_r)$, which is defined by counting maps $\gamma : \mathbb{R} \to B$ s.t. $\dot{\gamma}(s) = -\nabla f_{\phi(s)} \gamma(s)$, where $\phi : \mathbb{R} \to [r, r']$ monotone is fixed before hand with $\phi|_{(-\infty,0]} = r'$ and $\phi|_{[1,\infty)} = r$.

By chain rule, $\frac{d}{ds} f_{\phi(s)}(\gamma(s)) = |\nabla f_{\phi(s)}(\gamma(s))|^2 + \frac{d\phi(s)}{ds} \frac{\partial f_r(x)}{\partial r}|_{r=\phi(s), x=\gamma(s)}$. The second factor of the last term is bounded by C as $[r, r'] \times X$ is compact.

If p' and p critical points of $f_{r'}$ and f_r respectively, suppose $\langle \Phi(p'), p \rangle \neq 0$, namely the cofficient of p in $\Phi(p')$ is nontrivial, then integrating, $f_{r'}(p') \geq f_r(p) - C(r' - r)$. Plus the other direction, we have Lipschitz, thus continuous. □

We can show by differentiation and definition of $(A(r), \psi(r))$, that

$$(\odot) \quad \frac{d\, CSD}{dr}(A(r), \psi(r)) = -\frac{1}{2}\mathcal{E}(A(r)).$$

We only need to show $\mathcal{E}$ bounded, by analyzing the growth rates in subsequences.

13.6. **Dichotomy.** We have the following dichotomy:

- $\exists r_n \to \infty$, $\mathcal{E}(A(r_n)) < C$ for all n.
- $\exists r_n \to \infty$, $\mathcal{E}(A(r_n)) \geq C r_n$ and then $CS(A(r_n)) \geq C r_n^2$.

Proof. Denote $CS(r)$, $\mathcal{E}(r)$ and $CSD(r)$ for the respective corresponding quantities with $(A(r), \psi(r))$ substituted in. Can assume $\mathcal{E}(r) > 1$ for $r >> 0$, or the first case holds. Fix $\epsilon_0 \in (0, 1/5)$.

Case A: $\exists r_n \to \infty$ with $CS(r_n) \geq \epsilon_0 r_n \mathcal{E}(r_n)$ for all n.

We claim as a blackbox: $(\oslash)$ $\exists C$, if (A, ψ) satisfies $(\ddagger)_r$ with $\mathcal{E}(A) > 1$, then $|CS(A)| \leq C r^{2/3} \mathcal{E}(A)^{4/3}$.

From this and hypothesis of Case A, we are in the second case of the dichotomy.

Case B: we have $CS(r) < \epsilon_0 r \mathcal{E}(r)$ (labelled as (a)).

Define $v(r) := \mathcal{E}(r) - \frac{CS(r)}{r} = 2\frac{2CSD(r)}{r}$.

$(\odot)$ implies $\frac{dv}{dr} = \frac{CS}{r^2}$ (labelled (b)). (a) is equivalent to $\mathcal{E} < \frac{v}{1-\epsilon_0}$ (labelled (c)).

(a), (b) and (c) imply that $\frac{dv}{dr} < \frac{\epsilon v}{r}$ with $\epsilon := \frac{\epsilon_0}{1-\epsilon_0} < \frac{1}{4}$.

From these, we can deduce $v < C_1 r^\epsilon$ for some C_1 (labelled as (d)).

($\oslash$), (c) and (d) imply that $CS < C_2 r^{\frac{2}{3}+\frac{4}{3}\epsilon}$. Putting this into (b), we get that $\frac{dv}{dr} < C_2 r^{\frac{4}{3}(\epsilon-1)}$, and as $\epsilon < \frac{1}{4}$, the exponent $\frac{4}{3}(\epsilon-1) < -1$. Integrating, we have v bounded from above.

Then by (c), $\mathcal{E} < \frac{v}{1-\epsilon_0} < C$. □

But then we have the following result:

Proposition 13.9. *$\exists \kappa > 0$ s.t. for $r >> 0$, (A, ψ) solution to (‡), we have $|\deg(A, \psi) - \deg(A_{triv}, \psi_{triv}) + \frac{1}{4\pi^2} CS(A)| < \kappa r^{31/16}$, where it is important that the exponent is less than* 2.

Suppose we are not in the first scenario in the dichotomy statement, then we have $CS(r_n) \geq C r_n^2$ for some subsequence $r_n \to \infty$; however, by the above inequality where two degrees are constant, we have $CS(r_n)$ grows less than r_n^2, which is a contradiction.

We have obtained a sequence (A_{r_n}, ψ_{r_n}) with finite uniform energy bound (condition (2)) and nontrivial specified in condition (1), by the statement at the beginning of this section, we have a nontrivial orbit set, in particular a closed Reeb orbit.

14. Lecture 14: Detecting volume in 3d contact geometry

Let (Y, λ) be a closed connected contact 3-manifold. Recall from last time that, λ is a 1-form such that $\lambda \wedge d\lambda$ nowhere 0, thus a volume form.

For any $\Gamma \in H_1(Y)$, which (using the volume form/orientation) is under Poincaré duality corresponds to $PD(\Gamma) \in H^2(Y)$ which corresponds to a (iso class of) line bundle denoted still by $PD(\Gamma)$, and a spin^c structure $\mathfrak{s}_\xi \otimes PD(\Gamma)$. Here, recall the spin bundle of $\mathfrak{s}_\xi$ is $\underline{\mathbb{C}} \oplus \xi$ with $\xi = \ker \lambda$ the contact structure. We assume $c_1(\mathfrak{s}_\xi \otimes PD(\Gamma)) = c_1(\xi) + 2PD(\Gamma)$ is torsion. Thus we have absolute $\mathbb{Z}$ grading.

Let $\{\sigma_k\}_k$ be a sequence of nonzero homogeneous classes in $\widehat{HM}^{-*}$, $* \in \mathbb{Z}$ with $\lim\limits_{k\to\infty} -\mathrm{gr}(\sigma_k) = \infty$. Here, in the monopole Floer cohomology, we count trajectories from right to left, namely, $\partial[\mathfrak{b}] = \sum_{[\mathfrak{a}]} \#\check{M}([\mathfrak{a}], [\mathfrak{b}])[\mathfrak{a}]$. Then, the topic of today's lecture is about the following volume detecting property:

$$\lim_{k\to\infty} \frac{c_{\sigma_k}(Y, \lambda)^2}{-\mathrm{gr}(\sigma_k)} = \mathrm{vol}(Y, \lambda) := \int_Y \lambda \wedge d\lambda.$$

Now we explain the undefined term c_σ for a nonzero class:

There is a filtered version of $\widehat{HM}^{-*}$ as follows: $\widehat{C}_L^{-*} := \{a \in \hat{C}^{-*} \mid \mathcal{E}(a) \leq L\}$, with $a = \sum_i n_i c_i = \sum_i n_i[(A_i, \psi_i)]$ and $\mathcal{E}(a) := \sum_i n_i (i \int_Y \lambda \wedge F_{A_i})$, is a subcomplex of $\widehat{C}^{-*}$, inducing $\iota_L : \widehat{HM}_L^{-*} \to \widehat{HM}^{-*}$.

Given $\sigma \in \widehat{HM}^{-*} \backslash \{0\}$, define

$$c_\sigma(Y, \lambda) = \inf_L \{L \mid \sigma \in \mathrm{im}(\iota_L)\}.$$

This volume detecting result is proved by Cristofaro-Gardiner–Hutchings–Ramos, in their "The asymptotics to ECH capacities" paper.

14.1. Proof of "$\leq$".

$$(\ddagger)_r \begin{cases} *F_A = (r\langle c(\cdot)\psi, \psi\rangle - i\lambda) + (i * d\mu + \pi\overline{w}) \\ D_A\psi = 0. \end{cases}$$

Here π is the universal constant.

Last time, (†) $|\mathrm{gr}(A, \psi) + \frac{1}{4\pi^2} CS(A)| < \kappa r^{31/16}$ for $r > r_*$. (We can shift the degree to get rid of one grading term from last time.)

Fix $\delta \in (0, \frac{1}{16})$. Given $j \in \mathbb{N}$, define r_j to be the largest real number s.t.

$$(\circledast) \quad j = \frac{1}{16\pi^2} r_j^2 \mathrm{vol}(Y, \lambda) - r_j^{2-\delta}.$$

Since $r \gg 0$, no generators from reducibles (using a spectral flow estimate). So for j,

$s_j := \{r \mid \exists$ a generator with grading $\geq -j$ associated to reducible solution to $(\ddagger)_r\} < \infty$.

Claim: $s_j < r_j$ if $j \gg 0$. (This is one of significance of r_j.)

($\odot$) (Taubes): For (A, ψ) solution to $(\dagger)_r$, $\mathcal{E}(A) := i \int_Y \lambda \wedge F_A \leq \frac{r}{2}\mathrm{vol}(Y, \lambda) + C$.

$CSD(A, \psi) = \frac{1}{2}(CS(A) - r\mathcal{E}(A)) + I \int_Y \mu \wedge F_A + r \int_Y \langle D_A \psi, \psi \rangle d\mathrm{vol}$.

Fix $\sigma \in \widehat{HM}^{-*}$ nonzero with $\mathrm{gr}(\sigma) \geq -j$ (h defined last time using minmax for homology $\widehat{HM}_*$, here we have cohomolohgy, so maxmin)

$$h(r) = \max_{\sum_i n_i^r c_i^r \in \sigma} \min_i CSD(c_i^r) = CSD(c^r_{\mathrm{maxmin}}).$$

$(A(r), \psi(r)) := c^r_{\mathrm{maxmin}}$, and $h(r)$ piecewise smooth. Define $CSD(r) = CSD(A(r), \psi(r))$ and $\mathcal{E}(r) := \mathcal{E}(A(r))$.

$\frac{d}{dr} CSD(r) = -\frac{1}{2}\mathcal{E}(r)$ for all $r > s_j$ (no reducible) s.t. $\hat{C}^*$ is defined.

Proposition 14.1. $\lim\limits_{r\to\infty} \mathcal{E}(r) = 2\pi c_\sigma(Y, \lambda)$.

Lemma 14.2 (ⓐ)**.** $r \gg 0$, $\begin{cases} \mathcal{E}(A) > 2\pi L_0 \text{ \textit{a universal threshold}} \\ |CS(A)| \leq C r^{2/3} \mathcal{E}^{4/3} \text{ \textit{appeared in the last lecture}} \end{cases}$ *for* (A, ψ) *to* $(\dagger)_r$ *that is not* (A_{triv}, ψ_{triv}).

Fix $\gamma \in (0, \frac{\delta}{4})$ and $\delta \in (0, \frac{1}{16})$.

Lemma 14.3 (ⓑ)**.** *For all* j, $\exists$ ρ *s.t.* $r \gg \rho$ *and* $\hat{C}^{-*}$ *defined,* (A, ψ) *non-trivial solution to* $(\dagger)_r$ *of grading* $-j$, *then* $|CS(A)| \leq r^{1-\gamma}\mathcal{E}(A)$.

Proof. (of Lemma ⓑ) If not, $\exists$ such a (A, ψ) with

$$r^{1-\gamma}\mathcal{E}(A) < |CS(A)| \leq C r^{2/3} \mathcal{E}(A)^{4/3}$$

if $r \gg 0$. This implies that $r^{\gamma-1}|CS(A)| > \mathcal{E}(A) > C^{-3} r^{1-3\gamma}$, thus $|CS(A)| > C^{-3} r^{2-4\gamma}$ and this exponent is bigger than $\frac{31}{16}$ by choice ($4\gamma < \delta < \frac{1}{16}$) and and thus contradicts to $(\dagger)$ as $r \gg 0$. $\square$

Proof. (of Proposition) Assume $r \gg 0$, ⓐ and ⓑ ($\mathrm{gr}(\sigma) = -j$) and non-trivial solution in $\mathrm{gr}(\sigma)$ irreducible with energy $\mathcal{E} > 0$.

$$\begin{cases} \int |F_A| \overset{\text{Taubes}}{\leq} \kappa(\mathcal{E}(A) + 1) \\ \text{Lemma ⓐ} \end{cases} \Rightarrow (\text{ⓐ})\ |i \textstyle\int_Y \mu \wedge F_A| \leq \kappa \mathcal{E}(A).$$

As $CSD(A, \psi) = \frac{1}{2}(CS(A) - r\mathcal{E}(A)) + i \int \mu \wedge F_A$, by Lemma ⓐ and Inequality ⓒ, we have:

$$\begin{cases} (1 - r^{-\gamma} - 2\kappa r^{-1})\mathcal{E}(A) \leq \frac{-2}{r} CSD(A, \psi) \leq (1 + r^{-\gamma} + 2\kappa r^{-1})\mathcal{E}(A) \\ \lim\limits_{r\to\infty} \mathcal{E}(A_{triv}) = \lim\limits_{r\to\infty} \frac{CSD(A_{triv}, \psi_{triv})}{r} = 0. \end{cases}$$

This implies the proposition by the definition of c_σ. $\square$

To see the "$\leq$" part of the statement, heuristically:
Suppose σ_j nonzero of gr $-j$. Then,

$$\begin{aligned}
&\lim_j \frac{4\pi^2 c_{\sigma_j(Y,\lambda)^2}}{j} \\
\overset{\text{Prop}}{=}&\lim_j \frac{\mathcal{E}(A(r_j))^2}{j} \\
\underset{\circledast}{\overset{\odot}{\leq}}&\frac{(\frac{r_j}{2}\mathrm{vol}(Y,\lambda)+C)^2}{\frac{1}{16\pi^2}r_j^2\mathrm{vol}(Y,\lambda)+r_j^{2-\delta}} \\
=&4\pi^2\mathrm{vol}(Y,\lambda).
\end{aligned}$$

14.2. **Proof of "$\geq$".** Let $(Y_\pm, \lambda_\pm)$ closed oriented contact 3-manifolds. A strong symplectic cobordism $(Y_+, \lambda_+) \to (Y_-, \lambda_-)$ is a compact symplectic 4-manifold (X, ω) (where 2-form ω satisfies $d\omega = 0$ and ω^2 nowhere zero) with $\partial X = Y_+ \sqcup \bar{Y}_-$ s.t. $\omega|_{Y_\pm} = d\lambda_\pm$.

A weakly exact symplectic cobordism is a strong symplectic cobordism with $\omega = d\lambda$ exact. This gives rise to a morphism

$$\Phi_L(X,\omega) : \widehat{HM}_L^{-*}(Y_+, \lambda_+, 0) \to \widehat{HM}_L^{-*}(Y_-, \lambda_-, 0).$$

Let $A \in H_2(X, \partial)$, $\partial A = \Gamma_+ - \Gamma_-$. For a strong cobordism, the filtration level might get shifted but between the limits is well-defined $\widehat{HM}^{-*} := \lim \widehat{HM}_L^{-*}$. Here Γ_+ is identified as $PD(\Gamma_+)$ the line bundle to tensor with $\mathfrak{s}_\xi$. Thus, we have $\Phi_L(X,\omega) : \widehat{HM}^{-*}(\Gamma_+) \to \widehat{HM}^{-*}(\Gamma_-)$. This is defined by counting SW trajectories in the (completed) cobordism with limits in the $\pm$ pieces. We have the following properties:

- Trivial cobordism $\Rightarrow \Phi = \mathrm{Id}$.
- $\phi(X_r) \circ \phi(X_l) = \phi(X_l \circ X_r)$.
- $\phi(X_1 \sqcup X_2) = \phi(X_1) \otimes \phi(X_2)$.
- We have U map increasing $\widehat{HM}^{-*} \to \widehat{HM}^{-*+2}$, corresponding to cup product with c_1 of tautological line bundle in coupled homology picture. We have $\phi(X) \circ U_+ = U_- \circ \phi(X)$.

Proof. (of "$\geq$") Step 1: $\widehat{HM}^{-*}$ finitely generated and U is isomorphism.

$*$ large enough, $\Rightarrow \exists$ finite collection of sequence satisfying (§) $U\sigma_{k+1} = \sigma_k$ for all k, s.t. every nonzero homogeneous σ_k of sufficiently large k is in one of these sequences in (§). Thus suffice to prove "$\geq$" for (§).

Step 2: $([-a, 0] \times Y, d(e^s\lambda))$. For $\epsilon > 0$, $B(r) := \{z \in \mathbb{C}^2 \mid \pi|z|^2 \leq r\}$ with standard ω_0. Let $\varphi_i : B(r_i) \to [-a, 0] \times Y$, $i = 1, \cdots, N$, with disjoint images, s.t. $\omega^2([-a, 0] \times Y \setminus \bigsqcup_i \varphi_i(B(r_i))) < \epsilon$ (using Darboux charts). Denote $X := [-a, 0] \times Y \setminus \bigsqcup_i \varphi_i(B(r_i))$.

Weakly exact cobordism $X : (Y, \lambda) \to (Y, e^{-a}\lambda) \sqcup \bigsqcup_i \partial B(r_j)$.

Step 3: $\Phi(X)$ formula¿

$\widehat{HM}^{-*}(\partial B(r_i))$ has basis $\{\zeta_k\}_{k\leq 0}$. $\zeta_0 = [(A_{triv}, \psi_{triv})]$ and $U\zeta_i = \zeta_{i+1}$.

For any $\sigma \in \widehat{HM}^{-*}$, $\Phi(\sigma) = \sum_{k\leq 0} \sum_{k_1+\cdots+k_N=k} U^k\sigma \otimes \zeta_{k_1} \otimes \cdots \otimes \zeta_{k_N}$.

Step 4: $\sigma_k, k \leq 0$ with $U\sigma_k = \sigma_{k+1}$.

$$\begin{aligned}
c_{\sigma_k}(Y,\lambda) &\geq c_{\Phi(\sigma_k)}\big((Y,e^{-a}\lambda)\sqcup \bigsqcup_{i=1}^{N}\partial B(r_i)\big)\\
&= \max_{U^{\bar{k}}\sigma_k\neq 0}\max_{\bar{k}=k_1+\cdots+k_N}\big(c_{U^{\bar{k}}\sigma_k}(Y,e^{-a}\lambda)+\sum_{i=1}^{N}c_{\zeta_{k_i}(\partial B(r_i)}\big)\\
&\geq \max_{k_1+\cdots+k_N=k-1}\sum_{i=1}^{N}c_{\zeta_{k_i}}(\partial B(r_i)).
\end{aligned}$$

Here $c_{\zeta_{k_i}}(\partial B(r_i)) = dr_i$ where d is the unique nonnegative integer s.t. $\frac{d^2+d}{2}\leq k_i\leq \frac{d^2+3d}{2}$.

This implies $\liminf \frac{c^2_{\sigma_k}}{k}\geq 4\sum_i \mathrm{vol}(B(r_i))^2 = \frac{1-e^{-a}}{2}\mathrm{vol}(Y,\lambda)-\epsilon$, with $a>0$ and ϵ arbitrary. This immediately implies "$\geq$". □

15. Lecture 15: Smooth closing Lemma + non-simplicity

15.1. **Smooth closing lemma in $3d$ contact dynamics.** Recall (Y,ξ) contact 3-manifold, where contact structure $\xi=\ker\lambda$ for some 1-form λ with $\lambda\wedge d\lambda$ nowhere zero. λ is called contact form. We also sometimes call (Y,λ) contact manifold. Can associate Reeb vector field X_λ by $\iota_{X_\lambda}d\lambda=0$ and $\lambda(X_\lambda)=1$.

For $f>0$, $f\lambda$ is another contact form defining the same ξ, but dynamics (Reeb flow φ^t, periodic orbits etc) of $X_{f\lambda}$ can be quite different to X_λ. (Ex: write $X_{f\lambda}$ in terms of X_λ and f.) Here, we quickly recall the idea of flow: We associates a 1-parameter family $\{\varphi^t\}_{t\in\mathbb{R}}$ of diffeomorphism for a vector field X_λ on closed Y as follows: $\dot\gamma = X_\lambda\circ\gamma$ is an ODE and solution $\gamma_x:\mathbb{R}\to Y$ exists all time (as Y is closed) and unique upon specifying the initial condition $\gamma_x(0)=x$, and we define $\varphi^t(x)=\gamma_x(t)$.

Theorem 15.1 (Kei Irie)**.** *(Y,λ) closed contact 3-manifold.*

$\{f\in C^\infty(Y,\mathbb{R}^{>0})\mid$ the union of periodic Reeb orbits of $X_{f\lambda}$ is dense in $Y\}$ is residual in $C^\infty(Y,\mathbb{R}^{>0})$ (with C^∞ topology). (Here, residual means it contains a countable intersection of open dense subsets.)

Denote $\mathcal{P}(Y,\lambda)=\{\gamma:\mathbb{R}/T_\gamma\mathbb{Z}\to Y\mid T_\gamma>0,\dot\gamma=X_\lambda\circ\gamma\}$ the set of closed Reeb orbits.

$\mathcal{A}:\mathcal{P}(Y,\lambda)\to\mathbb{R}^{>0}$, $\gamma\mapsto\mathcal{A}(\gamma):=T_\gamma=\int\gamma^*\lambda$, action. $\mathcal{A}(\lambda)=\mathrm{Im}(\mathcal{A})$ action spectrum of (Y,λ).

Define $\mathcal{A}(\lambda)_+$ set of actions for orbits sets, namely, let $\mathcal{A}(\lambda)_0=\{0\}$, $\mathcal{A}(\lambda)_m=\{\sum_{i=1}^m T_i\mid T_i\in\mathcal{A}(\lambda)\}$, and $\mathcal{A}(\lambda)_+=\bigcup_{m\geq 0}\mathcal{A}(\lambda)_m$.

Fact: $\mathcal{A}(\lambda)_+$ is a closed set of (Lebesgue) measure 0 in $\mathbb{R}^{\geq 0}$.

($\mathcal{A}(\lambda)\subset\{$critical values of a fixed real smooth function$\}\Rightarrow\mathcal{A}(\lambda)_+$ is measure 0; $\mathcal{A}(\lambda)$ closed, $\min\mathcal{A}>0\Rightarrow\mathcal{A}(\gamma)_+$ is closed.)

Choose $\Gamma\in H_1(Y)$, s.t. $c_1(\xi)+2PD(\Gamma)$ torsion like last time.

For a nonzero $\sigma\in\widehat{HM}^{-*}(\Gamma)$, define $c_\sigma(Y,\lambda)=\inf\{L\mid\sigma\in\mathrm{Im}(\iota_L)\}$ where $\iota_L:\widehat{HM}^{-*}_L\to\widehat{HM}^{-*}$.

Here remark, although the filtration is defined as

$$\mathcal{E}(\sum_i n_i[(A_i,\phi_i)]):=\sum_i n_i(i\int_Y\lambda\wedge F_{A_i})\leq L,$$

as $r \to \infty$, and $\phi^r = (\alpha^r_\phi, \beta^r_\phi)$ with $(\alpha^r_\phi)^{-1}(0)$ converges to a Reeb orbit set $\alpha := (\alpha^\infty_\phi)^{-1}(0)$. (Hope the overuse of α will not be confusing as we will denote α an orbit set and α_ϕ the first component of ϕ w.r.t. the splitting $PD(\Gamma) \oplus (PD(\Gamma) \otimes \xi)$.) And in this (subsequential) limiting process for solution (A^r, ϕ^r) to $(\ddagger)_r$ with bounded energy $\mathcal{E}$, $\mathcal{E}(A^r)$ converges to $\mathcal{A}(\alpha)$. So this the following which values $c_\sigma(Y, \lambda)$ makes sense.

$c_\sigma(Y, \lambda)$ is called spectral invariant, because:

- $f \geq 1$, $c_\sigma(Y, f\lambda) \geq c_\sigma(Y, \lambda)$.
- $a \in \mathbb{R}^{>0}$, $c_\sigma(Y, a\lambda) = ac_\sigma(Y, \lambda)$.
- $(f_j)_{j \in \mathbb{N}}$ in $C^\infty(Y, \mathbb{R}^{>0})$ with $f_j \underset{C^0}{\to} 1$, we have $\lim_{j\to\infty} c_\sigma(Y, f_j\lambda) = c_\sigma(Y, \lambda)$.
- (Action selector, spectrality) $\sigma \in \widehat{HM}^{-*}$, $c_\sigma(Y, \lambda) \in \mathcal{A}_+$.

Before the proof, let me quickly recap a standard notion in contact geometry which we have not used/needed so far.

λ is nondegenerate if for all $x \in \mathrm{Fix}(\varphi^T)$ for all $T > 0$, $d\varphi^T := \xi_x \to \xi_{\varphi^T(x)} = \xi_x$ (preserves the contact structure direction) has no eigenvalue 1, this implies that T-periodic orbits are isolated among such (T-periodic orbits). We also say $x \in \mathrm{Fix}(\varphi^T)$ is non-degenerate, where $(\varphi^t(x))_{t\in[0,T]}$ is an embedded periodic orbit, if $\varphi^T : \xi_x \to \xi_x$ and all the iterates do not have 1 as eigenvalue.

Proof. (spectrality) λ is non-degenerate. If $c_\sigma := c_\sigma(Y, \lambda) \notin \mathcal{A}(\lambda)_+$, as the latter being closed, $\exists \epsilon > 0$ s.t. $(c_\sigma - \epsilon, c_\sigma + \epsilon) \cap \mathcal{A}(\lambda)_+ = \emptyset$. $\Rightarrow \widehat{HM}^{-*}_{c_\sigma - \epsilon}(\Gamma) = \widehat{HM}^{-*}_{c_\sigma + \epsilon}$ (recall again, Taubes showed $\mathcal{E}(A^r)$ (which is tied to $\phi^r = (\alpha^r_\phi, \beta^r_\phi)$ through SW equation) converges to $\mathcal{A}(\alpha)$ with $\alpha := (\alpha^\infty_\phi)^{-1}(0)$.) $\Rightarrow \mathrm{im}(\iota_{c_\sigma - \epsilon}) = \mathrm{im}(\iota_{c_\sigma + \epsilon})$, which contradicts to the definition of c_σ.

λ degenerate, $\exists\ (f_j)_{j\geq 1}$ with $f_j > 0$, $f_j \underset{C^1}{\to} 1$ and $f_j\lambda$ non-degenerate. Each $c_\sigma(Y, f_j\lambda) \in \mathcal{A}(f_j\lambda)_{m(j)}$ where total multiplicity $m(j)$ is defined as $m(j) := \sum_i n_i$ (where $\alpha = \bigsqcup_i (\gamma_i, n_i)$ where γ_i is the underlying embedded closed Reeb orbit and n_i is its multiplicity). We have $\sup_j m(j) < 0$, since $\inf_j \min \mathcal{A}(f_j\lambda) > 0$ bounded away from zero (the number of max allocation to create $m(j)$ is bounded above). Thus, (up to subsequence) $m(j) = m$ constant, $c_\sigma(Y, f_j\lambda) = a^1_j + ... + a^m_j$, $a^k_m = \lim_{j\to\infty} a^k_j$ exists $1 \leq k \leq m$ and $a^k_j \in \mathcal{A}(f_j\lambda)$. As $f_j \underset{C^1}{\to} 1 \underset{\text{Reeb orbit sets converge}}{\Rightarrow} a^k_\infty \in \mathcal{A}(Y, \lambda)$ for all k. Thus, $c_\sigma(Y, \lambda) = a^1_\infty + \cdots + \infty^m_\infty \in \mathcal{A}(\lambda)_m \subset \mathcal{A}(\lambda)_+$. □

Theorem 15.2 (volume detecting, CG-H-R, covered last time)**.** *Let (Y, λ) closed connected contact 3-manifold. $\Gamma \in H_1(Y; \mathbb{Z})$ s.t. $c_1(\xi) + 2PD(\Gamma)$ torsion. Let σ_k be sequence of nonzero homogeneous classes in $\widehat{HM}^{-*}(\Gamma)$ s.t. $-gr(\sigma_k) \to \infty$. (We always have such thanks to the existence result in monopole homology that we have covered.) Then the volume detecting property holds:*

$$\lim_{k\to\infty} \frac{c_{\sigma_k}(Y, \lambda)^2}{-gr(\sigma_k)} = \int_Y \lambda \wedge d\lambda =: vol(Y, \lambda).$$

As an important corollary:

Corollary 15.3. *Let λ, λ' be contact forms with same $\xi = \ker \lambda = \ker \lambda'$. Suppose for any $\Gamma \in H_1(Y; \mathbb{Z})$ s.t. $c_1(\xi) + 2PD(\Gamma)$ torsion and any $\sigma \in \widehat{HM}^{-*}(\Gamma) \backslash \{0\}$ thereof, we have $c_\sigma(Y, \lambda) = c_\sigma(Y, \lambda')$, then $vol(Y, \lambda) = vol(Y, \lambda')$.*

Proof. Due to the calculation 2 lectures ago, there exists σ_k nonzero with $-\mathrm{gr}(\sigma_{k+1}) = -\mathrm{gr}(\sigma_k) + 2$. With hypothesis applied to this sequence, the conclusion follows by volume detecting. $\square$

Define $\|f\|_{C^\infty} := \sum_{l=0}^{\infty} 2^{-l} \frac{\|f\|_{C^l}}{1+\|f\|_{C^l}}$ which is always ≤ 2.

Lemma 15.4 ($\mathcal{C}^\infty$ closing lemma, Kei Irie)**.** *For any non-empty open set U in Y, $\epsilon > 0$, $\exists\ f \in C^\infty(Y)$ s.t. $\|f-1\|_{C^\infty} < \epsilon$ and $\exists$ non-degenerate $\gamma \in \mathcal{P}(Y, f\lambda)$ which intersects U.*

Proof. Take any $h \in C^\infty(Y, \mathbb{R}^{\geq 0})$ s.t. $\mathrm{supp}(h) \subset U$, $\|h\|_{C^\infty} < \epsilon$ and $h \not\equiv 0$. Then a calculation shows ($\oslash$) $\mathrm{vol}(Y, (1+h)\lambda) > \mathrm{vol}(Y, \lambda)$.

Claim: $\exists\ t \in [0,1]$ and $\gamma \in \mathcal{P}(Y, (1+th)\lambda)$ which intersects U.

Proof. (of Claim) Suppose not, then for any $t \in [0,1]$, for all $\gamma \in \mathcal{P}(Y, (1+th)\lambda)$, γ avoids U. Then $\mathcal{P}(Y, (1+th)\lambda) = \mathcal{P}(Y, \lambda)$ for all t, since closed Reeb orbits for $(1+th)\lambda$ and λ coincide in $Y \backslash U$. $\Rightarrow \mathcal{A}((1+th)\lambda)_+ = (A)(\lambda)_+$.

For any $\Gamma \in H_1(Y)$ with $c_1(\xi) + 2PD(\Gamma)$ torsion, and for all $\sigma \in \widehat{HM}^{-*} \backslash \{0\}$, $c_\sigma(Y, (1+th)\lambda) \in \mathcal{A}((1+th)\lambda)_+ = \mathcal{A}(\lambda)_+$, the latter is measure 0 in $\mathbb{R}^{\geq 0}$, and $c_\sigma(Y, (1+th)\lambda)$ is continuous in t, thus constant. $\underset{\text{Corollary}}{\Rightarrow}$ $\mathrm{vol}(Y, \lambda) = \mathrm{vol}(Y, (1+h)\lambda)$, which is contradicting to $\oslash$. [Claim] $\square$

Take $g \in \mathcal{C}^\infty(Y)$ with $\|g\|_{C^\infty}$ sufficiently small, $g|_{\mathrm{im}\gamma} \equiv 0$, and $dg|_{\mathrm{im}\gamma} \equiv 0$, so $\gamma \in \mathcal{P}(Y, e^g(1+th)\lambda)$ s.t. γ is moreover nondegenerate ($\varphi^T : \xi_x \circlearrowleft$ is twisted by $\mathrm{Hess}(g)$). $f := e^g(1+th)$ is the desired. [Closing lemma] $\square$

Proof. (Theorem of $\{f$ s.t. Reeb orbits of $X_{f\lambda}$ is dense$\}$ is residual)

Let $U \subset Y$ be nonempty open. Let

$$\mathcal{F}_U := \{f \in C^\infty(Y, \mathbb{R}^{>0}) \mid \exists \text{ nondegenerate } \gamma \in \mathcal{P}(Y, f\lambda) \text{ that intersects } U\}.$$

$\mathcal{F}_U$ is open (due to open condition) and dense (closing lemma) in $C^\infty(Y, \mathbb{R}^{>0})$.

Take a countable basis $(U_i)_{i \in \mathbb{N}}$ of open sets in Y. Then $\bigcap_i \mathcal{F}_{U_i}$ is residual and any f in it has Reeb orbits of $X_{f\lambda}$ dense in Y. (For any open neighborhood of any point, there will be a point on closed periodic orbits in it.) [Theorem] $\square$

Remark 15.5. There is a variant (same argument) for closed geodesics on a closed Riemannian surface (as closed geodesics on (Σ, g) correspond to periodic Reeb orbits of the associated contact 3-manifold cosphere bundle).

15.2. **Simplicity conjecture.** Exercise session is converted into the final session of lecture to cover a recent exciting progress partly using ideas/tools in this lecture series.

Volume detecting $\lim\limits_{k\to\infty} \frac{c_{\sigma_k}(Y,\lambda)^2}{-\mathrm{gr}(\sigma_k)} = \int_Y \lambda \wedge d\lambda$.

"$\leq$" uses structure and intrinsic property of Seiberg-Witten $(\ddagger)_r$ very deeply via intricate analysis.

"$\geq$" needs Seiberg-Witten, but potentially replaceable. Because the other approach (e.g. embedded contact homology, which is more direct at set-up with orbit set as generators of homology) still needs to associate a map $\phi(\mathrm{cob})$ for a cobordism cob which is used essentially in the calculation for "$\geq$", and at the moment, this is not well-defined (except in Seiberg-Witten Floer homology and via isomorphism $ECH_*(\Gamma) \overset{\text{Taubes}}{\cong} \widehat{HM}^{-*}(\Gamma)$ through it) due to geometric transversality issue.

For the discussion in our "exercise session", we follow the "Simplicity conjecture" paper by Cristofaro-Gardiner–Humilière–Seyfaddini.

(S, ω) surface with area 2-form (symplectic 2-manifold).

A homeomorphism $\phi : S \to S$ is an area-preserving homeomorphism if it preserves the measure (from ω defined as $\omega(A) := \int_A \omega$): $\omega(\phi^{-1}(U)) = \omega(U)$.

(D, ω) a unit disk with boundary with standard area form. $\mathrm{Homeo}_c(D, \omega)$ group of area-preserving homeomorphisms of 2-disk that are identity near the boundary (i.e., compactly supported).

Definition 15.6. A group G is simple if any normal subgroup (denoted by $H \triangleleft G$, meaning that the subgroup H satisfies $gHg^{-1} \subset H$ for $g \in G$) is either $\{e\}$ or G. I.e., it does not have a non-trivial (not $\{e\}$) proper (not G) normal subgroup.

Question: (Fathi, '80) Is $\mathrm{Homeo}_c(D, \omega)$ simple?

Why is it a good/interesting question?

- dim ≥ 3 understood by Fathi, SIMPLE.
- ICM 2006. 4 different mathematicians in different areas ask about this.
- Motivation for C^0-symplectic topology.

Theorem 15.7. *(CG-H-S) $\mathrm{Homeo}_c(D, \omega)$ is NOT simple.*

Remark 15.8.
- No natural (possibly discontinuous) homeomorphism analogous to flux, Calabi, mass-flow.
- Le Roux fragmentation $\Rightarrow$ If not simple, then lots of proper normal subgroups. (This might partly explain why it is hard to find one.)

Why we end the course with this topic and what are the connections between various topics?

Thom conj. solved by KM — SW Floer homology $\widehat{HM}$ etc

using SW on W^4 w. $\partial = Y^3$ $\xrightarrow{\text{valued in}}$ SW on $\mathbb{R} \times Y$

$\big\downarrow$ calculation and existence

Taubes' equiv of counts $\longrightarrow$ 4d SW on $\mathbb{R} \times Y : (A^r, (\alpha^r, \beta^r)) \overset{r\to\infty}{\to}$

betw. 4d SW & holo. curves — $(\alpha^\infty)^{-1}(0)$ holo. curve w. orbit sets as limits $\longrightarrow$ Weinstein conj.

Hutchings' $ECH_* \cong$ KM's $\widehat{HM}^{-*}$ — Taubes' proof

$\big\downarrow$ Hutchings's c_σ Taubes' $ECH_*(\Gamma) \cong \widehat{HM}^{-*}(\Gamma)$

C^∞ closing lemma $\xleftarrow{\text{Irie}}$ volume detecting $\xleftarrow[\text{“}\leq\text{” \& “}\geq\text{”}]{\text{CG-H-R}}$ properties on spectral invariant

$\big\downarrow$ analogue betw. $PFH = ECH$

non-simple $\mathrm{Homeo}_c(D, \omega)$ $\xleftarrow{\text{CG-H-S}}$ "$\geq$" part of Calabi detecting

15.2.1. *Stable Hamiltonian structure.* $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$, $\omega_{S^2} = \frac{1}{4\pi} d\theta \wedge dz$, $\mathrm{Area}(S^2) = 1$.

$D = \{(x, y) \mid x^2 + y^2 \leq 1\}$, $S^+ = \{(x, y, z) \in S^2 \mid z \geq 0\}$.

$\iota : D \to S^2$, $(r, \theta) \mapsto (\theta, 1 - r^2)$ with the image S^+.

$\omega = \iota^*\omega_{S^2} = \frac{1}{2\pi} r dr \wedge d\theta$. Disk (D, ω) has area $1/2$.

More general notion to contact structure (odd dimension in general, here we focus on 3dim)

Stable Hamiltonian structure (SHS) (α, Ω) on a 3-manifold Y is a pair, where α 1-form, Ω closed 2-form, and $\alpha \wedge \Omega$ nowhere zero, and (stability) $\ker \Omega \subset \ker d\alpha$ (equivalent to $d\alpha = g\Omega$ for some $g \in C^\infty(Y, \mathbb{R})$, here $\ker \Omega$ means kernel of the map $\iota_\bullet \Omega = \Omega(\bullet,) : TM \to T^*M$). Reeb vector field $X = X_{\alpha,\Omega}$ is defined by $\iota_X \Omega = 0$ and $\alpha(X) = 1$.

Example 15.9.

- Contact $(\lambda, d\lambda)$.
- Mapping torus. (S, ω) closed surface with area 2-form, φ area-preserving diffeomorphism $\varphi^*\omega = \omega$. Mapping torus $Y_\varphi := \frac{S \times [0,1]_r}{(x,1) \sim (\varphi(x), 0)}$. $\alpha = dr$ and $\Omega = \mathrm{pr}_1^* \omega / \sim = \omega_\varphi$. SHS. Reeb $X = \frac{\partial}{\partial r}$. Reeb orbits=periodic orbits of φ.

15.2.2. *Periodic Floer homology and c_d.* Floer homology (has the spirit of HM) $PFH(\varphi, h)$ periodic Floer homology. Here $h \in H_1(Y_\varphi) \backslash \{0\}$.

- Generators (over $\mathbb{Z}/2$) $\alpha = \{(\alpha_i, m_i)\}_i$ orbit set. α_i embedded periodic Reeb orbits, m_i positive integer, 1 if α_i hyperbolic (linearized return map $\varphi_x^T : \xi_x \to \xi_x$ has real eigenvalues). $h = \sum_i m_i [\alpha_i]$.
- There is relative grading $I(\alpha, \beta, Z)$ for orbit sets α, β and Z a spanning surface homology class from α to β, which calculate the dimension of the moduli space below.
- $\mathbb{R}_s \times Y_\varphi$. The tangent bundle $TY_\varphi = \mathbb{R}X \oplus \xi$ with X Reeb and $\xi = \ker \alpha$. Almost complex structure (90^o rotation) $J : T(\mathbb{R} \times Y_\varphi) \to T(\mathbb{R} \times Y_\varphi)$, $J : \frac{\partial}{\partial s} \mapsto X$, and $\xi \to \xi$ with $\Omega(\cdot, J\cdot)$ metric.

  $u : (\Sigma, j) \to (\mathbb{R} \times Y_\varphi, J)$ mapping from a Riemann surface Σ with j (almost) complex structure is (J-)holomorphic if $du \circ i = J(u) \circ du$, namely, du intertwines almost complex structures. This equation gives a Fredholm problem and forms a moduli space.
- $M_J^1(\alpha, \beta)$ denotes the space of holomorphic embedded curve/current C (already mod out domain reparametrization) of dimension $I(\alpha, \beta, [C]) = 1$ with asymptotics to α as $s \to +\infty$ and to β as $s \to -\infty$, then modulo $\mathbb{R}$ translation in $\mathbb{R} \times Y_\varphi$. This index $I(\alpha, \beta, [C])$ has the property that C in the moduli space with $I(\alpha, \beta, [C]) = 1$ (if transverse as a point in the moduli space) is automatically embedded.

  (Taubes 4d SW on $\mathbb{R} \times Y_\varphi$ with first component of spinor having the zero set $(\alpha^r)^{-1}(0) \overset{r \to \infty}{\to}$ holomorphic curves with ends Reeb orbit sets.)
- $\partial \alpha = \sum_\beta \#_2 M_J^1(\alpha, \beta) \beta$. We have $\partial^2 = 0$ (this is super non-trivial involving elaborate definition and argument of gluing). The homology is called PFH. (This is an SHS variant of embedded contact homology for Y contact.)

Specialize to $(S, \omega) = (S^2, \omega_{S^2})$, $\varphi \in \mathrm{Diff}(S^2, \omega_{S^2})$ supported in S^+. $Y_\varphi \cong S^2 \times S^1$. $h \in H_1(Y_\varphi) = \mathbb{Z}$. $\pi : Y_\varphi \to S^1$, $d = \deg(h) = \#_{\mathrm{inters}}([\mathrm{fiber}], h)$.

Choose a cycle γ_0 in Y_φ s.t. $\pi|_{\gamma_0} : \gamma_0 \to S^1$ is an orientation preserving.

- generator for $\widetilde{PFC}$ which comes with an action filtration. (α, Z) with Z spanning surface homology class between α and $d\gamma_0$.
- absolute grading: $I(\alpha, Z)$ with $I(\alpha, \beta, Z - Z') = I(\alpha, Z) - I(\beta, Z')$.
- differential counts J-holomorphic current C from α (from generator (α, Z)) to β (from generator (β, Z')) s.t. $Z' + [C] = Z$, as follows: $M_J^1((\alpha, Z), (\beta, Z'))$

is the moduli spaces of such with index $I(\alpha,\beta,[C]) = I(\alpha, beta, Z - Z') = I(\alpha, Z) - I(\beta Z') = 1$, then quotiented by $\mathbb{Z}$ translation in $\mathbb{R}\times Y_\varphi$. Define $\partial(\alpha, Z) = \sum_{(\beta,Z')} \#_2 M^1_J((\alpha, Z),(\beta, Z'))(\beta, Z')$.

$\partial^2 = 0$ gives rise to $\widetilde{PFH}$.

- Action $\mathcal{A}(\alpha, Z) = \int_Z \omega_\varphi$. $\widetilde{PFC}^L = \{(\alpha, Z) \mid \mathcal{A}(\alpha, Z) \leq L\}$. $\partial : \widetilde{PFC}^L \circlearrowleft$. $\Rightarrow \iota_L : \widetilde{PFH}^L \to \widetilde{PFH}$.
- $c_\sigma(\varphi) = \inf\{L \mid \sigma \in \operatorname{im}(\iota_L)\}$.

$\widetilde{PFH}_*(Y_\varphi, d)$ depends on the homotopy class of φ.

We calculate $\widetilde{PFH}_*(Y_\varphi, d) = \begin{cases} \mathbb{Z}/2 \text{ if } * = d \bmod 2 \\ 0 \text{ otherwise.}\end{cases}$ by using φ irrational rotation $(z,\theta)\mapsto(z,\theta+\alpha)$ with α irrational.

For every (d, k), $k = d(\text{mod}2)$, there exists a unique class $\sigma_{d,k}$ in grading k. Define $c_{d,k}(\varphi) := c_{\sigma_{d,k}}(\varphi)$, $c_d(\varphi) := c_{d,-d}(\varphi)$.

15.2.3. *Finite energy homeomorphism.* Define $H \in C^\infty_c(S^1_t \times D)$ time-dependent function on D^2 supported in $\mathring{D}$.

There exists a unique X_H such that $\omega(X_H, \cdot) := dH$ as ω is an area form. Flow of X_H, φ^t_H, Hamiltonian flow.

Every area preserving diffeomorphism $\phi \in \text{Diff}_c(D, \omega)$ is φ^1_H for some $H \in C^\infty_c(S^1 \times D)$.

Hofer norm/energy: $\|H\|_{(1,\infty)} = \int_0^1 \big(\max_{x\in D} H(t, x) - \min_{x\in D} H(t, x)\big)dt$.

Definition 15.10. $\phi \in \text{FHomeo}_c(D, \omega)$ is called finite energy homeomorphism if $\phi \in \text{Homeo}_c(D, \omega)$ and $\exists\, H_i \in C^\infty_c(S^1 \times D)$ s.t. $\|H_i\|_{(1,\infty)} \leq C < \infty$ and $\varphi^1_{H_i} \underset{C^0}{\to} \phi$.

One can show that the $\text{FHomeo}_c(D, \omega)$ is a normal subgroup.

The main focus of the lecture is to show:

Theorem 15.11. *$FHomeo_c(D, \omega)$ is a proper normal subgroup of $Homeo_c(D, \omega)$. (Namely, $\exists\ \phi \in Homeo_c(D, \omega)\setminus FHomeo_c(D, \omega)$ s.t. any $\{H_i\}$ with $\varphi^1_{H_i} \underset{C^0}{\to} \phi$ has $\|H_i\|_{(1,\infty)} \to \infty$.) Therefore, $Homeo_c(D, \omega)$ is not simple.*

We will construct an example of such ϕ:

Let $f : (0, 1] \to \mathbb{R}$ smooth, vanishes near 1, decreasing. $\lim_{r\to 0} f(r) = \infty$.

Define ϕ_f with $\phi_f(0) = 0$, $\phi_f(r, \theta) = (r, \theta + 2\pi f(r))$. $\phi_f \in \text{Homeo}_c(D, \omega)$ called ∞-twist.

We first define Calabi invariant to motivate the condition below. Let $\theta \in \text{Diff}_c(D, \omega)$, $\exists\ H \in C^\infty_c(S^1 \times D)$ s.t. $\theta = \varphi^1_H$. Define $\text{Cal}(\theta) = \int_{S^1}\int_D H\omega dt$. $\text{Cal} : \text{Diff}_c(D, \omega) \to \mathbb{R}$ thus defined is a non-trivial group homomorphism independent of such H.

We can construct ∞-twist with a f such that $\int_0^1\int_r^1 sf(s)dsrdr = \infty$.

Remark 15.12. $\omega = \frac{1}{2\pi} rdr \wedge d\theta$, ϕ_f smooth, defined by $f : [0, 1] \to \mathbb{R}$ (defined at 0 now) as in the above formula. Then $\text{Cal}(\phi_f) = \int_0^1\int_r^1 sf(s)dsrdr$, which explains the above quantity.

15.2.4. *Spectral invariant c_d and Calabi detecting for monotone twist.* Claim: c_d defined above has the following properties:

- $c_d(\mathrm{Id}) = 0$.
- $H \leq G$ in $C_c^\infty(S^1 \times D) \Rightarrow c_d(\varphi_H^1) \leq c_d(\varphi_G^1)$ for all d.
- Hofer-continuous: $|c_d(\varphi_H^1) - c_d(\varphi_G^1)| \leq d\|H - G\|_{(1,\infty)}$.
- Spectrality: $\varphi_d(\varphi_H^1) \in \mathrm{Spec}_d(H)$ analogous to $\mathcal{A}_+$, which is the image of critical points of the action functional in the current setting.
- (C^0) $c_d : \mathrm{Diff}_c(D,\omega) \subset \mathrm{Diff}_c(S^2, \omega_{S^2}) \to \mathbb{R}$ continuous w.r.t. C^0 on $\mathrm{Diff}_c(D,\omega)$ extends continuously to $\mathrm{Homeo}_c(D,\omega)$.

Theorem 15.13. *For a monotone twist $\varphi = \phi_f$, namely, ϕ_f defined by the same formula above but with smooth function $f : [0,1] \to \mathbb{R}$. Then $\lim\limits_{d\to\infty} \frac{c_d(\varphi)}{d} = Cal(\varphi)$. (We only need the "$\geq$" part.)*

15.2.5. *∞-twist is not of finite energy and proof of non-simplicity.*

Lemma 15.14 (($\odot$), linear growth for FHomeo_c)**.** *$\psi \in FHomeo_c(D,\omega) \Rightarrow \exists\ C = C(\psi)$ s.t. $\frac{c_d(\psi)}{d} \leq C$ for all d.*

Proof. By definition, $\exists\ H_i \in C_c^\infty(S^1 \times D)$ with $\|H_i\|_{(1,\infty)}$ bounded by C.
Hofer continuous and $c_d(\mathrm{Id}) = 0$, $\varphi_{H_i}^1 \underset{C^0}{\to} \psi$. $\Rightarrow c_d(\varphi_{H_i}^1) \leq d\|H_i\|_{(1,\infty)} \leq dC$.
c_d extends to $\mathrm{Homeo}_c(D,\omega)$ continuously $\Rightarrow c_d(\psi) = \lim\limits_{i\to\infty} c_d(\varphi_{H_i}^1) \leq dC\ \forall d$. □

Lemma 15.15 (($\circledast$), ∞-twist has super-linear growth)**.** *$\exists\ \phi_{f_i} \in Diff_c(D,\omega)$.*

- *$\phi_{f_i} \underset{C^0}{\to} \phi_f$ ∞-twist.*
- *$\exists\ F_i$ supported in $\mathring{D}$ s.t. $\varphi_{F_i}^1 = \phi_{f_i}$ and $F_i \leq F_{i+1}$.*
- *$\lim\limits_{i\to\infty} Cal(\phi_{f_i}) = \infty$.*

Proof. Choose smooth $f_i : [0,1] \to \mathbb{R}$. $f_i = f$ on $[\frac{1}{i}, 1]$ and $f_i \leq f_{i+1}$.
ϕ_{f_i} defined using twist. $\phi_{f_i} = \phi_f$ outside $D(\frac{1}{i})$, so $\phi_f^{-1}\phi_{f_i} \underset{C^0}{\to} \mathrm{Id}$.
ϕ_{f_i} is $\varphi_{F_i}^1$ for $F_i(r,\theta) = \int_r^1 sf_i(s)ds$, $F_i \leq F_{i+1}$.
By definition of Cal,

$$\begin{aligned}
\mathrm{Cal}(\phi_{f_i}) &= \int_0^1 \int_r^1 sf_i(s)dsrdr \\
&\geq \int_{\frac{1}{i}}^1 \int_r^1 sf_i(s)dsrdr \\
&= \int_{\frac{1}{i}}^1 \int_r^1 sf(s)dsrdr \to \infty.
\end{aligned}$$

So, $\lim\limits_{i\to\infty} \mathrm{Cal}(\phi_{f_i}) = \infty$. □

Proof. (∞-twist is not a finite energy homeomorphism, thus $\mathrm{FHomeo}_c(D,\omega)$ is proper, and $\mathrm{Homeo}_c(D,\omega)$ is not simple.)
$c_d(\phi_{f_i}) \leq c_d(\phi_{f_{i+1}})$.
$\phi_{f_i} \underset{C^0}{\to} \phi$, $c_d(\phi) = \lim\limits_{i\to\infty} c_d(\phi_{f_i})$.
So $c_d(\phi_{f_i}) \leq c_d(\phi_f)\ \forall i\ \forall d$.
Thus, $\lim\limits_{d\to\infty} \frac{c_d(\phi_f)}{d} \geq \lim\limits_{d\to\infty} \frac{c_d(\phi_{f_i})}{d} \underset{\text{“}\geq\text{'' of Calabi detecting}}{\geq} \mathrm{Cal}(\phi_i) \underset{\circledast}{\to} \infty$.
Thus, $\lim\limits_{d\to\infty} \frac{c_d(\phi_f)}{d} = \infty$ and ($\odot$) $\Rightarrow \phi_f \notin \mathrm{FHomeo}_c(D,\omega)$. □

You reach the end of the lecture series. Thank you very much for following along. If you want to take the example to get credit, please email me.