HIGHER STRUCTURES IN GEOMETRY AND MODULI SPACES

DINGYU YANG

1. Lecture 1 (version 1.1)

The material of this introductory lecture borrows from Part 1 of Vallette's very readable `https://arxiv.org/abs/1202.3245`, but is phrased in the cochain convention for the strict associativity of the de Rham (or the singular) cochains. The series of lecture notes will hopefully be improved and slightly expanded over time, so please feel free to let me know if any typos, errors, remarks, or any important omissions from the content delivered in the actual lectures. Glad the appearance of this note meets the 2-week lag promise.

Definition 1.1 (Homotopy retract)**.** Suppose (A, d_A) and (H, d_H) are differential graded (DG) vector spaces over a field k (or DG modules over a commutative unital ring k), where the differentials are of degree $+1$[1]. Suppose we have chain maps $p : (A, d_A) \to (H, d_H)$ and $\iota : (H, d_H) \to (A, d_A)$, and a linear map/endomorphism $h : (A, d_A) \to (A, d_A)$ of degree -1. It can be pictured as

$$h \circlearrowright (A, d_A) \underset{\iota}{\overset{p}{\rightleftarrows}} (H, d_H)$$

(p, i, h) is called a *homotopy retract from* (A, d_A) *to* (H, d_H) if

$$\mathrm{Id}_A - \iota \circ p = d_A \circ h + h \circ d_A$$

and ι is a quasi-isomorphism between DG vector spaces (namely, the induced map $\underline{\iota} : H^*(H, d_H) \to H^*(A, d_A)$ between the cohomologies is an isomorphism, e.g. $p \circ \iota = \mathrm{Id}_H$ in addition to the above).

Definition 1.2 (DGA)**.** A *differential graded (k-)algebra* (or *k-DGA*, or just *DGA*) $(A, d_A, \nu : A \otimes A \to A)$ is a tuple where (A, d_A) is a differential graded k-vector space with the differential of degree $+1$, and $\nu : A \otimes A \to A$ is linear of degree 0 and associative

$$\nu \circ (\nu \otimes \mathrm{Id}_A) = \nu \circ (\mathrm{Id}_A \otimes \nu)$$

(not necessarily graded commutative), such that d_A is a derivation with respect to ν, namely, it satisfies the Leibniz rule: $d_A \circ \nu = \nu \circ (d_A \otimes \mathrm{Id}_A) + (-1)^{\deg x_1} \nu \circ (\mathrm{Id}_A \otimes d_A)$ when acting on a homogeneous $x_1 \otimes x_2$.

One can define $d_A \nu := d_A \circ \nu - \nu \circ (d_A \otimes \mathrm{Id}_A) - (-1)^{\deg x_1} \nu \circ (\mathrm{Id}_A \otimes d_A)$. (The Koszul sign rule (for maps) will be explained in Lecture 3 and the signs in the identities can be read as ± 1 for now.) The derivation property is just $d_A \nu = 0$.

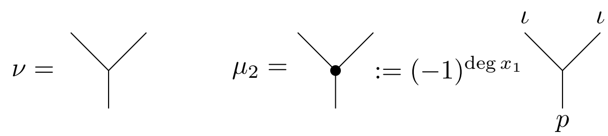
Date: November 1, 2019.

[1]Here the degree of a map f is $\deg f(x) - \deg x$.

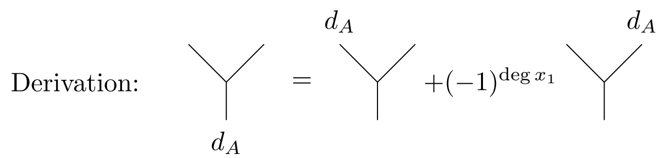
Suppose (A, d_A, ν) is a DGA and (H, d_H) is a DG vector space. (p, i, h) is a homotopy retract from (A, d_A) to (H, d_H). How can we transfer the algebra structure ν from (A, d_A) to (H, d_H)?

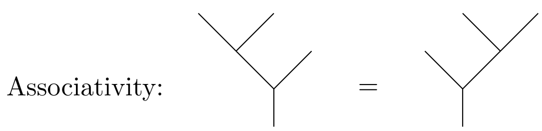
For $\mu_2 : H^{\otimes 2} \to H$, not much choice is given: $\mu_2 := p \circ \nu \circ (\iota^{\otimes 2})$.

We use a tree in the direction from top to bottom to pictorially indicate the input-output relation:

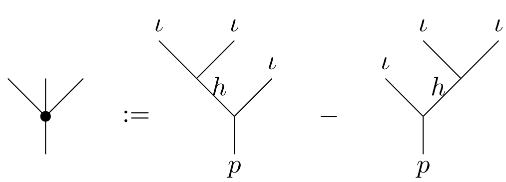


Using this device, the derivation and associativity properties of ν are depicted as:





One sees that μ_2 is not associative because $\iota \circ p$ is the identity only up to homotopy. Is there a way to rectify this? The answer is yes. One can define μ_3 by the following:



Here the internal edge being labeled by h means that before we connect a pair of an input and an output, we insert the action of h.

If we define analogously

$$
\begin{aligned}
d_H\mu_3 &:= d_H \circ \mu_3 - \mu_3 \circ (d_H \otimes \mathrm{Id}_H^{\otimes 2}) - (-1)^{\deg x_1}\mu_3 \circ (\mathrm{Id}_H \otimes d_H \otimes \mathrm{Id}_H) \\
&\quad - (-1)^{\deg x_1 + \deg x_2}\mu_3 \circ (\mathrm{Id}_H^{\otimes 2} \otimes d_H),
\end{aligned}
$$

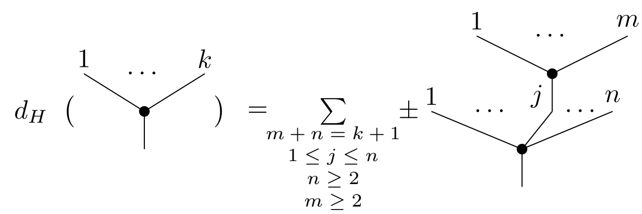
the failure of the associativity for μ_2 is the (co)boundary of μ_3 (as a map):

$$\mu_2 \circ (\mu_2 \otimes \mathrm{Id}_H) - \mu_2 \circ (\mathrm{Id}_H \otimes \mu_2) = d_H\mu_3.$$

Having introduced higher multiplications, there will be a measure of failure of higher associativity and a still 1-level-up higher multiplication whose coboundary will offset the failure.

In general, the higher multiplication μ_k is defined by a signed sum over the planar binary trees with k leaves and 1 root, where leaves are precomposed by ι, the root is (post)composed with p and internal edges are labeled by h. Here planar means embedded into the plane, binary means only two branches can appear at one time when ‘growing’ from the root upwards, and only the finitely many combinatorial (not geometric) types are summed over.

With this definition of μ_k which is depicted by a k-to-1 tree with a black dot at the branch (just as μ_2 and μ_3 above), we have the following identity where the right hand side is the k-th higher associator (the failure of higher associativity).



Denote $\mu_1 = \pm d_H$ (see Lecture 3 for the exact sign(s)), the above picture is precisely the defining A_∞ (associativity) relation for A_∞ *algebra* $(H, \{\mu_k\}_{k\geq 1})$. Moving the left hand side of the picture to the right, one can see the A_∞ relation is about a quadratic relation among pairs of (μ_p, μ_q) with $p+q = k+1$ for each k; and it can also be viewed as it is where μ_k provides a primitive of the ‘higher associativity’ of $\{\mu_d\}_{2\leq d\leq k-1}$ being a coboundary (as a map). The process of obtaining an A_∞ algebra on H from an (A_∞) algebra structure on A via a homotopy retract is called a *homotopy transfer*, and $\{\mu_k\}_{k\geq 1}$ is the *transferred structure*.

We specialize (A, d_A, μ) to the de Rham cochain complex $(\Omega^*(M), d, \wedge)$ (or the singular cochain complex with the also chain-level associative product structure), and $(H, d_H = 0)$ being the cohomology $H(\Omega^*(M), d)$, and we can make a choice of homotopy retract as follows:

$d_A : A^n \to B^{n+1} \subset A^{n+1}$ onto the subspace $B^{n+1} = d_A(A^n)$ of coboundaries with the kernel Z^n consisting of cocycles. Thus one can make a choice of an isomorphism $A^n \cong Z^n \oplus B^{n+1}$. As $H^n := Z^n/B^n$, we can make another choice of an identification $Z^n \cong B^n \oplus H^n$. After making choices for each n, $A^n \cong B^n \oplus H^n \oplus B^{n+1}$, and d_A under these identifications becomes $\mathrm{Id}_{B^{n+1}} \circ \mathrm{pr}_{B^{n+1}}$. Define

$$h := \mathrm{Id}_{B^n} \circ \mathrm{pr}_{B^n} : A^n \to A^{n-1}$$

via these identifications and $\iota = \mathrm{Id}_{H^n} : H^n \to A^n$ and $p := \mathrm{pr}_{H^n} : A^n \to H^n$. One has $\mathrm{Id}_A - \iota \circ p = d_A \circ h + h \circ d_A$ and $p \circ \iota = \mathrm{Id}_{H^n}$. Thus (p, ι, h) is a homotopy retract.

Using the homotopy transfer via (p, ι, h) as above, we have an A_∞ algebra $(H, \{\mu_k\}_{k\geq 1})$ with $\mu_1 = \pm d_H = 0$. Here, μ_k is a higher Massey product defined via a choice of (p, ι, h). The usual cohomological product μ_2 is associative, because $\mu_1 = 0$; but μ_3 might be non-trivial. See Exercise sheet 1 for the Borromean ring example with a non-trivial μ_3.

In fact, a result says that a stronger version of A_∞ algebra on the cochain complex that respects the shuffle products detects the rational homotopy type for simply connected spaces. So the usual cohomology product which truncates the higher information is not the right way to capture the product structure on cochains. This result that C_∞ algebra on cohomology is a complete invariant for simply connected rational homotopy types is due to Kadeishvili in 2008, see `https://arxiv.org/abs/0811.1655` (and the journal version can be found via mathscinet).

To see that μ_3 really agrees with the classical Massey product, we first recall the latter. The Massey product is a secondary operation defined for a triple of cohomology classes $\underline{x}$, $\underline{y}$ and $\underline{z}$ with $\underline{\nu}(\underline{x},\underline{y}) = 0$ and $\underline{\nu}(\underline{y},\underline{z}) = 0$, where the underlines denote the quotients in cohomology. So taking representatives without the underlines, we have $\nu(x,y) = du$ and $\nu(y,z) = dv$, note that $-\nu(u,z) + (-1)^{\deg x}\nu(x,v)$ is a cocycle due to associativity of ν. We define $m_3(\underline{x},\underline{y},\underline{z}) := \underline{-\nu(u,z) + (-1)^{\deg x}\nu(x,v)}$ where the last quotient lives in

$$H^{\deg x+\deg y+\deg z-1}/(\underline{\nu}(H^{\deg x+\deg y-1}, H^{\deg z}) + \underline{\nu}(H^{\deg x}, H^{\deg y+\deg z-1})),$$

for any $(\underline{x},\underline{y},\underline{z}) \in H^{\deg x} \times H^{\deg y} \times H^{\deg z}$ satisfying the pair of vanishing conditions and is well-defined independent of choices. Using the definition μ_3 in the transfer, we have $\mu_3 := p\big(h(x\cdot y)\cdot z) - x\cdot h(y\cdot z)\big)$, where $x\cdot y := (-1)^{\deg x}\nu(x,y)$ for clarity. Then

$$\begin{aligned}\mu_3(\underline{x},\underline{y},\underline{z}) &= p\left((-1)^{\deg x+\deg x+\deg y-1}\nu(u,z) + (-1)^{\deg y+\deg x}\nu(x,v)\right)\\ &= (-1)^{\deg y}p(-\nu(u,z) + (-1)^{\deg x}\nu(x,v)).\end{aligned}$$

Here $x = \iota(\underline{x})$ etc, $u = h(\nu(x,y))$ and $v = h(\nu(y,z))$. Here, after making a choice of a homotopy retract, μ_3 is defined, its value lifts to H, it does not need the vanishing condition for its definition, and the expression before precomposing p is not closed in general. But in the case where $\nu(x,y) = du$, the homotopy retract condition applying to $\nu(x,y)$ gives $\nu(x,y) = d_A(h(\nu(x,y)))$, so $h(\nu(x,y))$ is a possible choice for u; similarly $h(\nu(y,z))$ for v (while the well-definedness of m_3 is done by taking the quotient in the image); and the projection by p just takes the associated cohomology class and lands in H prior to taking the further quotient by decomposables. Therefore, the form of μ_3 agrees with m_3 up to a sign. A similar discussion can be had for the higher Massey products, and the readers are encouraged to try it for μ_4.