Relative Proportionality
on Picard and Hilbert Modular Surfaces

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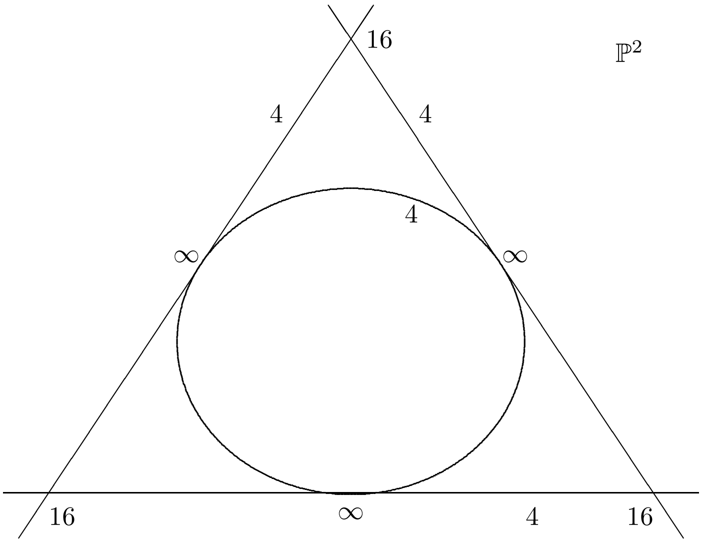
Contents

1 Preface

We start with a simple example of an Picard orbiface. The ***Apollonius configuration*** consists of a quadric together with three tangent lines on the complex projective plane $\mathbb{P}^2$. Explicitly, for instance, we can take the projective curve described by the equation

$$XYZ(X^2 + Y^2 + Z^2 - 2XY - 2XZ - 2YZ) = 0$$



(1)

We endow the points of the plane with weights

- ∞ for the 3 touch points;
- 16 for the 3 intersection points of lines;
- all other points on the 4 curves get weight 4;
- the remaining points of the plane have ***trivial weight*** 1.

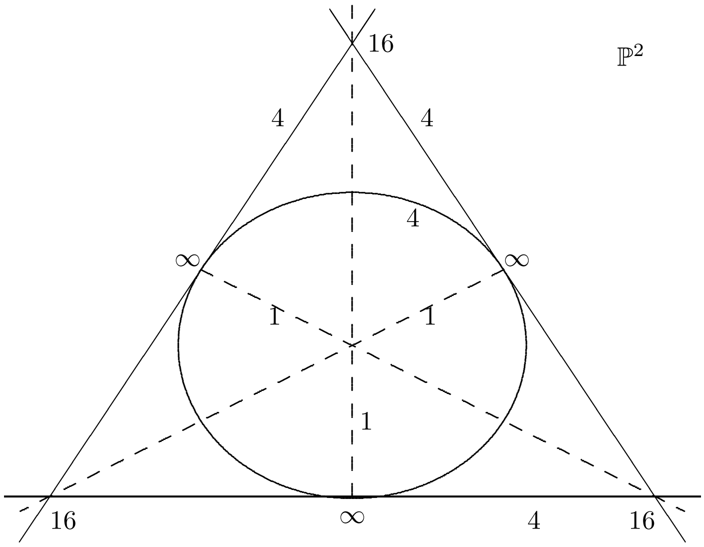
In the mean time it is known that the picture represents an orbital Picard modular plane together with weighted branch locus of a hyperbolic (complex ball) uniformization. A ***Picard modular plane*** is a Baily-Borel compactified Picard modular surface, which is a projective plane. Without weights, but in connection with ball uniformization, I saw the Apollonius configuration first in the paper [Y-S]. The corresponding Picard modular group acting on the complex 2-ball has been determined precisely first in the HU-preprint [HPV] as congruence subgroup of the full Picard modular group of the imaginary quadratic field $\mathbb{Q}(\sqrt{-1})$ of Gauß numbers. It has been published in [HV].

Surprisingly, in a pure algebraic geometric manner and with a finite number of steps, we are able to read off almost from this weighted projective plane the

Fourier series

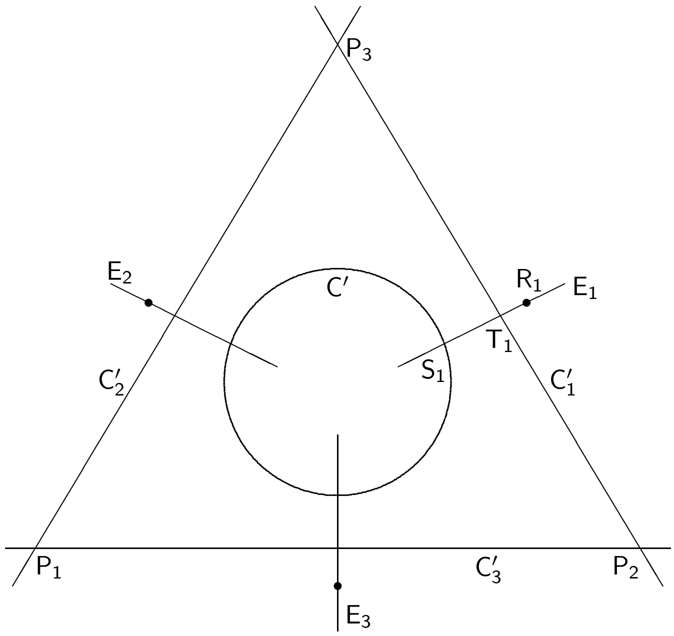
$$\begin{aligned}\mathbf{Heeg_C}(\tau) &= \sum_{N=0}^{\infty}\left(\left(\frac{3N}{2}-\frac{1}{8}\right)a_2(N)+3\sum_{m=1}^{N}\sigma(m)a_2(N-m)\right)q^N \in \mathcal{M}_3(4,\chi),\\ &= -\frac{1}{8}\cdot\vartheta^6-\frac{17}{2}\cdot\vartheta^2\theta,\\ &q=\exp(2\pi i\tau),\ \tau\in\mathbb{H}\ \text{(complex upper halfplane).}\end{aligned} \tag{2}$$

with Jacobi's modular form ϑ and Hecke's modular form θ described in the appendix. This is an elliptic modular form of certain level, weight and Nebentypus χ. The N-th coefficient counts (with intersection multiplicities) the arithmetic curves of norm N on the Picard-Apollonius orbiplane. Thereby $\mathbf{C}$ is the orbital arithmetic curve (with above weights) sitting in the Apollonius cycle, where $\sigma(m)$ denotes the sum of divisors of m, and $a_2(k)$ is the number of $\mathbb{Z}$-solutions of $x^2+y^2=k$, and $\chi=\chi_8$ is the Dirichlet character on $\mathbb{Z}$ extending multiplicatively $\left(\frac{2}{p}\right)=(-1)^{(p^2-1)/8}$ for odd primes $p\in\mathbb{N}$ and 0 for even numbers. More precisely, let us extend the cycle and consider



(3)

where all arithmetic curves of smallest norms 1 and 2 are drawn. One has the most dif and only ificulties with the algebraic geometric calculation of the constant coefficient of the Heegner series. For this purpose one has to consider rational orbital self-intersections on the so-called released Picard-Apollonius orbiplane. We draw the released Apollonius cycle:

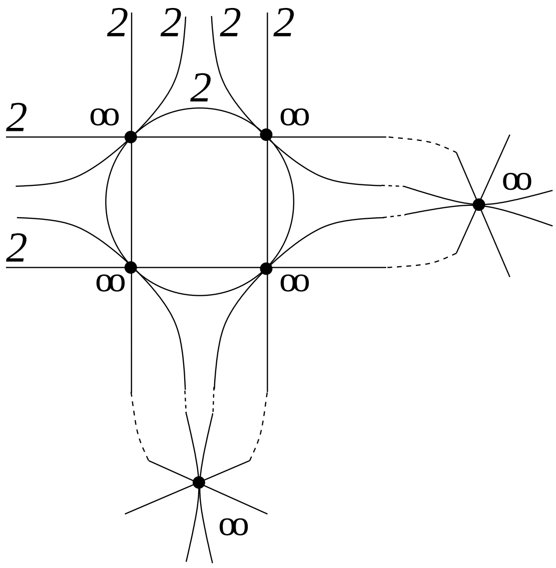


(4)

In the mean time we found another simple example. We call it the Hilbert-Cartesius orbiplane. The supporting ***Cartesius configuration*** lies also on $\mathbb{P}^2$. It consists of three quadrics touching each other (and nowhere crossing) together with four lines, each of them joining three of the six touch points. Take, for example, the projective closure of the affine curve described by the equation

$$(X^2 + Y^2 - 2)(XY - 1)(XY + 1)(X^2 - 1)(Y^2 - 1) = 0$$

consisting in the real affine plane of a circle, two hyperbolas and four lines parallel two the axis through pairs of the points $(\pm 1, \pm 1)$. Not visible are two further intersection points of same quality (two quadrics meet two lines) at infinity. It is easy to draw these seven curves.



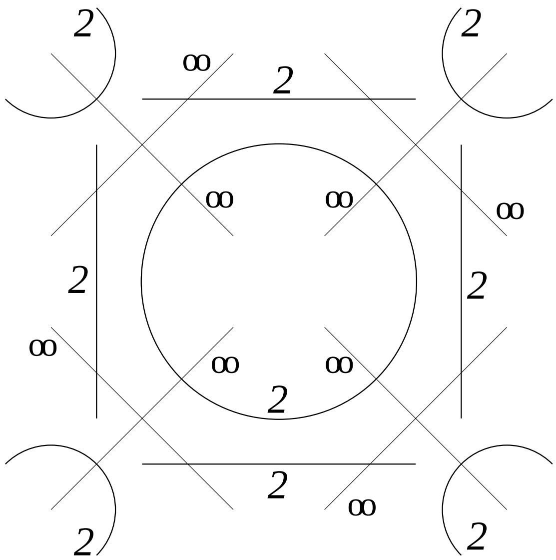
We endow the points of the plane with the following weights:

- ∞ for the six intersection points;
- all other points on the 7 curves get weight 2;
- the remaining points of the plane have "trivial weight" 1.

The picture represents an orbital Hilbert modular plane together with weighted branch locus of a bi-disc uniformization. A ***Hilbert modular plane*** is a Baily-Borel compactified Hilbert modular surface, which is a projective plane. The corresponding Hilbert modular group acting on the bi-product $\mathbb{H}^2$ of the upper half plane $\mathbb{H}$ can be found in [Hir1], [Hir2], [vdG]. It is commensurable with the full Hilbert modular group of the real quadratic number field $\mathbb{Q}(\sqrt{2})$. Also in this case it is possible in almost the same purely algebraic geometric manner in finite steps to read of the Fourier series

$$\mathbf{Heeg_C}(\tau) = -1 + 2 \cdot \sum_{N=1}^{\infty} \left(\sum_{d|N} \chi(d) d \right) q^N \in \mathcal{M}_2(8, \chi), \tag{5}$$

connected with the plane quadric $C : X^2 + Y^2 - 2Z^2 = 0$. This is again an elliptic modular form of certain level, weight and Nebentypus χ. More precisely, this is an Eisenstein series. The coefficients again count (with degree multiplicities) the arithmetic curves of fixed norms and of Humbert type. We also need for the calculation of the rational constant coefficient of the series an orbital curve self-intersection on the released Hilbert-Cartesius orbiface, whose non-trivially weighted curves we draw in the following picture:



2 Introduction

We only consider normal surfaces, analytic or algebraic over the complex numbers. We need a precise geometric language. More precisely, we need objects, which are a little finer than varieties but not so abstract as Shimura varieties over number fields or stacks. The latter notions make much sense for fine number theoretic considerations. But here we are near only to the original geometric Shimura varieties over $\mathbb{C}$, especially to Picard and Hilbert modular surfaces, called Shimura surfaces in common, and modular or Shimura curves on them, which we call also arithmetic curves. For example, a Picard modular surfaces is a ball quotient $\Gamma\backslash\mathbb{B}$ by a lattice Γ, which is a discrete subgroup Γ of $\mathbb{PU}((2,1),\mathbb{C})$ acting properly discontinuously on the 2-dimensional complex unit ball $\mathbb{B}$. Let $\mathbb{D}$ be a linear disc on the ball $\mathbb{B}$ such that $\Gamma\backslash\mathbb{D}$ is an algebraic curve on $\Gamma\backslash\mathbb{B}$. Invariant metrics on the ball $\mathbb{B}$ (Bergmann) go down to metrics on the surfaces with degeneration cycles (join of irreducible curves and points). These curves and points are endowed with natural weights coming from ramification indices of the locally finite covering $\mathbb{B} \to \Gamma\backslash\mathbb{B}$. It can be proved that on canonical surface models (in Shimura's sense) these curves are defined over a ring the integers of a number field. At the same time these disc quotient curves are geodesics with respect to the degenerate metric. In the mean time it was proved (A.M. Uludag,[Ul]) that there are infinitely many Picard modular projective planes.

Especially, it would be very interesting to find plane equations with integral coefficients for the above arithmetic geodesics. At the moment we are only able to give a numerical characterization of such curves. Looking back to history we know Plücker's formula for plane curves C with only cusps or transversal self-intersection points joining the numbers of them, the curve degree and the genus. Together with the idea of proof we remember Plücker's explicit relation

$$d(d-1) - 2\delta - 3\kappa = d^* = 2d + (2g-2) - \kappa, \tag{6}$$

where g denotes the genus, d the degree of C, d^* the number of tangents of C through a fixed general point of $\mathbb{P}^2$, δ the number of double points and κ the number of cusps of the curve. Forgetting d^* we get a relation between the degree d, the genus g, δ and κ.

We will define orbifaces, especially Picard and Hilbert modular orbifaces, and orbital curves $\mathbf{C}$ on the latter, all with help of branch weights. We will introduce orbital invariants for them, which are rational numbers explicitly expressed in terms of algebraic geometry. Of special interest are the two orbital invariants $\mathbf{E}$ and $\mathbf{S}$, which are singularity modifications of the Euler number and the selfintesection of the curve algebraically calculable on a special surface model, the so-called release model, which we must introduce. Restricting, for example to Picard orbifaces we get in analogy to (6) the relations

$$\mathbf{Eul}(\mathbf{C}) = \mathbf{vol}(\mathbf{\Gamma}_{\mathbb{D}}) = \mathbf{2} \cdot \mathbf{Self}(\mathbf{C}), \tag{7}$$

with supporting curve $C = \Gamma \backslash \mathbb{D}$. Forgetting the Euler-Bergman volume $vol(\Gamma_{\mathbb{D}})$ of a fundamental domain of the subgroup $\Gamma_{\mathbb{D}}$ of Γ of all elements operating on $\mathbb{D}$ one gets a characteristic relation between the orbital Euler invariant and the orbital self-intersection of $\mathbf{C}$:

Theorem 2.1 *With the above notations (and definitions of orbital invariants below) it holds that*

$$\mathbf{Eul}(\mathbf{C}) = \mathbf{2} \cdot \mathbf{Self}(\mathbf{C}).$$

The relation (7) generalizes the Relative Proportionality for modular curves and Shimura curves on neat ball quotient surfaces, see [H98]. For their importance we show that the relative orbital Euler invariants are the constant terms of Fourier series of modular quality joining infinitely many orbital invariants. Moreover, the (orbital invariant) coefficients count (with multiplicity depending on $\mathbf{C}$ embedded on $\mathbf{X}$) the number of arithmetic geodesics on X, see citeH02.

It is not difficult to prove and write down the relative Proportionality Relation

$$\mathbf{Eul}(\mathbf{C}) = \mathbf{Self}(\mathbf{C}) \tag{8}$$

for orbital modular or Shimura curves on any Hilbert orbiface (including also symmetric Hilbert modular groups). It is also clear that the orbital invariant in (8) is the constant coefficient of a Hirzebruch-Zagier modular form [HZ]. The aim of the next section is to develop a common language for Hilbert and Picard modular surface with a common algebraic geometric understanding of the involved elliptic modular Hirzebruch-Zagier series [HZ] and Kudla-Cogdell series (ball case) [Cog]. To use this fine geometric language for common proofs and explicit calculations is the aim of the whole paper.

For a clear definition of orbital invariants we need a category $\mathcal{C}$ with a multiplicatively closed set of "finite coverings" $\mathbf{D}/\mathbf{C}$ and "degrees" $[\mathbf{D}:\mathbf{C}]$ of the latter, satisfying the degree formula

$$[\mathbf{E}:\mathbf{C}] = [\mathbf{E}:\mathbf{D}] \cdot [\mathbf{D}:\mathbf{C}]$$

for all double coverings $\mathbf{E}/\mathbf{D}/\mathbf{C}$. A ***rational orbital invariant*** on $\mathcal{C}$ is simply a non-constant map

$$\mathbf{h} : \mathbf{Ob}\,\mathcal{C} \longrightarrow \mathbb{Q}$$

from the objects of our "orbital category" satisfying the "orbital degree formula"

$$\mathbf{h}(\mathbf{D}) = [\mathbf{D}:\mathbf{C}] \cdot \mathbf{h}(\mathbf{C})$$

for all finite coverings $\mathbf{D}/\mathbf{C}$. Part of the work is to clarify, which orbital categories we can construct. In the paper [Ul] by A.M. Uludag I saw him working with weighted surfaces $\mathbf{X} = (X, w)$, $w: X \to \mathbb{N}_+$ a map from the surface X to the natural numbers. These object we will basically use for the construction of our orbital categories. After the definition procedure we are able to find explicitly infinitely many orbital invariants. We combine the rational intersection theory and Heegner cycles. But all these invariants are "modular dependent", which means, that they are connected by modular forms of known and fixed type with each other. It follows that it suffices only to know finitely many of the orbital invariants to determine the others together with the corresponding modular form. The interpretation of counting arithmetic curves on Picard or Hilbert modular surfaces is then general. Until now the series were only known in neat cases. So it was e.g. until now not possible to count arithmetic curves on modular planes. The extension from neat to general cases is the progress described in this paper. In contrast, in the neat Picard case the corresponding quotient surfaces are never rational. Explicit calculations there seem to be very difficult. But in the plane case the situation is much better, especially with a view to coding theory on explicit arithmetic curves.

Remark 2.2 *: Volumes of fundamental domains of Picard or Hilbert modular groups with respect to fixed volume forms on the the uniformizing domains are obviously orbital invariants. If we take the Euler volume form, then these volumes coincide with the orbital Euler invariants of the corresponding orbital surfaces. This is a theorem (Holzapfel, [H98]), in the ball case, to be written down in the Hilbert case). The same is true for the signature volume form, which leads to the corresponding orbital signature invariants. The Euler and the signature forms are distinguished by a factor. This leads to a proportionality relation between the orbital invariants with different factors in the Picard and Hilbert surface cases.*

For explicit calculations it is important to know that the orbital invariants of the modular surfaces of the full lattices can be expressed by special values of Zeta-functions (or L-series) of corresponding number fields. (Maass in the Hilbert case, Holzapfel [H98] in the Picard case.)

3 The Language of Orbifaces

3.1 Galois weights

A ***weighted (algebraic) surface*** $\mathbf{X} = (X, w)$ is defined by:

- an irreducible normal complex algebraic surface X
- and a ***weight map*** $w: X \longrightarrow \mathbb{N}_+ \cup \{\infty\}$

with conditions

- almost everywhere (i.e. up to a proper closed algebraic subvariety of X) one has ***absolute trivial weight*** 1:
- w is almost constant on each closed irreducible curve C of X (t.m. up to a finite set of points $P_1, ..., P_r$ on C);

The corresponding constant $w(C) := w(C \setminus \{P_1, ..., P_r\})$ is called the ***weight*** of C on $\mathbf{X}$.

- If $P \in C$, C an irreducible curve on X, then $w(C)$ divides $w(C)$.

A point P on X is ***relatively non-trivial weighted*** if $w(P) > w(C)$ for all irreducible curves C though P. Otherwise it is called ***relatively trivial weighted***. The formal (finite) double sum

$$B = B(X, w) := \sum w(C)C + \sum w(P)P = B_1 + B_0$$

over all irreducible curves C with non-trivial weight (> 1) respectively all relatively non-trivial weighted points P on X is called the ***weight cycle*** of $\mathbf{X}$. If the double sum is restricted to (all) finite non-trivial weights, then we call it the ***finite weight cycle*** of $\mathbf{X}$, denoted by $B^{fin} = B^{fin}(X, w)$. As above we have two partial sums, one over the curves, the other over the points of finite non-trivial weights:

$$B^{fin} = B_1^{fin} + B_0^{fin}.$$

Complementarily, we define the ***infinite cycle*** of $\mathbf{X}$ by

$$B^{\infty} = B^{\infty}(X, w) := B - B^{fin} = B_1^{\infty} + B_0^{\infty}$$

together with its $1, 0$ - dimensional decomposition in obvious manner.

As usual we call the union of component sets of a cycle D on X the ***support*** of D and denote it by $supp\, D$. The weighted surface $\overset{o}{\mathbf{X}} = (\overset{o}{X}, \overset{o}{w})$ with $\overset{o}{X} = X \setminus supp\, B^{\infty}$ and $\overset{o}{w} = w|_{\overset{o}{X}}$ is called ***open finite part*** of $\mathbf{X}$. We get the first examples of open subsurfaces $\overset{o}{\mathbf{X}} \subseteq \mathbf{X}$ and open embeddings $\overset{o}{\mathbf{X}} \hookrightarrow \mathbf{X}$ of weighted surfaces in this way. Generally, we take open subsurfaces U of X instead of $\overset{o}{X}$ and define in analogous manner ***open weighted subvarieties*** $\mathbf{U} \subseteq \mathbf{X}$ and ***open embeddings*** $\mathbf{U} \hookrightarrow \mathbf{X}$ using restrictions of the weight map w.

Let C be an irreducible curve on X. The system of open neighbourhood U of C defines the surface germ of X along C. We imagine it as a small open neighbourhood of C or, more precisely, as refinement (equivalence) class of such neighbourhoods. Working additionally with weight restrictions we define the ***weighted surface germ*** $\mathbf{C}$ of $\mathbf{X}$ ***along*** C as refinement class of all $\mathbf{U} = (U, w|_U)$ with open neighbourhoods U of C. We write $\mathbf{C} \leftharpoonup \mathbf{X}$ in this situation and consider it as ***closed embedding*** of weighted objects (supported by the closed embedding of C into X in the usual sense).

The same can be done with open neighbourhoods of a point $P \in X$. Working with weight restrictions again we define the ***weighted surface germ*** $\mathbf{P}$ of $\mathbf{X}$ ***at*** P as refinement class of all $\mathbf{U} = (U, w|_U)$ with open neighbourhoods U of P. We write $\mathbf{P} \,\varepsilon\, \mathbf{X}$ in this situation. If P is a point on C, then we write $\mathbf{P} \,\varepsilon\, \mathbf{C}$ instead of the pair $(\mathbf{C}, \mathbf{P})$ and consider it as (closed) embedding of weighted surface germs.

Remark 3.1 *We left it to the reader to work with complex or with Zariski topology. In the later definitions of orbital invariants there will be no difference. But there will be numerical differences between the embedded germs* $\mathbf{P} \,\varepsilon\, \mathbf{X}$*,* $\mathbf{P} \,\varepsilon\, \mathbf{C}$*,* $\mathbf{P} \,\varepsilon\, \mathbf{D}$*, where D is another irreducible curve through P. This is not surprising because already for unweighted surfaces it can happen that P is a singularity of one or two of the objects X, C or D, but may be a regular point of the other one(s).*

An ***isomorphism*** of weighted surfaces $\mathbf{f} : \mathbf{X} \xrightarrow{\sim} \mathbf{Y}$, $\mathbf{Y} = (Y, v)$, $\mathbf{X} = (X, w)$ as above, is nothing else but a surface isomorphism $f : X \xrightarrow{\sim} Y$, which is weight compatible, that means $v \circ f = w$. If the isomorphism sends the irreducible curve C to D and the point P to $Q \in Y$, then it induces weighted surface germ ***isomorphisms*** $\mathbf{C} \xrightarrow{\sim} \mathbf{D}$ and $\mathbf{P} \xrightarrow{\sim} \mathbf{Q}$. They are not globally depending on $\mathbf{X}$ or $\mathbf{Y}$ but only on small open neighbourhoods U respectively V of the curves or points and on the isomorphic compatible weights around. For $P \in C$ we have $Q \in D$ and ***isomorphisms*** of embedded objects $(\mathbf{C}, \mathbf{P}) \xrightarrow{\sim} (\mathbf{D}, \mathbf{Q})$. As in scheme theory we visualize the situation by a commutative diagram:

$$\begin{array}{ccccccc}
\mathbf{P} & \varepsilon & \mathbf{C} & \leftharpoonup & \mathbf{U} & \hookrightarrow & \mathbf{X} \\
\downarrow \wr & & \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\
\mathbf{Q} & \varepsilon & \mathbf{D} & \leftharpoonup & \mathbf{V} & \hookrightarrow & \mathbf{Y}
\end{array}$$

Trivially weighted surfaces $(Y, \mathbf{1})$, $\mathbf{1}$ the constant weight map (with weight 1 for each point), are identified with the surfaces Y themselves. So we will write Y again instead of $(Y, \mathbf{1})$. A smooth surface Y together with a finite covering $p : Y \to X$ is called a ***finite uniformization*** of $\mathbf{X} = (X, w)$, if and only if $p = p_G$ is the Galois covering with Galois group $G \subseteq Aut\, Y$, $X \cong Y/G$, $w(P) = \#G_Q$ (number of elements of the stabilizer subgroup of G of all elements fixing Q) for any p-preimage point $Q \in Y$ of P. We write also $\mathbf{p} : Y \to \mathbf{Y/G}$ in this situation and consider $\mathbf{p}$ as a ***uniformization*** morphism in the category of weighted surfaces. The weighted surface $\mathbf{X}$ is called (finitely) ***uniformizable*** if and only if a (finite) uniformization $Y \to \mathbf{X}$ exists. The weights $w(P)$ are called ***Galois weights***. The weight of an irreducible curve on X coincide with the corresponding rami-fication index. It is also called ***branch weight***. The branch weight of such a curve is non-trivial if and only if the curve belongs to the branch locus of p_G.

Point surface germs $\mathbf{P}$ on uniformizable weighted surfaces are called ***orbital quotient points***. This definition is extended to isomorphic objects. With the following local-global diagram we introduce further notations.

$$\begin{array}{cccccccc} & \mathbf{Q} & \varepsilon & V & \hookrightarrow & Y & \\ & \downarrow{\scriptstyle G_Q} & & \downarrow{\scriptstyle G_Q} & & \downarrow{\scriptstyle G} & \\ \mathbf{Q/G_Q} = & \mathbf{P} & \varepsilon & \mathbf{U} & \hookrightarrow & \mathbf{X} & = \mathbf{Y/G} \end{array} \tag{9}$$

using small open neighbourhoods and quotient maps by G_Q or G in the columns (local and global uniformizations). The left column (together with the middle) is called a ***uniformization of*** $\mathbf{P}$. The orbital point $\mathbf{P}$ is called ***smooth*** if and only if the supporting point P is a smooth surface point of X (or U). Orbital quotient points realized as above by abelian groups G_Q are called ***abelian orbital points***. The others are called ***non-abelian***.

A finitely weighted surface $\mathbf{X}$ is called ***orbiface*** (or finitely weighted orbiface), if each of its point surface germs $\mathbf{P}$ is an orbital quotient point. If thereby X is not compact, we call it an ***open orbiface***. Let $Y \xrightarrow{H} \mathbf{Z}$ be a uniformization of the orbiface $\mathbf{Z}$ with subgroup H of $G \subseteq Aut\, Y$. Then p_G factors through p_H defining a finite covering $f : Z \to X$ with $p_G = f \circ p_H$. We get a commutative orbital diagram

$$\begin{array}{ccc} Y & & \\ \downarrow{\scriptstyle H} & \searrow{\scriptstyle G} & \\ \mathbf{Z} & \xrightarrow{\mathbf{f}} & \mathbf{X} \end{array} \tag{10}$$

defining a ***uniformizable orbital covering*** or ***uniformizable finite orbital morphism*** $\mathbf{f} = \mathbf{f}_{G:H}$ on this way on the bottom. Working with Galois weight maps $w_H : Z \to \mathbb{N}_+$ and $w_G : X \to \mathbb{N}_+$ we define $f^*w_G := w_G \circ f$. This ***lifted weight map*** is obviously pointwise divisible by w_H, which means that $w_H(z)$ divides $f^*w_G(z) = w_G(f(z))$ for all $z \in Z$. We write $w_H \mid f^*w_G$ and define the ***quotient weight map*** $w_{G:H} := \frac{f^*w_G}{w_H} : Z \to \mathbb{N}_+$ pointwise. The curve weights on Z coincide with the corresponding ramification indices along f. If $w_{G:H}$ is constant on the fibres of f, then we can push it down to the weight map $w_f : X \to \mathbb{N}_+$. This happens, if H is a normal subgroup of G. Then we have $w_f = w_{G/H}$ on X and a new (reduced) kind of ***finite orbital covering*** $Z \to (X, w_f)$. We call it the ***reduction*** of $\mathbf{f} : (Z, w_H) \to (X, w_G)$ and w_f the ***reduction of*** w_G ***through*** Z. A geometric Galois problem for a given finite covering $f : Z \to X$ looks for a common uniformization Y as described in diagram (10). For this purpose it is worth to notice that

$$w_H = \frac{f^* w_G}{f^* w_f} \; (= \frac{f^* w_G}{w_{G:H}}).$$

So, if one knows w_G and f (hence w_f) one recognizes already the branch weights and curves of the possible uniformization p_H.

Weighted surface germs $\mathbf{C} \leftharpoonup \mathbf{X}$ along irreducible curves C on X are called ***(open) orbital curves*** if $\mathbf{X}$ is open orbital. The definition does not depend on X but, more precisely, on open neighbourhoods U of C on X. It may happen that $\mathbf{X}$ is not orbital but $\mathbf{U} \hookrightarrow \mathbf{X}$ is. Then $\mathbf{C}$ is orbital.

The notions of uniformization and of finite morphisms restrict to orbital curves and points. From scheme theory it is well-known that finite morphisms are surjective and open. So for orbital points, curves and orbifaces (open) we get restriction diagrams along finite coverings $\mathbf{f} : \mathbf{Y} \to \mathbf{X}$ via restrictions on germs

$$\begin{array}{ccccccc} \mathbf{Q} & \varepsilon & \mathbf{D} & \leftharpoonup & \mathbf{V} & \hookrightarrow & \mathbf{Y} \\ {\scriptstyle \mathbf{f}_{D,Q}}\big\downarrow & & \big\downarrow{\scriptstyle \mathbf{f}_D} & & \big\downarrow{\scriptstyle \mathbf{f}|_V} & & \big\downarrow{\scriptstyle \mathbf{f}} \\ \mathbf{P} & \varepsilon & \mathbf{C} & \leftharpoonup & \mathbf{U} & \hookrightarrow & \mathbf{X} \end{array} \tag{11}$$

with vertical finite orbital coverings. These are orbitalizations of well-known diagrams in scheme theory or complex algebraic geometry. Here we used pairs (V_Q, V) of small open neighbourhoods of Q or D, respectively, to define ***finite coverings*** $\mathbf{f}_{D,Q}$ ***of orbital points on orbital curves***. Working only with small open neighbourhoods V of Q we define ***finite orbital point coverings*** $\mathbf{f}_Q$ algebraically visualized in the diagram

$$\begin{array}{ccccc} \mathbf{Q} & \varepsilon & \mathbf{V} & \hookrightarrow & \mathbf{Y} \\ {\scriptstyle \mathbf{f}_Q}\big\downarrow & & \big\downarrow{\scriptstyle \mathbf{f}|_V} & & \big\downarrow{\scriptstyle \mathbf{f}} \\ \mathbf{P} & \varepsilon & \mathbf{U} & \hookrightarrow & \mathbf{X} \end{array} \tag{12}$$

In the special case of uniformizations we dispose on Diagram (9) for orbital points. Working with Galois group G again, the ***normalizer group*** (decomposition group)

$$N_G(D) \; := \{g \in G;\; g(D) = D\}$$

of D and $N_G(D)$-invariant small open neighbourhoods V of D. Orbital curves are said to be ***smooth*** if and only if the supporting curve is smooth and the supporting surface is smooth around the curve. Let D be a smooth curve on a trivially weighted surface Y with G-action such that the orbital curve $\mathbf{D} \leftharpoonup Y$ is smooth. Then we define an ***orbital curve uniformization*** $\mathbf{f}_D$ of $\mathbf{C}$ by the vertical arrow on the left-hand side of the diagram

$$\begin{array}{ccccc} \mathbf{D} & \leftharpoonup & V & \hookrightarrow & Y \\ \Big\downarrow{\scriptstyle N_G(D)} & & \Big\downarrow{\scriptstyle N_G(D)} & & \Big\downarrow{\scriptstyle G} \\ \mathbf{C} & \leftharpoonup & \mathbf{U} & \hookrightarrow & \mathbf{X} \end{array} \tag{13}$$

The weight of C is equal to the order of the cyclic ***centralizer group*** (inertia group)

$$Z_G(D) := \{g \in G;\ g|_D = id|_D\}.$$

The curve C is isomorphic to the quotient curve D/G_D, where G_D is the effectively on D acting group defined by the exact group sequence

$$1 \longrightarrow Z_G(D) \longrightarrow N_G(D) \longrightarrow G_D = N_G(D)/Z_G(D) \longrightarrow 1. \tag{14}$$

3.2 Orbital releases

We want to introduce special birational morphisms for orbital points, curves and orbifaces. Changing special curve singularities by numerically manageable surface singularities. These will be ***abelian singularities***, which are defined as supporting surface singularities of orbital abelian points $\mathbf{P}$. The latter are well-understood by linear algebra. Let G be a finite abelian subgroup of $\mathbb{G}l_2(\mathbb{C})$. It acts effectively on the complex affine plane $\mathbb{C}^2$ and around the origin $O = (0, 0)$ of $\mathbb{C}^2$, hence on the trivially weighted smooth orbital point $\mathbf{O} \,\varepsilon\, \mathbb{C}^2$. Working in the analytic category, that means with small open analytic neighbourhoods of points, it is true that for each orbital quotient point $\mathbf{P}$ there is uniformization $\mathbf{O} \to \mathbf{P} \cong \mathbf{O}/\mathbf{G}$ for a suitable finite subgroup G of $\mathbb{G}l_2(\mathbb{C})$ (H. Cartan). Let us call it a ***linear uniformization*** of $\mathbf{P}$. If G is not abelian, then there are precisely three G-orbits of eigenlines in $\mathbb{C}^2$ of non-trivial elements of G. Going down to $\mathbf{P} = \mathbf{O}/\mathbf{G}$ they define precisely three orbital curve germs through $\mathbf{P}$ called ***eigen germ triple*** at $\mathbf{P}$. There are precisely two of them, called ***eigen germ pair*** at $\mathbf{P}$ if and only if G is an abelian group not belonging to the center of $\mathbb{G}l_2(\mathbb{C})$. If G is central then we declare the germs of the images of any two different lines through O as eigen germ pair at $\mathbf{P}$. Different choices are isomorphic.

By the way, ***orbital curve germs*** on orbifaces $\mathbf{X}$ at a point P are defined in the same manner as orbital curves but working with orbital curve germs at P and small analytic open neighbourhoods of them instead of global curves and their open neighbourhoods. More precisely, it is the weighted analytic surface germ around a curve germ on X through P. Now let $\mathbf{P}$ be an orbital point on the orbital curve $\mathbf{C}$ on $\mathbf{X}$. If C is smooth at P, then $\mathbf{C}$ defines a unique orbital curve germ $\mathbf{C}_P$ at $\mathbf{P}$. Now let $\mathbf{P}$ be an abelian orbital point. We say that two orbital curve(germ)s $\mathbf{C}_1$ and $\mathbf{C}_2$ on $\mathbf{X}$ ***cross (each other)*** at $\mathbf{P} \,\varepsilon\, \mathbf{X}$ if and only if they form an eigen germ pair there. Necessarily C_1 and C_2 have to be smooth at P.

A curve C on a surface X is called ***releasable*** at $P \in C$ if and only if there is a birational morphism $\varphi_P :\ X' \to X$ such that the exceptional curve E_P

of φ_P is smooth, irreducible, $\varphi_P(E_P) = P$ and the proper transform C' of C on X' crosses E_P at any common point. Observe that C' must be smooth at these intersection points. So $X' \to X$ resolves the (releasable) curve singularity P. If P is thereby a curve singularity, then we call φ_P the (honest) ***release*** of C at P. If P is a smooth curve and surface point then one could take the σ-process at P, but this is not a honest release. Honest releases are only applied to curve singularities. Using the uniqueness of minimal singularity resolution for surfaces (here applied to the abelian surface singularities on $E_P \subset X'$) it is easy to see that this local release φ_P is uniquely determined by $P \in C \subset X$. The curve $C \subset X$ is called ***releasable*** if and only if it is releasable at each of its points. There are only finitely many honestly releasable points on each fixed curve. Therefore, if C is releasable, there exists a unique birational morphim $\varphi = \varphi_C : X' \to X$ releasing all singular points of C. This morphism is called the ***release*** of X ***along*** C.

Remark 3.2 *The surface singularities on X of released curve points P are of special type. They are contractions of one curve E_P supporting finitely many abelian singularities (of X'). So P has a surface singularity resolution consisting of a (central) irreducible curve (the proper transform of E_P) crossed by some disjoint linear trees of lines (that means isomorphic to $\mathbb{P}^1$) with negative self-intersections smaller than -1. The linear trees are minimal resolutions of abelian surface singularities. Such a singularity resolutions of P is called* ***released****. It can happen that it is bigger than the minimal singularity resolution of P; for instance, if we are forced to release a smooth surface point, an abelian singularity or, more generally, a quotient singularity P.*

Example 3.3 *Let P be an* ***ordinary singularity*** *of a curve C on a surface X smooth at P. By definition, the branches of C at P cross each other there. Then the curve singularity P is released by the σ-process at P. The curve branches appear as (transversal) intersection points of the proper transform of C with the exceptional line over P.*

Example 3.4 ***Hypercusp singularities*** *of curves at smooth surface points are defined by local equations $y^n = x^m$, $m, n > 0$. They are releasable by a line (smooth rational curve) supporting at most two abelian surface points. This releasing line cuts the proper transform of the curve in precisely $gcd(m, n)$ smooth points. These intersections are transversal.*

Proof idea: Stepwise resolution of the curve singularity by σ-processes. At the end one gets a tree of lines with negative self-intersections. One discovers that the proper transform of the curve crosses only one component of the tree. The two (or less) partial trees meeting this component contract to an abelian point. The resolution steps reduce the exponent pairs (m, n) following the euclidean algorithm. It stops by arriving equal exponents in the singularity equation, $y^d = x^d$, $d = gcd(m, n)$. This is the local equation of an ordinary curve singularity with d branches.

$\square$

Definition 3.5 *An orbital curve* $\mathbf{C}$ *is* ***releasable*** *if and only if the supporting surface embedded curve* C *is.*

The definition does not depend on the choice of orbiface $\mathbf{X}$ defining $\mathbf{C}$ by restriction.

3.6 Examples

- Release of a curve cusp at smooth surface point.
  Let $[-3,-1,-2]$ represent a linear tree of three smooth rational curves on a smooth surface with the indicated self-intersections. It is contractable (stepwise) to the regular surface point P. But let us contract the first and last line to cyclic singularities P_1, P_2 of types $< 3,1 >$ respectively $< 2,1 >$ on the middle line L. Now consider a curve C' intersecting L transversally at one point $P' \neq P_1, P_2$. Contracting L, the image point P is a curve cusp on the image curve C of C'. Altogether $L \to P$ is a release of (C, P) with exactly one branch point (C', P'), and $P' \in C'$ is totally smooth.

- A more complicated release of smooth point.
  Let P be a smooth surface point again. There is a release $L \to P$ with two honest cyclic singularities

$$\begin{array}{rcl} P' : < 93, 76 > & \leftarrow & [-2,-2,-2,-2,-4,-2,-2,-2,-2,-2,-2,-2], \\ P'' : < 106, 17 > & \leftarrow & [-7,-2,-2,-2,-5], \end{array}$$

  on L numerically resolved by continued fractions (Hirzebruch-Jung singularities). As in the previous example one has only to consider the composed linear resolution tree connected by a (-1)-line and its stepwise blowing down to a smooth point:

$$\begin{aligned} \tilde{E}' : & [-2,-2,-2,-2,-4,-2,-2,-2,-2,-2,-2,-2,-1,-7,-2,-2,-2,-5] \\ & \to \quad [-2,-2,-2,-2,-4,-1,-2,-2,-2,-5] \\ & \to \quad [-2,-2,-2,-2,-1,-5] \quad \to \quad [-1] \quad \to \quad P, \end{aligned} \tag{15}$$

Remark 3.7 *It is easy to see now that each abelian point has infinitely many different releases.*

- Hilbert cusps.
  An **irreducible neat Hilbert cusp curve** is a contractible rational curve H on a surface with a double point P as one and only curve singularity. Moreover, P has to be a cyclic surface point (including smooth ones), and the two branches of H cross each other at P. By the above remark each

irreducible Hilbert cusp curve has infinitely many releases which - by abuse of language - are also called called **releases** of neat Hilbert cusp points. The irreducible Hilbert cusp curves are also called **simple releases** of neat Hilbert cusp points. There are also infinitely many simple releases of one and the same cusp point. One has only to consider the minimal resolution of such cusp point with smooth transversally intersecting components. It consists of a cycle of smooth rational curves. **Orbital Hilbert cusp points** in general are finite quotients of neat Hilbert cusp points. A release of one of them is nothing else but the quotient of a neat Hilbert cusp point release.

We say that the birational morphism $Y' \to Y$ is a ***smooth release*** of the curve $D \subset Y$, if it is a release of D and Y' is a smooth surface. Thereby we allow also non-honest releases at some points. Let G be a finite group acting effectively on Y and assume that the action lifts to Y' permuting the released points. The (smooth) proper transform of D on Y' is denoted by D'. Let $C =: D/G$ and $C' =: D'/G$ be the image curve of D on $X := Y/G$ or of D' on $X' := Y'/G$, respectively. We say that the release $Y' \to Y$ is ***G-stable*** if and only if additionally the induced morphism $X' \to X$ is a release of C (with proper transform C'). We endow X' and X with Galois weights by means of orders of stabilizer groups at points. Then we get an orbiface $\mathbf{X}'$ and its contraction $\mathbf{X}$. Such contractions will shortly also be called orbifaces. The induced orbital curve $\mathbf{C}'$ contracts to the weighted weighted surface germ $\mathbf{C}$, which we will also call orbital curve. Altogether we get a commutative diagram of orbital curves

$$\begin{array}{ccc} \mathbf{D}' & \longrightarrow & \mathbf{D} \\ \big\downarrow{\scriptstyle N_G(D)} & & \big\downarrow{\scriptstyle N_G(D)} \\ \mathbf{C}' & \longrightarrow & \mathbf{C} \end{array} \qquad (16)$$

with trivially weighted release on the top, an orbital curve release on the bottom and an orbital curve uniformization on the left-hand side.

Definition 3.8 *The orbital curve* $\mathbf{C}$ *is called* ***uniform releasable*** *if and only if there exist a commutative diagram (16). The corresponding quotient release* $\mathbf{C}' \to \mathbf{C}$ *is called an* ***orbital release*** *of* $\mathbf{C}$*. The ambient map* $\mathbf{X}' \to \mathbf{X}$ *is called the* ***orbital release*** *of* $\mathbf{X}$ *along* C*. The morphisms* $\mathbf{D}' \to \mathbf{D}$, $Y' \to Y$ *are called* ***release uniformizations*** *of* $\mathbf{C}' \to \mathbf{C}$ *or of* $\mathbf{X}' \to \mathbf{X}$*, respectively.*

Notice that a release uniformization of $\mathbf{C}$ endows automatically the surface around C' with (Galois) weights. Therefore we get an orbital curve in this case. Starting from a smooth surface Y with G-action it is interesting to ask, which curves $D \subset Y$ have a releasable quotient curve $C = D/G \subset X = Y/G$? Keep in mind the example 3.10 below, because it will play a central role.

Definition 3.9 *The action of G on Y as above is* ***ordinary at*** *D if and only if the curve*

$$GD := \bigcup_{g \in G} g(D)$$

has at most ordinary singularities. The action is ***smooth at*** *D if and only if GD is smooth. The action is* ***separating at*** *D if and only if for all $g \in G$ the curve $g(D)$ is either equal to D or has no common point with D.*

Obviously, a smooth action at D must be separating at D. For smooth curves D both notions coincide.

Example 3.10 *Let Y be a smooth surface with G-action and D a smooth curve on X. If G acts ordinarily at D, then the orbital curve $\mathbf{C} = \mathbf{D}/\mathbf{G}$ is uniform releasable.*

Proof . We release simultaneously the ordinary singularities of GD by σ-processes. Let $Y' \to Y$ be this simultaneous releasing morphism, and denote the proper transform of D on Y' by D'. Then G acts on Y' and thereby smoothly at D'. With $C' = D'/G = D'/N_G(D) = D'/G_{D'}$ we get a uniform release diagram (16).

□

Example 3.11 *Let Y be a smooth surface with G-action separating at D, D a curve on X with at most hypercusp singularities. Then the orbital curve $\mathbf{C} = \mathbf{D}/\mathbf{G}$ is uniform releasable.*

Proof . Because of the separating property we can assume that $G = N_G(D)$. Then G acts on the set of singularities of D. In a G-equivariant manner we resolve stepwise the curve singularities as described in Example 3.4, each by a linear tree of lines such that the smooth proper transform D' of D on the arising surface Y' intersects precisely one (central) line of the tree. All intersections of tree lines and such lines with D' are empty or transversal. The G-action on Y transfers to a G-action on Y'. Since $N_G(D') = G$ the quotient curve $C' = D'/G$ is smooth, or in other words, G acts smoothly at D'. Let $Q \in D$ be a curve singularity and $E_Q \subset Y'$ the resolving linear tree over Q. Then the Q-stabilizing group G_Q acts on E_Q and especially on the D'-crossing central line L_Q of E_Q. Now it is clear that G acts (via G_Q) separately, hence smooth at L_Q. This property refers also to the other line components of E_Q. The isotropy groups $G_{Q'}$ at points $Q' \in E_Q$ must be abelian because of transversal intersections of the components of $E_Q \cup D'$. Therefore the image of E_Q on Y'/G is again a linear tree of lines intersecting at abelian quotient singularities. If we take the minimal singularity resolutions of them, then we get again a linear tree of lines crossing each other. Now blow down the two partial linear trees outside of the proper transform of the central line $L_Q/G = L_Q/G_Q$ crossing $C' = D'/G$ in at most abelian singularities. Altogether one gets a uniform release diagram (16) for the orbital curve $\mathbf{C} = \mathbf{D}/\mathbf{G}$. The upper release $\mathbf{D}' \to \mathbf{D}$ is that of hypercusps of curves locally described in Example 3.4.

□

3.3 Homogeneous points

Definition 3.12 *A **simple surface singularity** is a singularity, whose minimal resolution curve consists of one (smooth) irreducible curve only. A **simple surface point** is a simple singularity or a regular surface point.*

In the latter case we consider the exceptional line of the σ-process as resolution curve.

Now let G be a finite group acting on a surface Y with only simple points, and let $Q \in Y$ be one of them. The group action extends to the simultaneous minimal resolution Y' of all points of the orbit GQ, and the stabilizer group G_Q acts on the resolving curve $E_Q \subset Y'$. Moreover, G_Q acts on the normal bundle over E_Q respecting fibres. Therefore the stationary subgroups of G_Q at points on E_Q must be abelian (fibres and E_Q are diagonalizing). Take a G_Q-invariant open neighbourhood $V \subset Y$ of Q, smooth outside of Q. Locally the situation is described by the following commutative ***local coniform release diagram***:

$$\begin{array}{ccc} V' & \longrightarrow & V \\ \downarrow & & \downarrow \\ U' = V'/G_Q & \longrightarrow & U = V/G_Q \end{array} \qquad (17)$$

with vertical quotient morphisms and upper horizontal resolution. Let $P \in U$ be the image point of Q. We endow U', $U \setminus \{P\}$ with Galois weights coming from the finite G_Q-uniformizations $V' \to U'$, $V \setminus \{Q\} \to U \setminus \{P\}$. Finally, we set $w(P) := w(E_Q)$. The corresponding orbifaces are denoted by $\mathbf{U'}$ or $\mathbf{U}$, respectively. Our diagram can be written as

$$\begin{array}{ccc} V' & \longrightarrow & V \\ \downarrow & & \downarrow \\ \mathbf{U'} = \mathbf{V'}/\mathbf{G_Q} & \longrightarrow & \mathbf{U} = \mathbf{V}/\mathbf{G_Q} \end{array} \qquad (18)$$

Again, we have a refinement equivalence class of weighted open neighbourhoods of P denoted by $\mathbf{P} = \mathbf{Q}/\mathbf{G_Q} \in \mathbf{U}$.

Definition 3.13 *Weighted surface points* $\mathbf{P}$ *constructed on this way are called **(orbital) homogeneous points**.*

It is clear that orbital quotient points are homogeneous. The morphism $Q \to \mathbf{P}$ of orbital points, defined as refinement class of $V \to \mathbf{U}$, is called a ***coniformization*** of $\mathbf{P}$. It is a ***uniformization***, if the preimage point Q is regular on V. Homogeneous points which are coniformable but not uniformizable are called ***honest homogeneous*** points.

Remark 3.14 *The supporting surface point P of homogeneous point $\mathbf{P}$ is in any case a so-called* ***quasihomogeneous singularity****. The resolutions of these quasihomogeneous singularities are precisely known. We refer to [P], [D]. For graphical descriptions with weights see [H98].*

Remark 3.15 *Simple singularities are "cone like". Namely, up to isomorphy (look at normal bundle of E_Q), they are contractions of a section C of a line bundle over C with negative self-intersection contractible to a singularity Q. In our imagination the contracting surface looks like a cone around the singularity Q. Therefore we introduced the notion "coniformization".*

Remark 3.16 *Observe that the $\mathbf{U}'$ supports finitely many abelian points sitting on the smooth quotient curve E_Q/G. These simpler orbital points "release" the homogeneous point $\mathbf{P}$, which explains our calling. Notice also that an abelian point $\mathbf{P}$ can loose its original weight after a coniformization. The old one is "released" by the new "coniform weight".*

A global ***coniform release diagram*** looks like

$$\begin{array}{ccc} Y' & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \mathbf{X}' = \mathbf{Y}'/\mathbf{G} & \longrightarrow & \mathbf{X} = \mathbf{Y}/\mathbf{G} \end{array} \tag{19}$$

Thereby Y is a surface with only simple singularities, G a finite group acting effectively on Y, $Y' \to Y$ resolves minimally all simple singularities of Y and, possibly, G-orbits of finitely many regular points by σ-processes. The weights of $\mathbf{X}'$ are Galois weights. We push forward the weights of the quotients of the exceptional curves of $Y' \to Y$ to get a birational morphism of weighted surfaces in the bottom of the diagram.

Definitions 3.17 $Y \longrightarrow \mathbf{X}$ *supported by a a quotient morphism* $Y \longrightarrow X = Y/G$ *is called a (global)* ***coniformization****, if and only if one can extend it to a coniform release diagram (19). Morphisms in the bottom of (19) are called* ***coniform releases****, and* $Y' \longrightarrow Y$ *is a* ***coniformization*** *of* $\mathbf{X}' \longrightarrow \mathbf{X}$.

An ***(finitely weighted) orbiface*** is a weighted surface supporting (finitely weighted) homogeneous points only.

At the end of this section we want to describe orbital cusp points. A ***neat hyperbolic cusp point*** is a simple elliptic surface point $Q \in V$ endowed with weight ∞. Its resolution curve $C \subset V$ is, by definition, elliptic. We use the notations of the diagrams (17) and (18) to get the homogeneous quotient point $\mathbf{P}$ of the elliptic point Q. We change the finite Galois weight of P by ∞ to get a ***hyperbolic orbital cusp point*** $\mathbf{P}_\infty$. Notice that the weights of $\mathbf{U}$ outside of P will not be changed. Also the weights of all points of the preimage curves of Q and P in diagram (17) will be changed to ∞. Then we get local coniform release diagrams (18) for hyperbolic orbital cusp points $\mathbf{P}_\infty$.

A ***neat Hilbert cusp singularity*** Q is a normal surface singularity which has a cycle of transversally intersecting smooth rational curves as resolution curve. The minimal resolution curve E_Q is of the same type or a rational curve with only one curve singularity which is an ordinary self-intersection of this curve. Locally, we have diagrams of type (17) again with a finite group action around Q. Endowing Q with weight ∞ and all other points on V outside Q with trivial weight 1, we get a ***neat Hilbert cusp point*** Q_∞. As in the hyperbolic cusp case we endow the quotient point P, the points of E_Q and of its quotient curve also with weight ∞ and all other points with the usual finite Galois weights to get a diagram (18) in the category of weighted surfaces called ***local Hilbert cusp release diagrams***. The corresponding orbital point $\mathbf{P}_\infty$ itself is called a ***Hilbert orbital cusp point***. After pulling back the weight ∞ to the points on the preimage curve of P we get, with obvious notations, representatives $\mathbf{U}'_\infty \to \mathbf{U}_\infty$ called ***orbital cusp releases*** of $\mathbf{P}_\infty$.

Altogether, in the ***category of orbifaces*** $\mathbf{Orb}^2$ we dispose on ***orbital releases***, ***orbital release diagrams***, ***coniform releases***, ***finite orbital morphisms***, ***orbital open embeddings***, ***birational orbital morphisms*** by composition of orbital releases, ***orbital morphisms*** by composition of birational and finite orbital morphisms, and on ***orbital correspondence classes***, which consist of orbital objects connected by finite orbital coverings.

The notions restrict in obvious manner to orbital curves on orbifaces via representative neighbourhoods. So we dispose on ***category of orbital curves*** $\mathbf{Orb}^{2,1}$ (on orbifaces) with all the types of orbital morphisms above.

Remark 3.18 *Restricting to algebraic objects and morphisms one can work with Zariski-open sets only to define refinement classes and corresponding weighted (orbital) points. There will be no difference for the later definitions of orbital invariants.*

3.4 Picard and Hilbert orbifaces

Let $\mathbb{B} \subset \mathbb{C}^2$ be a bounded domain and Γ a group of analytic automorphisms of $\mathbb{B}$ acting properly discontinuously. Then the quotient $\Gamma\backslash\mathbb{B}$ together with Galois weights is a finitely weighted orbiface, which we denote by $\mathbf{\Gamma}\backslash\mathbb{B}$.

There are two symmetric subdomains of $\mathbb{C}^2$: The irreducible ***complex unit ball***

$$\mathbb{B}\ |z_1|^2 + |z_2|^2 < 1$$

and the product $\mathbb{D}^2$ of two unit discs. The latter is biholomorphic equivalent to the product

$$\mathbb{H}^2 = \mathbb{H} \times \mathbb{H}, \quad \mathbb{H} :\ Im\, z > 0,$$

of two upper half planes $\mathbb{H}$ of $\mathbb{C}$.

The automorphism groups are the projectivizations of the ***unitary group*** $\mathbb{U}((2,1),\mathbb{C}) \subset \mathbb{G}l_3(\mathbb{C})$ or of the symmetric extension (by transposition of coordinates) $\mathbb{G}S_2^+(\mathbb{R})$ of $\mathbb{G}l_2^+(\mathbb{R}) \times \mathbb{G}l_2^+(\mathbb{R})$, respectively. Both groups act transitively on the corresponding domain.

Now let $K = \mathbb{Q}(\sqrt{D})$ be a quadratic number field with discriminant $D = D_{K/\mathbb{Q}} \in \mathbb{Z}$ and ring of integers $\mathcal{O}_K$. In the ball case we let K be an imaginary quadratic field and in the splitting case a real quadratic field. The arithmetic groups acting (non-effectively) on $\mathbb{B}$ or $\mathbb{H}^2$,

$$\Gamma_K = \begin{cases} \mathbb{SU}((2,1),\mathcal{O}_K), & \text{if } K \text{ imaginary quadratic,} \\ \mathbb{S}l_2(\mathcal{O}_K), & \text{if } K \text{ real quadratic,} \end{cases}$$

are called ***full Picard modular*** or ***full Hilbert modular group*** of the field K, respectively. To be precise, we use, if nothing else is said, in the Picard case the hermitian metric on $\mathbb{C}^3$ of signature $(2,1)$ represented by the diagonal matrix $\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right)$. The action on $\mathbb{B}$ restricts the $\mathbb{G}l_3$-action on $\mathbb{P}^2$ in obvious manner. In the Hilbert case we have to restrict the action on $\mathbb{P}^1 \times \mathbb{P}^1$ of

$$\mathbb{G}l_2^+(K) \ni g: \ (z,w) \mapsto (g(z), g'(w)),$$

where $'$ denotes the non-trivial field automorphism of K applied to each coefficient of g.

Definitions 3.19 *A* ***Picard modular group*** *(of the imaginary quadratic field K) is a subgroup of $\mathbb{G}l_3(\mathbb{C})$ commensurable with Γ_K.*
A ***Hilbert modular group*** *(of the real quadratic field K) is a subgroup of $\mathbb{G}S_2^+(\mathbb{R})$ commensurable with Γ_K.*

Definitions 3.20 *The finitely weighted orbital quotient surfaces*

$$\overset{o}{\mathbf{X}}_\Gamma = \begin{cases} \mathbf{\Gamma}\backslash\mathbb{B} = \mathbb{P}\mathbf{\Gamma}\backslash\mathbb{B} \\ \mathbf{\Gamma}\backslash\mathbb{H}^2 = \mathbb{P}\mathbf{\Gamma}\backslash\mathbb{H}^2 \end{cases}$$

are called the ***open Picard orbiface*** *of Γ, if Γ is a Picard modular group, respectively the* ***open Hilbert orbiface*** *of Γ, if Γ is a Hilbert modular group.*

Forgetting Galois weights, the surfaces $\overset{o}{X}_\Gamma = \Gamma\backslash\mathbb{B}$ or $\Gamma\backslash\mathbb{H}^2$ are called ***open Picard modular*** or ***open Hilbert modular surfaces***, respectively. Each of them has a unique analytic ***Baily-Borel compactification*** $\hat{X}_\Gamma := \widehat{\Gamma\backslash\mathbb{B}}$ adding finitely many hyperbolic respectively Hilbert cusp "singularities" (which may be regular). These are projective normal surfaces. We extend the Galois weight map of $\mathbf{X}_\Gamma$ to $\hat{X}_\Gamma$ endowing the cusps with weight ∞ to get the ***orbital Baily-Borel model*** $\hat{\mathbf{X}}_\Gamma$ of $\overset{o}{X}_\Gamma$ or $\overset{\mathbf{o}}{\mathbf{X}}_\mathbf{\Gamma}$. Releasing all cusp points we get the ***cusp released models*** X_Γ with orbital versions $\mathbf{X}_\Gamma$.

Each arithmetic group Γ has a ***neat normal subgroup*** Δ of finite index. By definition, the eigenvalues of each element of a neat arithmetic linear group generate a free abelian subgroup of $\mathbb{C}^*$ (which may be trivial). Especially, because of absence of unit roots, a neat Picard and Hilbert modular group acts fixed point free on $\mathbb{B}$ or $\mathbb{H}^2$, respectively. Moreover, the cusp points of the corresponding modular surfaces are neat. With the groups Δ and $G := \Gamma/\Delta$ we get the global cusp release diagrams

$$\begin{CD} X_\Delta @>>> \hat{X}_\Delta \\ @V{/G}VV @VV{/G}V \\ \mathbf{X}_\Gamma @>>> \hat{\mathbf{X}}_\Gamma \end{CD} \tag{20}$$

Notice that X_Δ is a smooth projective surface. Moreover, the objects and the orbital release morphism on the bottom of the diagram do not depend on the choice of Δ. In the Picard case we refer to [H98] for the complete classification of orbital hyperbolic cusps and their local releases working on $\hat{\mathbf{X}}_\Gamma$ only. The analogous work for the Hilbert case has not been done until now, but seems to be not difficult. Since X_Δ is smooth, all finitely weighted orbital points on $\hat{\mathbf{X}}_\Gamma$ are quotient points, also well-classified in [H98]. The non-abelian ones have also unique releases. Locally, they come from σ-processes at their preimage points on X_Δ, not depending on the choice of Δ again. The complete release of non-abelian orbital quotient points of $\mathbf{X}_\Gamma$ is denoted by $\mathbf{X}'_\Gamma$. We get global commutative orbital release diagrams

$$\begin{CD} X'_\Delta @>>> X_\Delta \\ @V{/G}VV @VV{/G}V \\ \mathbf{X}'_\Gamma @>>> \mathbf{X}_\Gamma \end{CD} \tag{21}$$

Altogether we get orbital release diagrams

$$\begin{CD} X'_\Delta @>>> X_\Delta @>>> \hat{X}_\Delta \\ @VVV @VV{/G}V @VVV \\ \mathbf{X}'_\Gamma @>>> \mathbf{X}_\Gamma @>>> \hat{\mathbf{X}}_\Gamma \end{CD} \tag{22}$$

We can shorten them to one diagram

$$\begin{CD} X'_\Delta @>>> \hat{X}_\Delta \\ @VVV @VVV \\ \mathbf{X}'_\Gamma @>>> \hat{\mathbf{X}}_\Gamma \end{CD} \tag{23}$$

with the releases of precisely all non-abelian orbital points, because all hyperbolic orbital cusp points are homogeneous, hence releasable.

Visualization:

- 1) Released Picard modular Apollonius plane (4);
- 2) Released Hilbert modular Cartesius plane (**??**).

From $\mathbf{Orb}^2$ we single out the correspondence classes

- $\overset{o}{\mathbf{Pic}}{}^2_K$ of ***open Picard orbifaces*** of the field K,
  with objects $\overset{\mathbf{o}}{\mathbf{X}}_{\mathbf{\Gamma}}$, Γ a Picard modular group of the field K;
- $\widehat{\mathbf{Pic}}{}^2_K$ of ***Picard orbifaces*** of the field K,
  with objects $\hat{\mathbf{X}}_\Gamma$;
- $\mathbf{Pic}^{2'}_K$ of ***released Picard orbifaces*** of the field K,
  with objects $\mathbf{X}'_\Gamma$.

As surviving orbital morphisms in each of these correspondence classes we take only the finite ones coming from pairs $\Delta \subset \Gamma$ of Picard modular groups of the same field K.

In the same manner we dispose in $\mathbf{Orb}^2$ on correspondence classes

- $\overset{\mathbf{o}}{\mathbf{Hilb}}{}^2_K$ of ***open Hilbert orbifaces*** of the field K,
  with objects $\overset{\mathbf{o}}{\mathbf{X}}_{\mathbf{\Gamma}}$, Γ a Hilbert modular group of the field K;
- $\widehat{\mathbf{Hilb}}{}^2_K$ of ***Hilbert orbifaces*** of the field K,
  with objects $\hat{\mathbf{X}}_\Gamma$;
- $\mathbf{Hilb}^{2'}_K$ of ***released Hilbert orbifaces*** of the field K,
  with objects $\mathbf{X}'_\Gamma$.

In difference to the Picard case, the neat objects of $\mathbf{Hilb}_K$ come from minimal singularity resolution of the corresponding neat objects of $\widehat{\mathbf{Hilb}}_K$. One has to plug in at the Baily-Borel cusps a cycle of rational curves with negative self-intersection. In general we must plug in finite quotients of such cycles. For Picard and Hilbert cases we call it cusp released in common.

We denote by $\mathbf{Pic}^2$ or $\mathbf{Hilb}^2$ the complete subcategories of $\mathbf{Orb}^2$ with the above three types of Picard- or Hilbert objects, respectively. Both kinds of objects together form the complete subcategory $\mathbf{Shim}^2$ of irreducible Shimura orbifaces.

Aim:
Volumes of fundamental domains of Picard or Hilbert modular groups with respect to fixed volume forms on the uniformizing domains are obviously orbital invariants. If we take the Euler volume form, then these volumes coincide with the orbital Euler invariants of the corresponding orbifaces. This is a theorem (Holzapfel, [H98], in the ball case; to be written down in the Hilbert case). The same is true for the signature volume form, which leads to the corresponding orbital signature invariants. The Euler and the signature forms are distinguished by a factor. This leads to a proportionality relation between the orbital invariants with different factors in the Picard and Hilbert cases.

For explicit calculations it is important to know that the orbital invariants of the modular surfaces of the full lattices can be expressed by special values of Zeta-functions (or L-series) of corresponding number fields. (Maass in the neat Hilbert case; Holzapfel [H98] in the general Picard case.)

3.5 Orbital arithmetic curves

Let Γ be a Picard or Hilbert modular group of a quadratic number field K. We say that $\mathbb{D} \subset \mathbb{B}$ or $\mathbb{H}^2$ is a K***-arithmetic disc***, if and only if there is a holomorphic embedding of of the unit disc $\mathbb{D}^1 \hookrightarrow \mathbb{B}$ with image $\mathbb{D}$ such that $\mathbb{D}$ is closed in $\mathbb{B}$ and the $\mathbb{D}$***-normalizing subgroup*** (decomposition group) of Γ

$$N_\Gamma(\mathbb{D}) := \{\gamma \in \Gamma;\ \gamma(\mathbb{D}) = \mathbb{D}\}$$

is a $\mathbb{D}$-lattice, that means $N_\Gamma(\mathbb{D})\backslash\mathbb{D} = \Gamma_\mathbb{D}\backslash\mathbb{D}$ is a quasiprojective algebraic curve, where

$$\Gamma_\mathbb{D} = N_\Gamma(\mathbb{D})/Z_\Gamma(\mathbb{D}),\ \text{the effective decomposition group of } \mathbb{D},$$
$$Z_\Gamma(\mathbb{D}) = \{\gamma \in \Gamma;\ \gamma|_\mathbb{D} = id_\mathbb{D}\},\ \mathbb{D} - \text{centralizing (or inertia) group.}$$

To be more precise, we have commutative diagrams with algebraic groups defined over $\mathbb{Q}$ in the upper two rows

$$\begin{array}{ccccccc}
N_\Gamma(\mathbb{D}) & \longrightarrow & \mathbf{N} & \longrightarrow & \mathbf{G} & \longleftarrow & \Gamma \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathbb{S}N_\Gamma(\mathbb{D}) & \longrightarrow & \mathbb{S}\mathbf{N} & \longrightarrow & \mathbb{S}\mathbf{G} & \longleftarrow & \mathbb{S}\Gamma \\
 & & \downarrow & & \downarrow & & \\
 & & \mathbb{D} & \longrightarrow & \mathbb{B} & &
\end{array} \tag{24}$$

with algebraic Lie groups $\mathbf{N}$, $\mathbf{G}$ acting on the symmetric domains below. The algebraic groups in the middle are special: $\mathbb{S}N(\mathbb{R})$ isomorphic to $\mathbb{S}l_2(\mathbb{R})$ or

$\mathbb{SU}((1,1),\mathbb{C})$. The $\mathbb{Q}$-algebraic group embeddings are lifted from the bottom row.

We denote the image curve of $\mathbb{D}$ on $\overset{o}{X}_\Gamma = \Gamma\backslash\mathbb{B}$ by $\Gamma\backslash\mathbb{D}$ or $\overset{o}{D}_\Gamma$. Its closure on X_Γ or $\hat{X}_\Gamma$ is denoted by D_Γ or $\hat{D}_\Gamma$, respectively, and their proper transform on X'_Γ by D'_Γ. We have birational curve morhisms

$$\widehat{\Gamma_{\mathbb{D}}\backslash\mathbb{D}} \longrightarrow D'_\Gamma \longrightarrow D_\Gamma \longrightarrow \hat{D}_\Gamma.$$

with (unique) smooth compact model on the left.

The corresponding weighted orbital curves on $\overset{\mathbf{o}}{\mathbf{X}}_{\mathbf{\Gamma}}, \mathbf{X}'_\Gamma, \mathbf{X}_\Gamma$ or $\hat{\mathbf{X}}_\Gamma$ are denoted by $\overset{\mathbf{o}}{\mathbf{D}}_{\mathbf{\Gamma}}, \mathbf{D}'_\Gamma, \mathbf{D}_\Gamma$ or $\hat{\mathbf{D}}_\Gamma$, respectively. We have birational orbital curve morphisms

$$\mathbf{D}'_\Gamma \longrightarrow \mathbf{D}_\Gamma \longrightarrow \hat{\mathbf{D}}_\Gamma, \quad \overset{\mathbf{o}}{\mathbf{D}}_{\mathbf{\Gamma}} \hookrightarrow \mathbf{D}'_\Gamma, \quad \overset{\mathbf{o}}{\mathbf{D}}_{\mathbf{\Gamma}} \hookrightarrow \mathbf{D}_\Gamma.$$

If $\overset{o}{D}_\Gamma$ is non-compact, then $\overset{\mathbf{o}}{\mathbf{D}}_{\mathbf{\Gamma}}, \mathbf{D}'_\Gamma, \mathbf{D}_\Gamma, \hat{\mathbf{D}}_\Gamma$, are (weighted) ***orbital modular curves***. If it is compact, then $\overset{\mathbf{o}}{\mathbf{D}}_{\mathbf{\Gamma}}, \mathbf{D}'_\Gamma, \mathbf{D}_\Gamma, \hat{\mathbf{D}}_\Gamma = \overset{\mathbf{o}}{\mathbf{D}}_{\mathbf{\Gamma}}$ are ***orbital Shimura curves***. Altogether they are called ***orbital arithmetic curves***. Their supporting curves are defined over an algebraic number field because these are one-dimensional Shimura varieties.

Theorem 3.21 *Each open orbital arithmetic curve* $\overset{\mathbf{o}}{\mathbf{D}}_{\mathbf{\Gamma}}$ *on a Picard or Hilbert modular surface has a smooth model in its correspondence class. More precisely, one can find a finite uniformization of* $\overset{\mathbf{o}}{\mathbf{D}}_{\mathbf{\Gamma}}$.

In the Picard case also $\mathbf{D}_\Gamma$ has a finitely covering smooth model in its correspondence class.

Theorem 3.22 *All orbital arithmetic curves* $\mathbf{D}_\Gamma$ *are orbital releasable.*

Proof steps for the last two theorems (e.g. for $\mathbb{B}$):

Definition 3.23 *A* $\mathbb{B}$*-lattice* Γ *is called* neat *if and only if each stationary group* Γ_P*,* $P \in \mathbb{B}$*, is torsion free.*

It is well-known by a theorem of Borel, that each lattice Γ contains a normal sublattice Γ_0 (of finite index), which is neat. Especially, Γ_0 is torsion free.

Definition 3.24 *Let* $\mathbb{D}$ *be a* K*-disc on* $\mathbb{B}$*. A neat* $\mathbb{B}$*-lattice* Δ *is called* $\mathbb{D}$***-neat****, if and only if the implication*

$$\gamma(\mathbb{D}) \cap \mathbb{D} \neq \emptyset \Longrightarrow \gamma(\mathbb{D}) = \mathbb{D}$$

holds for all $\gamma \in \Delta$*.*

Since Δ is assumed to be neat there are no honest Galois weights, hence $\overset{o}{X}_\Delta = \overset{o}{\mathbf{X}}_\Delta$. Then our $\mathbb{D}$-neat condition is equivalent to the regularity of the image curve $\overset{o}{D}_\Delta = \Delta\backslash\mathbb{D}$ on $\overset{o}{X}_\Delta$. Thus Theorem 3.21 follows from the following

Proposition 3.25 Γ *has a normal* $\mathbb{D}$*-neat sublattice.*

It is well-known that the arithmetic group Γ has a neat subgroup of finite index. Therefore we can assume Γ to be neat, hence $\Gamma_{\mathbb{D}} = N_\Gamma(\mathbb{D})$. Without loss of generality we can also assume that all elements of Γ are special, hence

$$\Gamma_{\mathbb{D}} \subset \mathbb{S}\mathbf{N}(\mathbb{Z}) \subset \mathbb{S}\mathbf{G}(\mathbb{Z}) \supset \mathbf{\Gamma} \tag{25}$$

with the notations of diagram (24).

Proof . We call a point $Q \in \mathbb{D}$ a **Γ-singular point** on $\mathbb{D}$, if and only if it belongs to $\mathbb{D} \cap \gamma(\mathbb{D})$ for a $\gamma \in \Gamma$ not belonging to $\Gamma_{\mathbb{D}}$. The Γ-singular points on $\mathbb{D}$ are precisely the preimages on $\mathbb{D}$ of the singular points of $\Gamma\backslash\mathbb{D}$. Therefore there are only finitely many $\Gamma_{\mathbb{D}}$-equivalence classes of Γ-singular points on $\mathbb{D}$. The branch set at Q of the orbit curve $\Gamma\mathbb{D} \subset \mathbb{B}$ corresponds bijectively to the set of branches of $\Gamma\backslash\mathbb{D}$ at the image point P. Let $Q_1, ..., Q_s$ be a complete set of $\Gamma_{\mathbb{D}}$- representatives of Γ-singular points on $\mathbb{D}$ and $\mathbb{D}_{ij} = \gamma_{ij}(\mathbb{D})$, $\gamma_{ij} \in \Gamma_{\mathbb{D}}$, $j = 1, ..., k_i$, all different $\Gamma\mathbb{D}$-branches at Q_i excluding $\mathbb{D}$.

It is a fact of algebraic group theory, see e.g. [B], II.5.1, that $\mathbf{N}$ is the normalizing group of a line L in a faithful linear representation space E of $\mathbf{G}$, all defined over $\mathbb{Q}$. Each $X \in L(\mathbb{Q})$ defines, over $\mathbb{Q}$ again, a weight character $\rho = \rho_X = \rho_L : \mathbf{N} \rightarrow \mathbb{G}\mathbf{l_1}$.

$$\alpha : G \longrightarrow \mathbb{G}l(E), \quad \alpha(g)X = \rho(g)X, \ g \in \mathbf{N}(\mathbb{Q}),$$

Since $\mathbb{S}\mathbf{N}$ is simple, ρ restricts to the trivial character on $\mathbb{S}\mathbf{N}$, hence $\rho(\Gamma_{\mathbb{D}}) = \{1\}$. With the above chosen elements we define

$$E \ni X_{ij} := \gamma_{ij}X \neq X.$$

The latter non-incidence holds because each γ_{ij} does not normalize $\mathbb{D}$, hence does not belong to N, so it cannot normalize L. We can assume (by choice of $\mathbb{Q}$-base) that all X, X_{ij} belong to $E(\mathbb{Z})$. We find a natural number a such that

$$X \not\equiv X_{ij} \mod (a), \text{ for all } i = 1, ..., s, \ j = 1, ..., k_i.$$

Now we show that $\Gamma(a)$ is $\mathbb{D}$-neat:
Assume the existence of $\gamma' \in \Gamma(a)$ such that $\gamma'\mathbb{D}$ intersects $\mathbb{D}$ properly at P. The intersecting pair $(\mathbb{D}, \gamma'\mathbb{D})$ has to be $\Gamma_{\mathbb{D}}$-equivalent to one of the representative intersection pairs $(\mathbb{D}, \gamma_{ij}\mathbb{D})$, say

$$\delta\gamma'\mathbb{D} = \gamma_{ij}\mathbb{D}, \ \delta \in \Gamma_{\mathbb{D}}.$$

Therefore $\gamma_{ij}^{-1}\delta\gamma' \in N$ because it normalizes $\mathbb{D}$.

$$\gamma_{ij}^{-1}\delta\gamma'X = X, \ \delta\gamma'X = X_{ij}, \ \gamma_1 X = X_{ij},$$

with $\gamma_1 := \delta\gamma'\delta^{-1} \in \Gamma(a)$. Since $\gamma_1 \equiv id \mod (a)$, we get the contradiction

$$X = id(X) \equiv \gamma_1(X) = X_{ij} \mod (a).$$

□

4 Neat Proportionality

We want to define norms of orbital arithmetic curves $\overset{o}{\mathbf{D}}_{\mathbf{\Gamma}}$ and their birational companions on Picard or Hilbert modular surfaces. They sit in the specification of the discs $\mathbb{D} = \mathbb{D}_V \hookrightarrow \mathbb{B}$ or $\mathbb{H}^2$ defining the supporting curve as $\overset{o}{D}_\Gamma = \Gamma \backslash \mathbb{D}$, namely

$$\mathbb{D}_V = \mathbb{P}(V^\perp) \cap \mathbb{B} \text{ or } \mathbb{P} \times \mathbb{P}(V^\perp) \cap \mathbb{H}^2,$$

where in the Picard case : $V \in K^3$, $< V, V >$ positive, $\perp$ with respect to unitary $(2,1)$-metric on $\mathbb{C}^3$.

In the Hilbert case the situation is more complicated: Let $V \in \mathbb{G}l_2^+(K)$ be scew-hermitian with respect to the non-trivial $K/\mathbb{Q}$-isomorphism $'$, that means ${}^tV' = -V$, explicitly

$$V = \begin{pmatrix} a\sqrt{D} & \lambda \\ -\lambda' & b\sqrt{D} \end{pmatrix}, \tag{26}$$

$a, b \in \mathbb{Q}$, $\lambda \in K$. By abuse of language we define "orthogonality" with elements of $\mathbb{C}^2 \times \mathbb{C}^2$ in the following manner:

$$\mathbb{C}^2 \times \mathbb{C}^2 \ni (z_1, z_0; w_1, w_0) \perp V : \ (z_1, z_0) V \begin{pmatrix} w_1 \\ w_0 \end{pmatrix} = 0$$

and the "bi-projectivization" of $\mathbb{C}^2 \times \mathbb{C}^2$ by

$$\mathbb{P} \times \mathbb{P}(z_1, z_0; w_1, w_0) \ := (z_1 : z_0) \times (w_1 : w_0) \in \mathbb{P}^1 \times \mathbb{P}^1 \supset \mathbb{H} \times \mathbb{H}$$

$\mathbb{D}_V \subset \mathbb{P}(V^\perp)$ or $\mathbb{P} \times \mathbb{P}(V^\perp)$ is complex 1-dimensional and analytically isomorphic to the upper half plane $\mathbb{H}$.

For later use of Heegner divisors we define norms of subdiscs and their quotient curves on this place.

Definition 4.1 *of* ***norms***

Picard case: $N(V) \ := < V, V > \in \mathbb{N}_+$, $V \in \mathfrak{O}_K^{3,+}$,
Hilbert case: $N(V) \ := \det V \in \mathbb{N}_+$, $V \in Scew_2^+(\mathfrak{O}_K)$.

and of ***norm sets*** of arithmetic curves:

$$\mathcal{N}(\widehat{\Gamma \backslash \mathbb{D}}) = \mathcal{N}(\mathbb{D}) \ := \{N(V);\ \mathbb{D} = \mathbb{D}_V, V \text{ integral}\} \subset \mathbb{N}_+$$

Definition 4.2 *For $N \in \mathbb{N}_+$ the Weil divisor*

$$H_N = H_N(\Gamma) \ := \sum_{\substack{\mathbb{D} \\ \mathcal{N}(\mathbb{D}) \ni N}} \widehat{\Gamma \backslash \mathbb{D}}$$

is called the N-th ***Heegner divisor*** *on $\hat{X}_\Gamma$.*

The scew-symmetric elements (26) with fixed K form a quadratic vector space $\mathfrak{V}_\mathbb{Q} \ := (Scew_2(K), \det)$ with signature $(2,2)$ as well as $\mathfrak{V}_\mathbb{R} \cong \mathbb{R}^{2,2}$. The group $\mathbb{S}l_2(K)$ acts on $Scew_2(K)$ and on the positive part $Scew_2^+(K)$:

$$\mathbb{S}l_2(K) \ni g : \ V \mapsto {}^tg'Vg$$

It defines an embedding

$$\mathbb{H}^2 \longrightarrow Grass^+(2,\mathfrak{V}_\mathbb{R}) \subset Grass(2,\mathfrak{V}_\mathbb{R})$$
$$(z,w) \mapsto (z,w)^\perp := \{V \in \mathfrak{V}_\mathbb{R};\ (z_1,z_0;w_1,w_0)\perp V\}$$

and a homeomorphism

$$\mathbb{H}^2 \longleftrightarrow \mathbb{SO}_e(2,2)/(\mathbb{SO}(2)\times\mathbb{SO}(2)),$$

where the lower index $_e$ denotes the unit component. The normalizer of $\mathbb{D}_V$ in $\Gamma \subseteq \mathbb{S}l_2(\mathfrak{O}_K)$ is

$$N_\Gamma(\mathbb{D}_V) = \Gamma_V = \{g \in \Gamma;\ {}^t g' V g = \pm V\}$$

In the special case $V = \left(\begin{smallmatrix} 0 & \lambda \\ -\lambda' & 0 \end{smallmatrix}\right)$, $\lambda \in \mathfrak{O}_K$, primitive, $\lambda\cdot\lambda' = N$, $\Gamma = \mathbb{S}l_2(\mathfrak{O}_K)$ the action on $\mathbb{D}_V$ is (conjugation) equivalent to the action of

$$\left(\begin{smallmatrix} \lambda & 0 \\ 0 & 1 \end{smallmatrix}\right)\mathbb{S}l_2(\mathfrak{O})\left(\begin{smallmatrix} \lambda^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)$$

on the diagonal of $\mathbb{H}^2$. Then for $K = \mathbb{Q}(\sqrt{d})$

$$N_\Gamma(\mathbb{D}_V) \cong \begin{cases} \mathbb{S}l_2(\mathbb{Z})(N)_0 := \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right);\ a,b,c,d \in \mathbb{Z},\ N \mid c\},\ \text{if } \sqrt{d} \nmid \lambda \\ \text{index 2 extension of } \mathbb{S}l_2(\mathbb{Z})(N)_0,\ \text{if } \sqrt{d} \mid \lambda \text{ in } \mathfrak{O}_K. \end{cases}$$

Any hermitian symmetric domain $\mathbb{B}$ is embedded in its dual symmetric space $\check{\mathbb{B}}$, which is compact, hermitian and of same dimension as $\mathbb{B}$. For $\mathbb{B}$, $\mathbb{D} \cong \mathbb{H}$ or $\mathbb{H}^2$ the duals are simply:

$$\mathbb{B} \check{=} \mathbb{P}^2,\ \check{\mathbb{D}} = \check{\mathbb{H}} = \mathbb{P}^1\ \check{\mathbb{H}}^2 = \mathbb{P}^1 \times \mathbb{P}^1.$$

The Lie algebra of the Lie group $G_\mathbb{C}$ of the dual symmetric space $\check{\mathbb{B}}$ is the complexification of the Lie algebra corresponding to the Lie group G of $\mathbb{B}$. For the splitting case $\mathbb{H}^2$ we use the isomorphy of Lie algebras
$\mathfrak{so}(2,2) \cong \mathfrak{sl}_2(\mathbb{R}) \times \mathfrak{sl}_2(\mathbb{R})$ to get the pairs

$$\begin{array}{rcl} G & \subset & G_\mathbb{C} \\ \mathbb{SU}((2,1),\mathbb{C}) & \subset & \mathbb{S}l_3(\mathbb{C}), \\ \mathbb{SO}_e(2,2) & \subset & \mathbb{S}l_2(\mathbb{C}) \times \mathbb{S}l_2(\mathbb{C}) \end{array}$$

The conjugation classes of commutators of normalizing Lie group pairs $N' = [N,N]$ with complexifications $N'_\mathbb{C}$ corresponding to $\mathbb{D} \subset \mathbb{B}$ or $\mathbb{D} \subset \mathbb{H}^2$ are represented by:

$$\begin{array}{rcl} G \supset \quad N' & \subset & N'_\mathbb{C} \quad \subset G_\mathbb{C} \\ \mathbb{SU}((1,1),\mathbb{C}) & \subset & \mathbb{S}l_2(\mathbb{C}), \\ \mathbb{S}l_2(\mathbb{R}) & \subset & \mathbb{S}l_2(\mathbb{C}) \text{ diagonal in } \mathbb{S}l_2(\mathbb{C})^2. \end{array}$$

We have the following corresponding commutative embedding diagrams

$$\begin{array}{ccc} \check{\mathbb{B}} & \hookleftarrow & \mathbb{B} \\ \uparrow & & \uparrow \\ \check{\mathbb{D}} & \hookleftarrow & \mathbb{D} \end{array} \qquad\qquad \begin{array}{ccc} G_\mathbb{C} & \hookleftarrow & G \\ \uparrow & & \uparrow \\ N'_\mathbb{C} & \hookleftarrow & N' \end{array}$$

To be more explicit we consider the point-curve-surface flags

$$\begin{array}{rcl} 0 \times 0 = O \in 0 \times \mathbb{D} \subset \mathbb{B} & , & O \in \Delta \subset \mathbb{H} \times \mathbb{H} \text{ (diagonal)}; \\ O \in 0 \times \mathbb{P}^1 \subset \mathbb{P}^2 & , & O \in \Delta' \subset \mathbb{P}^1 \times \mathbb{P}^1 \text{ (diagonal)}. \end{array}$$

with embeddings

$$N'_{\mathbb{C}} \hookrightarrow G_{\mathbb{C}} : \begin{cases} \mathbb{S}l_2(\mathbb{C}) \ni h \mapsto \left(\begin{smallmatrix} 1 & o \\ o & h \end{smallmatrix}\right) & , \text{ Picard case} \\ \mathbb{S}l_2(\mathbb{C}) \ni g \mapsto (g, g) & , \text{ Hilbert case.} \end{cases}$$

The (special) compact stabilizer groups with complexifications are
Picard case:

$$\begin{array}{c} K = Stab_O(G) = \mathbb{S}(\mathbb{U}(2) \times \mathbb{U}(1)) \ , \ K_{\mathbb{C}} = \mathbb{S}(\mathbb{S}l_2(\mathbb{C}) \times \mathbb{G}l_1(\mathbb{C})); \\ k = Stab_O(N') = \mathbb{S}(\mathbb{U}(1) \times \mathbb{U}(1)) \ , \ k_{\mathbb{C}} = \mathbb{S}(\mathbb{G}l_1(\mathbb{C}) \times \mathbb{G}l_1(\mathbb{C})) \end{array}$$

Hilbert case:

$$\begin{array}{c} K = Stab_O(G) = \mathbb{SO}(2) \times \mathbb{SO}(2) \ , \ K_{\mathbb{C}} = \mathbb{G}l_1(\mathbb{C})) \times \mathbb{G}l_1(\mathbb{C}) \subset Stab_O \mathbb{G}_{\mathbb{C}}; \\ k = Stab_O(N') = \mathbb{SO}(2) \ , \ k_{\mathbb{C}} = \mathbb{S}(\mathbb{G}l_1(\mathbb{C}) \times \mathbb{G}l_1(\mathbb{C})) \subset Stab_O N'_{\mathbb{C}}, \end{array}$$

Lie group diagram for $O \in \mathbb{D} \subset \mathbb{B}$:

$$\begin{array}{ccc} K = G_O & \longrightarrow & G \\ \uparrow & & \uparrow \\ k = N'_O & \longrightarrow & N' \end{array}$$

Complexification diagram for $O \in \check{\mathbb{D}} = \mathbb{P}^1 \in \check{\mathbb{B}} = \mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$:

$$\begin{array}{ccccc} K_{\mathbb{C}} & \longrightarrow & P_+ \cdot K_{\mathbb{C}} = G_{\mathbb{C},O} & \longrightarrow & G_{\mathbb{C}} \\ \uparrow & & \uparrow & & \uparrow \\ k_{\mathbb{C}} & \longrightarrow & p_+ \cdot k_{\mathbb{C}} = N'_{\mathbb{C},O} & \longrightarrow & N'_{\mathbb{C}} \end{array}$$

with suitable stabilizer splitting complex parabolic groups $P_+ \subset \mathbb{G}_{\mathbb{C}}$ or $p_+ \subset N'_{\mathbb{C}}$, respectively. These are the unipotent radicals of the corresponding stabilizer groups.

Example 4.3 *2-ball case:*

$$P_+ = \{ \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & 1 \end{smallmatrix}\right) ; \ a, b \in \mathbb{C} \} \ , \ p_+ = \{ \left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right) ; \ c \in \mathbb{C} \}$$

Now take a G-vector bundle $\overset{o}{E}$ on $\mathbb{B}$. The stabilizer $K = G_O$ acts on the fibre E_O of E, and $\overset{o}{E}$ together with the G-action is received by extension of the K-action along the G/K-transport: $\overset{o}{E}= E_O \times_K G$.

We extend the (really represented, bi-unitary or bi-orthogonal) K-action on $\overset{o}{E}$ by complexification to a $K_{\mathbb{C}}$-action. Putting together with the trivially defined action of P_+ on E_O we get the $G_{\mathbb{C}}$-bundle

$$\check{E} \ := E_O \times_{P_+K} G_{\mathbb{C}} \text{ on } \check{\mathbb{B}}.$$

Now let Γ be a neat arithmetic subgroup of G, and $\overline{\Gamma\backslash\mathbb{B}} \longrightarrow \widehat{\Gamma\backslash\mathbb{B}}$ a singularity resolution of the Baily-Borel compactification of $\Gamma\backslash\mathbb{B}$ (already smooth) with (componentwise) normal crossing compactification divisor. The G-bundle $\overset{o}{E}$ goes down to the quotient bundle $E \ := \Gamma\backslash \overset{o}{E}$.

We endow $\overset{o}{E}$ with a G-equivariant hermitian metric $\overset{o}{h}$. It extends along the above $\mathbb{G}_{\mathbb{C}}/P_+K_{\mathbb{C}}$ transport from E_O to a hermitian $\mathbb{G}_{\mathbb{C}}$-equivariant metric $\check{h}$ on $\check{E}$. On the other hand it goes down along the quotient map $\mathbb{B} \to \Gamma\backslash\mathbb{B}$ to a hermitian metric h on E.

Theorem 4.4 *(Mumford). Up to isomorphy there is a unique hermitian vector bundle $\bar{E}$ extending E on $\overline{\Gamma\backslash\mathbb{B}}$, such that h is "logarithmically restricted" around the (smooth) compactification divisor X_Γ^∞.*

This means: Using coordinates z_i on a small polycylindric neighbourhood $\bar{U} = \mathbb{D}^{a+b}$ around $Q \in X_\Gamma^\infty$ with finite part $U = (\mathbb{D} \setminus 0)^a \times \mathbb{D}^b$, and a basis $\mathfrak{e}_j$ of $\bar{E}$ over $\bar{U}$, then

$$|h(\mathfrak{e}_j, \mathfrak{e}_k)|, |\det(h(\mathfrak{e}_j, \mathfrak{e}_k))| \leq C \cdot \left(\sum_{i=1}^{a} \log |z_i| \right)^{2N}$$

with constants $C, N > 0$. Altogether we get bundle diagrams

$$\begin{array}{ccccccc} \check{E} & \longleftarrow & \overset{o}{E} & \longrightarrow & E & \longrightarrow & \bar{E} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \check{\mathbb{B}} & \longleftarrow & \mathbb{B} & \longrightarrow & \Gamma\backslash\mathbb{B} & \longrightarrow & \overline{\Gamma\backslash\mathbb{B}} \end{array} \tag{27}$$

Let

$$1 + c_1(F) + .. + c_r(F) \in H^{even}(V, \mathbb{R})$$

be the total Chern class of a holomorphic vector bundle of rank r on the compact smooth complex algebraic variety V of dimension n, say. By Hodge theory we interpret $c_j(F)$ (uniquely up to exact forms) as a differential form $\gamma_j = \gamma_j(F)$ of degree $2j$ on X. We have the differential forms

$$\gamma_{\mathfrak{j}}(F) \ := \gamma_{j_1} \wedge ... \wedge \gamma_{j_k}, \ \mathfrak{j} = (j_1, ..., j_k), \ \sum_i j_i \leq n;$$

in particular the ***Chern forms***, if $\sum_i j_i = n$. In the latter cases the ***Chern numbers*** are defined as

$$c_{\mathfrak{j}}(F) := \int_V \gamma_{\mathfrak{j}}(F).$$

Especially, the Chern number $c_n(V) = c_n(\mathcal{T}_V)$, where $\mathcal{T}_V$ is the tangent bundle on V, is the ***Euler number*** of V.

Coming back to our neat arithmetic quotient variety $\Gamma\backslash\mathbb{B}$ and vector bundle quadruples described in diagram (27), we come to the important

Theorem 4.5 *(Mumford's Proportionality Theorem). The Chern numbers of $\check{E}$ and $\bar{E}$ are related as follows:*

$$c_{\mathfrak{j}}(\bar{E}) \cdot c_n(\check{\mathbb{B}}) = c_{\mathfrak{j}}(\check{E}) \cdot c_n(\Gamma\backslash\mathbb{B}),$$

where $c_n(\Gamma\backslash\mathbb{B})$ denotes the Euler volume (with respect to Euler-Chern volume form of the Bergmann metric on $\mathbb{B}$) of a Γ-fundamental domain on $\mathbb{B}$.

Now take a symmetric subdomain $\mathbb{D}$ of $\mathbb{B}$ such that $\Gamma_{\mathbb{D}}$ is a neat arithmetic $\mathbb{D}$-lattice. We have an extended commutative diagram with vertical analytic embeddings

$$\begin{array}{ccccccc} \check{\mathbb{B}} & \longleftarrow & \mathbb{B} & \longrightarrow & \Gamma\backslash\mathbb{B} & \longrightarrow & \overline{\Gamma\backslash\mathbb{B}} \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \check{\mathbb{D}} & \longleftarrow & \mathbb{D} & \longrightarrow & \Gamma_{\mathbb{D}}\backslash\mathbb{D} & \longrightarrow & \overline{\Gamma_{\mathbb{D}}\backslash\mathbb{D}} \end{array} \qquad (28)$$

satisfying the

Absolute and relative normal crossing conditions:

- All varieties in the diagram are smooth;
- the compactification divisor $X_\Gamma^\infty = \overline{\Gamma\backslash\mathbb{B}} \setminus (\Gamma\backslash\mathbb{B})$ is normal crossing;
- the compactification divisor $X_{\Gamma_{\mathbb{D}}}^\infty = \overline{\Gamma_{\mathbb{D}}\backslash\mathbb{D}} \setminus (\Gamma_{\mathbb{D}}\backslash\mathbb{D})$ is normal crossing,
- at each common point, the (small) compactification divisor $X_{\Gamma_{\mathbb{D}}}^\infty$ crosses transversally the big one X_Γ^∞.

As described in the beginning of this section we work with the (special) Lie group $N' \subset G$ acting on $\mathbb{D}$ and containing $\Gamma_{\mathbb{D}}$. Starting with a N'-equivariant hermitian vector bundle $\overset{o}{e}$ on $\mathbb{D}$ we get a commutative diagram as (27)

$$\begin{array}{ccccccc} \check{e} & \longleftarrow & \overset{o}{e} & \longrightarrow & e & \longrightarrow & \bar{e} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \check{\mathbb{D}} & \longleftarrow & \mathbb{D} & \longrightarrow & \Gamma\backslash\mathbb{D} & \longrightarrow & \overline{\Gamma\backslash\mathbb{D}} \end{array} \qquad (29)$$

Mumford's Proportionality Theorem yields the Chern number relations

$$c_{\mathfrak{j}}(\bar{e}) \cdot c_m(\check{\mathbb{D}}) = c_{\mathfrak{j}}(\check{e}) \cdot c_m(\Gamma\backslash\mathbb{D}) \tag{30}$$

with $m = \dim \mathbb{D}$, $\mathfrak{j} = (j_1, ..., j_k)$, $\sum\limits_i j_i = m$.

If we start with a G-bundle $\overset{o}{E}$ on $\mathbb{B}$ and restrict it to the N'-bundle $\overset{o}{e} = \overset{o}{E}|_{\mathbb{D}}$ on $\mathbb{D}$, then we get two quadruples of bundles described in the diagrams (27) and (29). By construction, it is easy to see that the bundles $\check{e}$, e, $\bar{e}$ are restrictions of the bundles $\check{E}$, E or $\bar{E}$, respectively. The relations (30) specialize to the

Theorem 4.6 *(Relative Proportionality Theorem). For* $\overset{o}{e} = \overset{o}{E}|_{\mathbb{D}}$*, with the above notations, it holds that*

$$c_{\mathfrak{j}}(\bar{E}\,|_{\overline{\Gamma\backslash\mathbb{D}}}) \cdot c_m(\check{\mathbb{D}}) = c_{\mathfrak{j}}(\check{E}\,|_{\check{\mathbb{D}}}) \cdot c_m(\Gamma\backslash\mathbb{D}). \tag{31}$$

Corollary 4.7 *In particular, for* $\dim \mathbb{B} = 2$ *and* $\dim \mathbb{D} = 1$ *we get with the conditions of the theorem the relations*

$$c_1(\bar{E}\,|_{\overline{\Gamma\backslash\mathbb{D}}}) \cdot c_1(\check{\mathbb{D}}) = c_1(\check{E}\,|_{\check{\mathbb{D}}}) \cdot c_1(\Gamma\backslash\mathbb{D}). \tag{32}$$

Knowing $\check{\mathbb{D}} = \mathbb{P}^1$ and its Euler number $c_1(\mathbb{P}^1) = 2$ we get

$$2 \cdot c_1(\bar{E}\,|_{\overline{\Gamma\backslash\mathbb{D}}}) = c_1(\check{E}\,|_{\check{\mathbb{D}}}) \cdot vol_{EP}(\Gamma_{\mathbb{D}}), \tag{33}$$

where vol_{EP} denotes the Euler-Poincaré volume of a fundamental domain of a $\mathbb{D}$-lattice.

Now we work with canonical bundles $\mathcal{K} = \mathcal{T}^* \wedge \mathcal{T}^*$, $\mathcal{T}$ the tangent bundle on a smooth analytic variety with dual cotangent bundle $\mathcal{T}^*$. The above construction yields

$$\overset{o}{E} = \mathcal{K}_{\mathbb{B}},\ \check{E} = \mathcal{K}_{\check{\mathbb{B}}},\ E = \mathcal{K}_{\Gamma\backslash\mathbb{B}},$$

For the restriction of $\check{E}$ to $\check{\mathbb{D}}$ we get

$$c_1(\check{E}\,|_{\check{\mathbb{D}}}) = (K_{\check{\mathbb{B}}} \cdot \check{\mathbb{D}}) = \begin{cases} (K_{\mathbb{P}^2} \cdot L) = -3(L^2) = -3, & \text{Picard case} \\ (K_{\mathbb{P}^1 \times \mathbb{P}^1} \cdot \Delta') = ((-2H - 2V) \cdot \Delta') = -4, & \text{Hilbert case} \end{cases}$$

Thereby, $H = \mathbb{P}^1 \times 0$ (horizontal line), $V = 0 \times \mathbb{P}^1$ (vertical line), Δ' the diagonal on $\mathbb{P}^1 \times \mathbb{P}^1$ and L is an arbitrary projective line on $\mathbb{P}^2$. Substituting in (33) we receive

$$c_1(\bar{E}\,|_{\overline{\Gamma\backslash\mathbb{D}}}) = \begin{cases} -\frac{3}{2} \cdot vol_{EP}(\Gamma_{\mathbb{D}}), & \text{Picard case} \\ -2 \cdot vol_{EP}(\Gamma_{\mathbb{D}}), & \text{Hilbert case} \end{cases} \tag{34}$$

One knows that for cotangent bundles that $\bar{\mathcal{T}}^*$ is the extension of $\mathcal{T}^*_{\Gamma\backslash\mathbb{B}}$ by logarithmic forms along the compactification divisor X_Γ^∞ (allowing simple poles there). Wedging them we see that the Mumford-extended canonical bundle is nothing else but the logarithmic canonical bundle

$$\bar{E} = \Omega^2_{\overline{\Gamma\backslash\mathbb{B}}}(\log X_\Gamma^\infty)$$

corresponding to the ***logarithmic canonical divisor*** $K_{\overline{\Gamma\backslash\mathbb{B}}} + X_\Gamma^\infty$, where the first summand is a canonical divisor of $\overline{\Gamma\backslash\mathbb{B}}$ and the second summand is the compactification divisor (reduced divisor with the compactification set as support). Therefore we get

$$c_1(\bar{E}\mid_{\overline{\Gamma\backslash\mathbb{D}}}) = ((K_{\overline{\Gamma\backslash\mathbb{B}}} + X_\Gamma^\infty)\cdot\overline{\Gamma\backslash\mathbb{D}}).$$

Together with (34) we get

$$(K_{\overline{\Gamma\backslash\mathbb{B}}}\cdot\overline{\Gamma\backslash\mathbb{D}}) + (\overline{\Gamma\backslash\mathbb{D}}\cdot X_\Gamma^\infty) = -\begin{cases}\frac{3}{2}\cdot vol_{EP}(\Gamma_{\mathbb{D}}), & \text{Picard case}\\ 2\cdot vol_{EP}(\Gamma_{\mathbb{D}}), & \text{Hilbert case}\end{cases} \tag{35}$$

The adjunction formula for curves on surfaces relates the Euler number of $\overline{\Gamma\backslash\mathbb{D}}$ with intersection numbers as follows:

$$-eul(\overline{\Gamma\backslash\mathbb{D}}) = (\overline{\Gamma\backslash\mathbb{D}}^2) + (K_{\overline{\Gamma\backslash\mathbb{B}}}\cdot\overline{\Gamma\backslash\mathbb{D}}).$$

On the other hand, by a very classical formula, the Euler number can be read off from the volume of a fundamental domain and the number of compactification points:

$$eul(\overline{\Gamma\backslash\mathbb{D}}) = vol_{EP}(\Gamma_{\mathbb{D}}) + (\overline{\Gamma\backslash\mathbb{D}}\cdot X_\Gamma^\infty) = eul(\Gamma_{\mathbb{D}}\backslash\mathbb{D}) + (\overline{\Gamma\backslash\mathbb{D}}\cdot X_\Gamma^\infty). \tag{36}$$

Adding the last two relations we get

$$(K_{\overline{\Gamma\backslash\mathbb{B}}}\cdot\overline{\Gamma\backslash\mathbb{D}}) + (\overline{\Gamma\backslash\mathbb{D}}\cdot X_\Gamma^\infty) = -(\overline{\Gamma\backslash\mathbb{D}}^2) - vol_{EP}(\Gamma_{\mathbb{D}}). \tag{37}$$

We define and obtain by substitution in (35)

$$Self(\overline{\Gamma\backslash\mathbb{D}}) := (\overline{\Gamma\backslash\mathbb{D}}^2) = \begin{cases}\frac{1}{2}\cdot vol_{EP}(\Gamma_{\mathbb{D}}), & \text{Picard case}\\ 1\cdot vol_{EP}(\Gamma_{\mathbb{D}}), & \text{Hilbert case}\end{cases} \tag{38}$$

We call $Self(\overline{\Gamma\backslash\mathbb{D}})$ the ***orbital self-intersection*** and define also the ***orbital Euler number***

$$Eul(\overline{\Gamma\backslash\mathbb{D}}) := eul(\Gamma\backslash\mathbb{D}) = vol_{EP}(\Gamma_{\mathbb{D}})$$

of $\overline{\Gamma\backslash\mathbb{D}} \leftharpoonup \overline{\Gamma\backslash\mathbb{B}}$ in the in the case of $\mathbb{D}$-neat lattices Γ. Then, together with (36), one gets

$$vol_{EP}(\Gamma_{\mathbb{D}}) = eul(\Gamma\backslash\mathbb{D}) = Eul(\overline{\Gamma\backslash\mathbb{D}}) = \begin{cases}2\cdot(\overline{\Gamma\backslash\mathbb{D}}^2) = 2\cdot Self(\overline{\Gamma\backslash\mathbb{D}}),\\ 1\cdot(\overline{\Gamma\backslash\mathbb{D}}^2) = 1\cdot Self(\overline{\Gamma\backslash\mathbb{D}}),\end{cases} \tag{39}$$

in the Picard or Hilbert case, respectively.

5 The General Proportionality Relation

5.1 Ten rules for the construction of orbital heights and invariants

We introduce relative orbital objects, and the corresponding notations:
These are finite orbital coverings: $\mathbf{Y}/\mathbf{X}$ (surfaces), $\mathbf{D}/\mathbf{C}$ (curves) $\mathbf{Q}/\mathbf{P}$ (points) and also birational ones (relative releases, mainly) $\mathbf{X}'\to\mathbf{X}$, $\mathbf{C}'\to\mathbf{C}$, $\mathbf{P}'\to\mathbf{P}$.

The relative orbital morphisms joining these relative objects are commutative diagrams. We use the notations:

$\mathbf{Y}'/\mathbf{X}'\to\mathbf{Y}/\mathbf{X}$, $\mathbf{D}'/\mathbf{C}'\to\mathbf{D}/\mathbf{C}$, $\mathbf{Q}'/\mathbf{P}'\to\mathbf{Q}/\mathbf{P}$,

Definition 5.1 *A rational* ***orbital invariant*** *on* $\mathbf{Orb^{2,1}}$ *is a non-constant map* $\mathbf{h}$ *corresponding each orbital curve a rational number*

$$O \neq \mathbf{h}: \ \mathbf{Orb}^2 \longrightarrow \mathbb{Q} \ \textit{with}$$

R.1 deg(1/1)
$[\mathbf{D}:\mathbf{C}]\mathbf{h}(\mathbf{C})\mathbf{h}(\mathbf{D}) = [\mathbf{D}:\mathbf{C}]\cdot\mathbf{h}(\mathbf{C})$
for all finite orbital curve coverings $\mathbf{D}/\mathbf{C}$*. Thereby* $[\mathbf{D}:\mathbf{C}] := \frac{w_D}{w_C}[D:C]$ *is the* ***orbital degree*** *of the covering with usual covering degree* $[D:C]$*.*

If $0 \neq h: \ \mathbf{Orb}^2 \longrightarrow \mathbb{Q}$ *satisfies*

R.1 deg(1/1)
$h(\mathbf{D}) = [D:C]\cdot h(\mathbf{C})$,
for all finite orbital curve coverings $\mathbf{D}/\mathbf{C}$*, then we call it an* ***orbital height****.*

Remark 5.2 *It is easy to see that we get immediately from an orbital height* h *an orbital invariant* $\mathbf{h}$ *setting* $\mathbf{h}(\mathbf{C}) := w_{\mathbf{C}}\cdot h(\mathbf{C})$*.*

Convention: In this paragraph we restrict ourselves to compact curves or to open orbital curves $\overset{o}{\mathbf{C}}$ with cusp point compactification $\hat{\mathbf{C}}$. For simplicity we will write $\mathbf{C}$ instead of $\hat{\mathbf{C}}$ and set $h(\overset{o}{\mathbf{C}}) := h(\hat{\mathbf{C}}) = h(\mathbf{C})$ (same for $\mathbf{h}$).
For constructions we need:

- $h(\mathbf{P})$ local orbital invariants,
- relative orbital heights such that:
  - for orbital curves:

$$h(\mathbf{C}'\to\mathbf{C}) := h(\mathbf{C}') - h(\mathbf{C}) \, , \, h(\mathbf{D}/\mathbf{C}) := h(\mathbf{D}) - [D:C]\cdot\mathbf{h}(\mathbf{C})$$

- for orbital curve points:

  $$h(\mathbf{P'}\to\mathbf{P}) := h(\mathbf{P'}) - h(\mathbf{P}) \ , \ h(\mathbf{Q}/\mathbf{P}) := h(\mathbf{Q}) - [Q:P]\cdot h(\mathbf{P}),$$

  where $[Q:P]$ has to be defined later.

and the **Decomposition laws** (absolute and relative):

R.2 Dec(1,0)
$h(\mathbf{C}) = h_1(\mathbf{C}) + h_0(\mathbf{C}) \ , \ h_0(\mathbf{C}) = \sum_{\mathbf{P}\in\mathbf{C}} h(\mathbf{P})$

R.3 Dec(11,00)
$$h(\mathbf{C'}\to\mathbf{C}) = h_1(\mathbf{C'}\to\mathbf{C}) + h_0(\mathbf{C'}\to\mathbf{C})$$
$$h_0(\mathbf{C'}\to\mathbf{C}) = \sum_{\mathbf{P'}\to\mathbf{P}\in\mathbf{C'}\to\mathbf{C}} h(\mathbf{P'}\to\mathbf{P})$$

with finite sums h_0 and local incidence diagrams

$$\begin{array}{ccc} \mathbf{P'} & \longrightarrow & \mathbf{C'} \\ \downarrow & & \downarrow \\ \mathbf{P} & \longrightarrow & \mathbf{C} \end{array}$$

Relative orbital rules:

R.4 deg(11/11)

$$h(\mathbf{D'}\to\mathbf{D}) = [D:C]\cdot h(\mathbf{C'}\to\mathbf{C})$$

R.5 deg(00/00)

$$h(\mathbf{Q'}\to\mathbf{Q}) = [Q:P]\cdot h(\mathbf{P'}\to\mathbf{P})$$

Initial relations:

R.6 $\deg(1/1)_{sm}$
deg(00/00) holds for totally smooth **D**, **C**
(no curve and no surface singularities);

R.7 $\deg(00/00)_{sm}$
deg(00/00) holds for smooth releases of abelian point uniformizations
(background: stepwise resolution of singularities)

Shift techniques along releases:

R.8 $(\text{Shift})_{ab}$
shifts deg(1/1) along locally abelian releases;

R.9 $(\text{Shift})^{ab}_{*}$
shifts deg(1/1) to all finitely weighted releasable orbital curves;

R.10 $(\text{Shift})^{*}_{\infty}$
shifts deg(1/1) to all releasable orbital curves including infinite weights.

Implications (Geometric Local-Global Principle):

(Imp 1) Dec(11,00), deg(00/00) $\Longrightarrow$ deg(11/11) $\Longrightarrow$ $(\text{Shift})_{ab}$;

(Imp 2) $\deg(1/1)_{sm}$, $(\text{Shift})^{sm}_{ab}$ $\Longrightarrow$ $\deg(1/1)_{ab}$;

Later (via definitions)

(Imp 3) $(\text{Shift})^{ab}_{*}$, $\deg(1/1)_{ab} \Longrightarrow \deg(1/1)_{*}$;

(Imp 4) $(\text{Shift})^{*}_{\infty}$, $\deg(1/1)_{*} \Longrightarrow \deg(1/1)$;

5.2 Rational and integral self-intersections

We need rational intersections of curves on (compact algebraic) normal surfaces. Let $\nu : Y \longrightarrow X$ be a birational morphism of normal surfaces. We denote by $Div\, X$ the space of Weil divisors on X with coefficients in $\mathbb{Q}$. There is a rational intersection theory for these divisor groups together with canonically defined orthogonal embeddings $\nu^{\#} : Div\, X \longrightarrow Div\, Y$ extending the integral intersection theory on smooth surfaces and the inverse image functor. Via resolution of singularities the intersection matrices are uniquely determined by the postulate of preserving intersections for all $\nu^{\#}$-preimages. Namely, let $E = E(\nu)$ the exceptional (reduced) divisor on Y, $Div_E Y$ the $\mathbb{Q}$-subspace of $Div\, Y$ generated by the irreducible components of E. For any Weil divisor C on X the generalized inverse image $\nu^{\#}C$ satisfies the conditions

$$Div_E Y \perp \nu^{\#}C = \nu' C + \nu^{\#}_{E} C\ ,\ \nu^{\#}_{E} C \in Div_E Y,$$

where $\nu' C$ is the proper transform of C on Y. Before proofs one has to define the rational intersections on normal surfaces. Let ν be a singularity resolution, $(\ \cdot\ )$ the usual intersection product on smooth surfaces, here on Y, and $E = E_1 + \cdots + E_r$ the decomposition into irreducible components. There is only one divisor $\nu' C + \sum c_i E_i \in Div\, Y$ orthogonal to $Div_E Y$ because the system of equations

$$\sum c_i (E_1 \cdot E_i) = -(E_1 \cdot \nu' C),$$
$$\cdots\cdots$$
$$\sum c_i (E_r \cdot E_i) = -(E_r \cdot \nu' C),$$

has a regular coefficient matrix (negative definite by a theorem of Mumford). The unique $\mathbb{Q}$-solution determines $\nu^{\#}C$. Then the rational intersection product on X is well-defined by

$$< C \cdot D > := (\nu^{\#}C \cdot \nu^{\#}D)\ ,\ C, D \in Div\, X$$

Using orthogonality we get for self-intersections the relations

$$\begin{aligned} < C^2 > &= (\nu^{\#}C \cdot \nu^{\#}C) = ((\nu' C + \nu_E^{\#}C) \cdot \nu^{\#}C)) = (\nu' C \cdot \nu^{\#}C) \\ &= (\nu' C \cdot (\nu' C + \nu_E^{\#}C)) = (\nu' C \cdot \nu' C) + (\nu' C \cdot \nu_E^{\#}C), \end{aligned}$$

hence

$$(\nu' C)^2 := (\nu' C \cdot \nu' C) = < C^2 > -(\nu' C \cdot \nu_E^{\#}C). \tag{40}$$

For minimal singularity resolutions μ we write shortly $(C^2) := (\mu' C)^2$ and notice

$$(C)^2 = < C^2 > -(\mu' C \cdot \mu_E^{\#}C). \tag{41}$$

This number is called the **minimal self-intersection** of C. The difference $(C)^2 - < C^2 >$ splits into a finite sum of point contributions $(\mu' C \cdot \mu_E^{\#}C)_Q$ at intersection points Q of $\mu' C$ and E.

Now we compare self-intersections of locally abelian orbital curves and their releases. The release at $\mathbf{P}$ is supported by a birational surface morphism, also denoted by ρ, precisely $\rho : (X', C') \longrightarrow (X, C)$. The intersection point of the proper transform C' of C and the exceptional line L is denoted by P'. With the weights $w(\mathbf{C}') = w(\mathbf{C})$ and of $\mathbf{L}$ it supports a well-defined abelian point $\mathbf{P}'$ of $\mathbf{C}'$. Let $\mu : \tilde{X} \to X$, $\mu_1 : X'' \to X'$ be the minimal resolutions of P or P', respectively and $\mu_2 : \tilde{X}' \to X'$ that of X'. (We have to resolve in general two cyclic singularities lying on L). For further notations we refer to the following commutative diagram

$$\begin{array}{ccc} \tilde{X}' & \xrightarrow{\tilde{\rho}} & \tilde{X} \\ \downarrow{\scriptstyle \mu_2} & \searrow^{\nu} & \downarrow{\scriptstyle \mu} \\ X' & \xrightarrow[\rho]{} & X \end{array}$$

It restricts to morphisms along exceptional divisors

$$\begin{array}{ccc} \tilde{E}' & \xrightarrow{\tilde{\rho}} & \tilde{E} \\ \downarrow{\scriptstyle \mu_2} & \searrow^{\nu} & \downarrow{\scriptstyle \mu} \\ L & \xrightarrow[\rho]{} & P \end{array}$$

Since $< C^2 >$ is a birational constant we get from (40), (41) and by definition of minimal self-intersections the relation

$$(C'^2) - (C^2) = ((C'^2) - (C^2))_P := (\nu' C)^2 - (C)^2 = -(\nu' C \cdot \nu_E^{\#}C) + (\mu' C \cdot \mu_E^{\#}C)$$

with obvious notations. (The index E stands for the corresponding exceptional divisor).

Lemma 5.3 *The* **relative self-intersection**

$$s(P'\to P) := ((C'^2) - (C^2))_P \leq 0$$

does only depend on the local release $P'\to P$, not on the choice of C, C' crossing P or P', respectively. More precisely, it only depends on the exceptional resolution curves $E_P(\nu)$, $E_P(\mu)$, even only on the intersection graphs of $E(\nu) = E_P(\nu)$ and $E(\mu) = E_P(\mu)$.

Proof . Both linear trees resolve P, the latter minimally. Therefore $E(\nu) \to E(\mu)$ splits into a sequence of σ-processes. The stepwise contraction of a (-1)-line ascends $(\tilde{C}'^2)$ by 1 if and only if this exceptional line crosses $\tilde{C}'$. If not, then $(\tilde{C}'^2)$ is not changed. By stepwise blowing down (-1)-lines we see that $s(P'\to P)$ is equal to the number of steps contracting a line to a point on the image of $\tilde{C}'$. The curve $\tilde{C}'$ crossing the first line of $E(\nu)$ can be chosen arbitrarily.

□

Definitions 5.4 *A release* $\mathbf{P}'\to\mathbf{P}$ *of an abelian orbital curve point is called* **smooth***, if and only if P' is a smooth surface point. A release* $\mathbf{C}'\to\mathbf{C}$ *of a locally abelian orbital curve* $\mathbf{C}$ *is called* **smooth***, if and only if it is smooth at each released abelian point on* $\mathbf{C}$*. A* **smooth release** *of the finite covering* $\mathbf{Q}/\mathbf{P}$ *is a relative finite covering* $\mathbf{Q}'\to\mathbf{Q}/\mathbf{P}'\to\mathbf{P}$ *with smooth releases* $\mathbf{Q}'\to\mathbf{Q}$ *and* $\mathbf{P}'\to\mathbf{P}$*. A* **smooth release** *of the finite covering* $\mathbf{D}/\mathbf{C}$ *is a relative finite covering* $\mathbf{D}'\to\mathbf{D}/\mathbf{C}'\to\mathbf{C}$ *with only smooth local releases* $\mathbf{Q}'\to\mathbf{Q}/\mathbf{P}'\to\mathbf{P}$*.*

Example 5.5 *(stepwise resolution of cyclic singularities) . Let (D,Q) be a uniformization of the abelian curve point (C,P), the latter of cyclic type $< d,e >$. It is realized by cyclic group action of $Z_{d,e} := \left\langle \begin{pmatrix} \zeta_d & 0 \\ 0 & \zeta_e \end{pmatrix} \right\rangle \subset \mathbb{G}l_2(\mathbb{C})$, where ζ_g denotes a g-th primitive unit root. Blow e times up the point (D,Q), each time at intersection point of the exceptional line with the proper transform of C'. Blow down the arising $e-1$ exceptional (-2)-lines. Then we get a smooth release $(D',Q')\to(D,Q)$. The exceptional line E supports a cyclic surface singularity Q'' of type $< e, e-1 >$. Factorizing by $Z_{d.e}$ one gets a smooth release $P'\to P$ with singularity $P'' = Q''/Z_{d,e}$ of type $< e, d' >$ on the exceptional line L over P, where $d' \equiv -d \bmod e$. Altogether we get a smooth release $Q'\to Q/P'\to P$ of the uniformization Q/P of P.*

For the proof we enlarge $Z_{d,e}$ by the reflection group S generated by $< 1, \zeta_e >$, $< \zeta_e, 1 >$ to an abelian group A acting around $O \in \mathbb{C}^2$. We consider the σ-release $O'\to O$ by blowing up O to the (-1)-line N. The directions of x- or y-axis through O correspond to points O' and O'' on N, respectively. Easy coordinate calculations show that the double release $O'\to O\leftarrow O''$ goes down via factorization by S to the double release $\mathbf{Q}'\to\mathbf{Q}\leftarrow\mathbf{Q}''$. Thereby $\mathbf{Q}'\to\mathbf{Q}$ is a smooth release of the smooth abelian point $\mathbf{Q}$, and Q'' is of type $< e, e-1 >$. Furthermore, factorizing by A yields the double release $\mathbf{P}'\to\mathbf{P}\leftarrow\mathbf{P}''$ with smooth release $\mathbf{P}'\to\mathbf{P}$ and P'' of type $< e, d' > = < e, -d >$. Now forget the weights and the upper σ-double release to get $P'\to P\leftarrow P'' = Q'\to Q\leftarrow Q''/ < d, e >$.

□

Proposition 5.6 *Let* $\mathbf{D}/\mathbf{C}$ *be a finite cover of orbital curves with smooth* $\mathbf{D}$*. There exists a smooth relative release* $\mathbf{D}'\to\mathbf{D}/\mathbf{C}'\to\mathbf{C}$*.*

Proof . Essentially, we have only to find local smooth relative releases $\mathbf{Q}'\to\mathbf{Q}/\mathbf{P}'\to\mathbf{P}$ over $\mathbf{Q}/\mathbf{P}\in\mathbf{D}/\mathbf{C}$, if $\mathbf{P}$ is not smooth. If the weights around $\mathbf{Q}$ and $\mathbf{P}$ are trivial (equal to 1), then we refer to the above proof. Otherwise the weights $w(\mathbf{C})$ and $w(\mathbf{D})$ are Galois weights coming from a common local uniformization O of $\mathbf{Q}$, $\mathbf{P}$. Since $\mathbf{Q}$ is smooth it is a quotient point of O by an abelian reflection group Σ. We lift $Q'\to Q$ to a release $O'\to O$ by normalization along the Σ-quotient map around the exceptional line supporting Q'. Then we get the coverings $O'\to O/\mathbf{Q}'\to\mathbf{Q}/\mathbf{P}'\to\mathbf{P}$ with the original weights we need.

□

5.3 The decomposition laws

Definition 5.7 *Let* $\mathbf{P}'\to\mathbf{P}$ *be a release of the abelian cross point* $\mathbf{P}$ *of the abelian curve* $\mathbf{C}$ *of weight* w*, and* $<d',e'>$ *respectively* $<d,e>$ *the cyclic types of* P' *or* P*. The number*

$$h(\mathbf{P}'\to\mathbf{P}) = h_1(\mathbf{P}'\to\mathbf{P}) + h_0(\mathbf{P}'\to\mathbf{P}) \ := \frac{s(P'/P)}{w} + (\frac{e'}{wd'} - \frac{e}{wd}) \tag{42}$$

is called the **(relative local) orbital self-intersection** *of the (local) release.*

For global releases $\mathbf{C}'\to\mathbf{C}$ of locally abelian orbital curves $\mathbf{C}$ we take sums over the (unique) local branches $(\mathbf{C}',\mathbf{P}')$ pulled back from $(\mathbf{C},\mathbf{P})$ along local releases:

$$h_0(\mathbf{C}'\to\mathbf{C}) \ := \sum_{\mathbf{P}'\to\mathbf{P}} h_0(\mathbf{P}'\to\mathbf{P})$$
$$h_1(\mathbf{C}'\to\mathbf{C}) \ := \sum_{\mathbf{P}'\to\mathbf{P}} h_1(\mathbf{P}'\to\mathbf{P})$$

5.8 *Relative Decomposition Law Dec(11/00)$_{ab}$.*

$$h(\mathbf{C}'\to\mathbf{C}) \ := h_1(\mathbf{C}'\to\mathbf{C}) + h_0(\mathbf{C}'\to\mathbf{C}) = \sum_{\mathbf{P}'\to\mathbf{P}} h(\mathbf{P}'\to\mathbf{P})$$

Definition 5.9 *The rational number* $h(\mathbf{C}'\to\mathbf{C})$ *is called the* **(relative) self-intersection height** *of the release* $\mathbf{C}'\to\mathbf{C}$*.*

On this way we presented us the relative Decomposition Law Dec$(11/00)_{ab}$ for releases of locally abelian orbital curves by definition. Now we are well-motivated for the next absolute Decomposition Law given by definition again:

Definition 5.10 *Decomposition Law Dec(1,0)$_{ab}$.*
Let **C** *be a locally abelian orbital curve of weight $w = w(\mathbf{C})$. The* **signature height** *of* **C** *is*

$$h(\mathbf{C}) = h_1(\mathbf{C}) + h_0(\mathbf{C}) = h_1(\mathbf{C}) + \sum_{\mathbf{P}\in\mathbf{C}} h(\mathbf{P}) \ := \frac{1}{w}(C^2) + \sum_{\mathbf{P}\in\mathbf{C}} \frac{e_P}{wd_P},$$

For a further motivation we refer to the rather immediately resulting relative degree formula (43) below for local releases.

We shift this definition now to general (finitely weighted locally releasable) orbital curves **C**. By definition, there exists a (geometrically unique minimal) locally abelian release $\mathbf{C}'\to\mathbf{C}$. Splitting **C** at each blown up point **P** into finitely many orbital branch points $(\mathbf{C}', \mathbf{P}')$ we are able to generalize the above definition to the

Definition 5.11 *Decomposition Law Dec(1,0)$_*$ for releasable orbital curves.*
Set

$$h_1(\mathbf{C}) := h_1(\mathbf{C}') = \frac{1}{w}\cdot(C'^2),$$

$$h_0(\mathbf{C}) := \sum_{\mathbf{P}\in\mathbf{C}} h(\mathbf{P}) \quad \textit{with} \quad h(\mathbf{P}) := \sum_{\mathbf{P}'\to\mathbf{P}} (h(\mathbf{P}') + \delta^{rls}_{P'}),$$

$$h(\mathbf{C}) = h_1(\mathbf{C}) + \sum h(\mathbf{P}) = \frac{1}{w}\left((C'^2) + \sum_{\mathbf{P}'\to\mathbf{P}} (\delta^{rls}_{P'} + \frac{e_{P'}}{d_{P'}})\right)$$

with the **local release branch symbol**

$$\delta^{rls}_{P'} = \delta^{rls}_{P'}(\mathbf{C}') := \begin{cases} 1, & \textit{if } P' \in E_P \cap C',\ E_P \textit{ exceptional release curve over } P), \\ 0, & \textit{else.} \end{cases}$$

We call $\mathbf{h}(\mathbf{C})$ *the* **orbital self-intersection** *of* **C**.

Remark 5.12 *If $w = w(\mathbf{C}) > 1$, then each abelian point on* **C** *is automatically an abelian cross point of* **C**. *In this case do not release* **C** *or consider the identical map as trivial release. So $\delta^{rls}_{P'} \neq 0$ appears only in the trivial weight case $w = 1$.*

5.4 Relative local degree formula for smooth releases

Proof of deg$(00/00)_{sm}$.

We start with a uniformization of a cyclic singularity P of type $< d, e >$ unramified outside P, see Example 5.5. From the covering exceptional lines

$$(E; Q', Q'') \to Q :< 1, 0 > \ \text{over}\ (L; P', P'') \to P :< d, e >$$

supporting singular points $Q'' :< e, e-1 >$ respectively $P'' :< e, -d >$ we read off:

$$h(Q'/Q) = s(Q'/Q) + \frac{0}{1} - \frac{0}{1} = -e + 0 - 0 = -e,$$
$$h(P'/P) = s(Q'/Q) + \frac{0}{1} - \frac{e}{d} = 0 + 0 - \frac{e}{d} = -\frac{e}{d},$$

hence

$$h(Q'/Q) = d \cdot h(P'/P) = [Q : P] \cdot h(P'/P). \tag{43}$$

Now we allow $\mathbf{P} \in \mathbf{C}$ to come with honest weight $w = w(\mathbf{C}) > 1$. As demonstrated in the proof of Proposition 5.6 the situation is the same as above with additional weight w at $\mathbf{C}, \mathbf{C}', \mathbf{D}, \mathbf{D}'$. So we have only to divide the above identities by w to get

$$h(\mathbf{Q}'/\mathbf{Q}) = \frac{1}{w} \cdot d \cdot h(P'/P) = \frac{1}{w} \cdot [Q : P] \cdot h(P'/P) = [Q : P] \cdot h(\mathbf{P}'/\mathbf{P}).$$

□

5.5 Degree formula for smooth coverings

Proof of $\deg(1/1)_{sm}$.

Let $\mathbf{D}/\mathbf{C}$ be a finite covering of totally smooth orbital curves. By definition of orbital finite coverings and multiplicativity of covering degrees it suffices to assume that $\mathbf{D} =: D$ is trivially weighted and $\mathbf{C} = \mathbf{D}/\mathbf{G}$ with Galois group G. The supporting surfaces Y, X of D or C, respectively, are assumed to be smooth along D or C, and the Galois covering $D \to \mathbf{C}$ is the restriction of a global Galois covering $p: Y \to \mathbf{X} = \mathbf{Y}/\mathbf{G}$. We can assume that $G = N_G(D)$ because the self-intersection of a smooth curve on a smooth surface is locally defined as degree of its normal bundle restricted to the curve. Looking at the normal bundle surfaces we can also assume that D is the only preimage of C along p. The ramification index is equal to $w = w(\mathbf{C})$, hence $p^*C = w \cdot D$. Now we apply the well-known degree formula for inverse images of curves on smooth surface coverings to our situation:

$$w^2 \cdot (D^2) = (p^*C)^2 = [Y : X] \cdot (C^2) = \#N_G(D) \cdot (C^2) = [D : C] \cdot w \cdot (C^2)$$

Division by w^2, together with the definition of the orbital self-intersection invariants and absence of singularities, yields finally

$$h(D) = [D : C] \cdot \frac{1}{w} \cdot (C^2) = (\frac{1}{w} \cdot [D : C]) \cdot (w \cdot h(\mathbf{C})) = [\mathbf{D} : \mathbf{C}] \cdot \mathbf{h}(\mathbf{C}).$$

□

5.6 The shift implications and orbital self-intersection

Proof of Implication (Imp 1), first part.
Suppose Dec(11,00), deg(00/00) to be satisfied for locally abelian orbital curves. Consider relative releases $\mathbf{D'}\to\mathbf{D}/\mathbf{C'}\to\mathbf{C}$, locally $\mathbf{Q'}\to\mathbf{Q}/\mathbf{P'}\to\mathbf{P}$, supported by relative finite orbiface coverings $\mathbf{Y'}\to\mathbf{Y}/\mathbf{X'}\to\mathbf{X}$. For degree formulas it is sufficient to consider uniformizing orbital Galois coverings $D'\to D$ of $\mathbf{C'}\to\mathbf{C}$ with trivially weighted objects D', D (omitting fat symbols) and Galois weights on $\mathbf{C'}, \mathbf{C}$.

$$\mathbf{X} = \mathbf{Y}/\mathbf{G}\ ,\ \mathbf{X'} = \mathbf{Y'}/\mathbf{G}\ ,\ \mathbf{P} = \mathbf{Q}/\mathbf{G_Q}\ ,\ \mathbf{P'} = \mathbf{Q'}/\mathbf{G_{Q'}}$$

We use notations of the following orbiface and orbital curve diagrams around orbital points.

$$\begin{array}{ccccccccccc} Y' & \longrightarrow & Y & & D' & \longrightarrow & D & & Q' & \longrightarrow & Q \\ \downarrow & & \downarrow & \text{restricting to} & \downarrow & & \downarrow & \text{restricting to} & \downarrow & & \downarrow \\ \mathbf{X'} & \longrightarrow & \mathbf{X} & & \mathbf{C'} & \longrightarrow & \mathbf{C} & & \mathbf{P'} & \longrightarrow & \mathbf{P} \end{array}$$

with vertical quotient morphisms and horizontal releases. The joining incidence diagram on the released side can be understood as locally abelian Galois diagram:

$$\begin{array}{ccccccc} Q' & \longrightarrow & D' & & Q' & \longrightarrow & D' \\ \downarrow & & \downarrow & \cong & \downarrow & & \downarrow \\ \mathbf{P'} & \longrightarrow & \mathbf{C'} & & \mathbf{Q'}/\mathbf{A}(\mathbf{Q'}) & \longrightarrow & \mathbf{D'}/\mathbf{N_G}(\mathbf{D}) \end{array}$$

Especially we restrict to work along D, D' with

$$G = N_G(D),\ \text{abelian}\ A := G_Q = G_{Q'},$$
$$[Q:P] := \frac{\#A}{\#Z_A(D)} = \frac{\#A}{w(\mathbf{C})},$$

the number of preimage points of P on D around Q w.r.t. the local A-covering $(D,Q)\to(C,P)$.
For fixed $\mathbf{P}$ it holds that

$$\sum_{D\ni Q/\mathbf{P}} [Q:P] = [G:G_Q]\cdot\frac{\#G_Q}{w} = \frac{\#G}{w} \tag{44}$$

Applying Dec(11,00), deg(00/00) and (44) we get

$$\begin{aligned} h(D'\to D) &= \sum_{Q\in D} h(Q'\to Q) = \sum_{\mathbf{P}}\sum_{Q/\mathbf{P}} h(Q'\to Q) \\ &= \sum_{\mathbf{P}}\sum_{Q/\mathbf{P}} [Q:P]\cdot h(\mathbf{P'}\to\mathbf{P}) = \frac{\#G}{w}\sum_{\mathbf{P}} h(\mathbf{P'}\to\mathbf{P}) \qquad (45) \\ &= [\mathbf{D}:\mathbf{C}]\cdot h(\mathbf{C'}\to\mathbf{C}). \end{aligned}$$

□

Proof of Implication (Imp 1), second part.
Together with the definitions of h for relative objects one gets

$$\begin{aligned} h(\mathbf{D}'/\mathbf{C}') &= h(\mathbf{D}') - [D':C'] \cdot h(\mathbf{C}') = h(\mathbf{D}') - [D:C] \cdot h(\mathbf{C}') \\ &= (h(\mathbf{D}) + h(\mathbf{D}' \to \mathbf{D})) - [D:C] \cdot (h(\mathbf{C}) + h(\mathbf{C}' \to \mathbf{C})) \\ &= (h(\mathbf{D}) + h(\mathbf{D}' \to \mathbf{D}) - [D:C] \cdot h(\mathbf{C}) - h(\mathbf{D}' \to \mathbf{D}) \\ &= h(\mathbf{D}) - [D:C] \cdot h(\mathbf{C}) = h(\mathbf{D}/\mathbf{C}). \end{aligned}$$

The degree formula deg(1/1) translates to the vanishing of relative degrees, by definition. This vanishing condition is shifted by the above identity.

□

Remark 5.13 *Via stepwise resolutions and contractions it is not difficult to extend the relations deg(11/11)$_{sm}$, deg(00/00)$_{sm}$ for smooth releases to deg(11/11)$_{ab}$, deg(00/00)$_{ab}$ for all abelian releases. Since we don't need it for the proof of deg(1/1)$_{ab}$, the proof is left to the reader.*

Proof of deg$(1/1)_{ab}$ via implication (Imp 2).
Especially we dispose on the shifting principle $(\text{Shift})_{ab}^{sm}$ for smooth releases $D'/\mathbf{C}' \to D/\mathbf{C}$ with locally abelian objects $D, \mathbf{C}$. The implication (Imp 2) is a simple application of $(\text{Shift})_{ab}^{sm}$. Since we proved already deg$(1/1)_{sm}$, this degree formula shifts now to the covering $D/\mathbf{C}$.

□

Proof of deg(1/1) via implication (Imp 3), (Imp 4) in the coniform case.

We want to shift the main orbital property deg$(1/1)_{ab}$ to orbital curves supporting honest $*$-singularities for given coniform releasable orbital curve $\hat{\mathbf{C}} \subset \hat{\mathbf{X}}$. More precisely, we have globally the following situation:

$$\begin{array}{ccccc} Y' & \longrightarrow & Y & \longrightarrow & \hat{Y} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{X}' & \longrightarrow & \mathbf{X} & \longrightarrow & \hat{\mathbf{X}} \end{array} \qquad (46)$$

with horizontal releases, vertical quotient maps by a Galois group G, $Y \to \hat{Y}$ releases all honest cone singularities, such that Y is smooth. Let $\hat{D}$ be a component of the preimage of $\hat{C}$ on $\hat{Y}$ and D its proper transform assumed to be smooth. The next release $Y' \to Y$ takes care for a smooth action of G along the proper transform D' of D by equivariant blowing up of some points of Y. Take the minimal set of such σ-processes. Locally along the orbital curves we get the following commutative diagram:

$$\begin{array}{ccccc} D' & \longrightarrow & D & \longrightarrow & \hat{D} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{C}' & \longrightarrow & \mathbf{C} & \longrightarrow & \hat{\mathbf{C}} \end{array} \tag{47}$$

Already C is smooth at (honest) $*$-points (contraction points of $X \to \hat{X}$), and C' is smooth everywhere. The surface singularities of X and X' are cyclic. All of them around C' are abelian cross points of $\mathbf{C}$ with released exceptional curves as opposite cross germ. Remember that we already defined $h(\mathbf{C})$ in 5.11 via releases $\mathbf{C}' \to \mathbf{C}$ which are unique up to weights of released exceptional curves. These weights play no role in 5.11.

We verify the degree property

$$h(D) = [D : C] \cdot h(\mathbf{C}),$$

which is sufficient for our coniform category (defining orbital curve coverings via smooth releasable coniformizations). In the case of $w > 1$ it is easy to see that the minimal release $\mathbf{C}' \to \mathbf{C}$ is the identity because all points on $\mathbf{C}$ are already abelian cross points of $\mathbf{C}$, see Remark 5.12. The degree formula is already proved. So we can assume that $w = 1$, hence

$$G_D = N_G(D) = N_G(D'),\ \#N_G(D) = \#N_G(D') = \#G_D = [D : C].$$

Using the same counting procedure as in (45) we can also assume that $\mathbf{C}' \to \mathbf{C}$ releases only one point $\mathbf{P}$. Choosing a preimage Q of P on D we have

$$h(D') = (D'^2) = (D^2) - \#G \cdot Q, \text{ hence } h(D) = h(D') + \#G \cdot Q.$$

On the other hand, from (45) we get

$$h(C) = h(C') + b_P^{rls}(C' \to C), \quad b_P^{rls}(C' \to C) = \#\{\text{(released) branches of } C \text{ at } P\}$$

This number of branches multiplied with $\#N_G(D)$ coincides with $\#G \cdot Q$:

$$b_P^{rls} \cdot \#G_D = \#G \cdot Q, \text{ hence } h(D) = h(D') + b_P^{rls} \cdot [\mathbf{D} : \mathbf{C}].$$

Now divide the latter identity by $[\mathbf{D} : \mathbf{C}]$ to get

$$\frac{h(D)}{[D : C]} = \frac{h(D')}{[D : C]} + b_P^{rls} = h(C') + b_P^{rls} = h(C),$$

which proves the degree formula $\deg(1/1)_*$.

The last shift to $\deg(1/1)$ including infinitely weighted points is simply done by definitions. Observe that for the definition of orbital self-intersections of orbital curves we never needed weights of points and of released exceptional curves. For points only the singularity types (of curves and points) were important.

Definitions 5.14 *If the orbital point* $\mathbf{R} \in \hat{\mathbf{C}} \subset \hat{\mathbf{X}}$ *is not a quotient point, then we set* $w(\mathbf{R}) = \infty$*. The same will be done for any exceptional curve* $\mathbf{E}$ *releasing* $\mathbf{R}$*:* $w(\mathbf{E}) := \infty$*.*

We break the releases $\mathbf{X} \to \hat{\mathbf{X}}$ and $\mathbf{C} \to \hat{\mathbf{C}}$ - and of its coniform Galois coverings - in the diagrams (46), (47) into two releases starting with releases $\mathbf{X}^* \to \mathbf{X}$ at infinitely weighted points. Altogether we get commutative orbital diagrams

$$\begin{array}{ccccccc} Y' & \longrightarrow & Y & \longrightarrow & Y^* & \longrightarrow & \hat{Y} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathbf{X}' & \longrightarrow & \mathbf{X} & \longrightarrow & \mathbf{X}^* & \longrightarrow & \hat{\mathbf{X}} \end{array} \tag{48}$$

$$\begin{array}{ccccccc} D' & \longrightarrow & D & \longrightarrow & D^* & \longrightarrow & \hat{D} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathbf{C}' & \longrightarrow & \mathbf{C} & \longrightarrow & \mathbf{C}^* & \longrightarrow & \hat{\mathbf{C}} \end{array} \tag{49}$$

Definition 5.15 *With the above notations, the* **signature height** *of* $\hat{\mathbf{C}}$ *is defined to be*

$$h(\hat{\mathbf{C}}) := h(\mathbf{C}^*) = \frac{1}{w}(C'^2) + \sum_{\mathbf{P}\in\mathbf{C}} h(\mathbf{P}), \quad w = w(\mathbf{C}),$$
$$h(\mathbf{P}) = \sum_{\mathbf{P}'\to\mathbf{P}} (h(\mathbf{P}') + \delta_{P'}^{rls}) = b_P^{rls} + \sum_{\mathbf{P}'\to\mathbf{P}} h(\mathbf{P}'), \quad h(\mathbf{P}') = \frac{e_{P'}}{wd_{P'}},$$

b_P^{rls} *the* **number of (released) curve branches** *of* $\mathbf{C}$ *at* P*.*

For the general degree formula (deg(1/1) there is nothing new to prove. We can restrict ourselves to Galois coverings as described in the above diagrams. Then we get

$$h(\hat{D}) = h(D^*) = [D : C] \cdot h(\mathbf{C}^*) = [D : C] \cdot h(\hat{\mathbf{C}})$$

by definition.

□

For the signature height alone it makes not much sense to introduce infinite weights because it works only with the internal curve weights $w(\mathbf{C})$. But in the next section we will introduce orbital Euler invariants working with external weights around $\mathbf{C}$. Then infinite weights will become useful.

5.7 Orbital Euler heights for curves

Let first $\mathbf{C}'$ be an orbital curve with weight w having only abelian cross points. It follows that the supporting C' is a smooth curve. We follow the proof line of the ten rules. In detail it is then not difficult to follow the proof of degree formula of the signature height for orbital curves on orbifaces in the last subsection. Notice that we distinguish in this subsection h and $\hat{h}$ for local reasons.

Definition 5.16

$$h(\mathbf{C}') := h_1(\mathbf{C}') - h_0(\mathbf{C}'),$$
$$h_1(\mathbf{C}') := eul(C') \text{ (Euler number)}, \; h_0(\mathbf{C}') := \sum_{\mathbf{P}'\in\mathbf{C}'} h(\mathbf{P}'),$$
$$h(\mathbf{P}') := 1 - \frac{1}{d_{P'}v_{P'}},$$

where $< d_{P'}, e_{P'} >$ is the type of the cyclic singularity P' and $v_{P'}$ is the weight of curve germ at P' opposite to $\mathbf{C}'$. The proof is given in [H98] by the same procedure as for orbital self-intersections through the first eight commandments. Basically, Hurwitz genus formula for the change of Euler numbers along finite curve coverings has to be applied.

Now we shift the definition as above along $\mathbf{C}' \to \mathbf{C}$ along a coniform orbital release as described in diagrams (48), (49), to the finitely weighted orbital curve $\mathbf{C}$ setting

$$h(\mathbf{C}) := h(\mathbf{C}') = h_1(\mathbf{C}) - h_0(\mathbf{C}),$$
$$h_1(\mathbf{C}) := h_1(\mathbf{C}') = eul(C'), \quad h_0(\mathbf{C}) := \sum_{\mathbf{P}\in\mathbf{C}} h(\mathbf{P}),$$
$$h(\mathbf{P}) := \sum_{\mathbf{C}'\ni\mathbf{P}'\to\mathbf{P}} h(\mathbf{P}') = b_P^{rls} - \frac{1}{v_P}\sum \frac{1}{d_{P'}}, \tag{50}$$

where v_P is the released weight of P defined as weight of the exceptional release curve $\mathbf{E}_P$ over P and b_P^{rls} is the number of exceptional curve branches of $\mathbf{C}$ at P.

(Shift)$^{ab}_*$: $\deg(1/1)_s m = \deg(1/1)_a b \Rightarrow \deg(1/1)_*$

Let $\mathbf{D}'/\mathbf{C}' \to \mathbf{D}/\mathbf{C}$ a locally abelian (coniform) release. Then

$$h(\mathbf{D}) = h(\mathbf{D}') = [D' : C'] \cdot h(\mathbf{C}') = [D : C] \cdot h(\mathbf{C}).$$

(Shift)$^*_\infty$: $\deg(1/1)_* \Rightarrow \deg(1/1) =: \deg(1/1)_\infty$.

We have only to check what happens at points R with new weight ∞. Changing to ∞ at some points we write $\hat{\mathbf{C}}$ instead of $\mathbf{C}$ and define $h(\hat{\mathbf{C}})$ as in (50) substituting the new weights ∞. So we get with obvious notations

$$
\begin{aligned}
\hat{h}(\hat{\mathbf{C}}) &= eul(C') - \sum_{\mathbf{P}\in\hat{\mathbf{C}}_{fin}} \hat{h}(\mathbf{P}) - \sum_{\mathbf{R}\in\hat{\mathbf{C}}_{\infty}} \hat{h}(\mathbf{R}) \\
&= eul(C') - \sum_{\mathbf{P}} \left(b_P^{rls} - \frac{1}{v_P} \sum_{\mathbf{P}'\to\mathbf{P}} \frac{1}{d_{P'}} \right) - \sum_{\mathbf{R}} b_R^{rls}
\end{aligned} \tag{51}
$$

defining $\hat{h}$ for orbital points and curves. In order to prove that $\hat{h}$ is orbital we have only to check the realtive local degree formula the following $\deg(00/00)^*_\infty$ over infinitely weighted points $\mathbf{R}$ for coniform coverings $D/\mathbf{C}$.

$$
\sum_{D\ni S/R} (\hat{h}(S) - h(S)) = \sum_{D\ni S/R} b_S
$$

$$
\hat{h}(\hat{\mathbf{R}}) - h(\mathbf{R}) = \frac{1}{v_R} \sum_{\mathbf{R}'\to\mathbf{R}} \frac{1}{d_{R'}}
$$

The weight $\#Z_G(D)$ of $\mathbf{C}$ doesn't play any role. So we can assume that

$$
G = N_G(D) = N_G(D') = G_D = G_{D'} = [D' : C'] = [D : C]
$$

is the Galois group acting smoothly on D', where we find all the curve branches of D at points S over R we need. With the above notations we get

$$
\sum_{D\ni S/R} (\hat{h}(S) - h(S)) = [G : G_S] \cdot b_S = [D : C] \cdot \frac{b_S}{\#G_S}
$$

$$
\hat{h}(\hat{\mathbf{R}}) - h(\mathbf{R}) = \sum_{\mathbf{R}'\to\mathbf{R}} \frac{1}{\#G_{S'}} = \sum_{i=1}^{b_P^{rls}} \frac{1}{\#G_{S'_i}},
$$

where S' is a (D-branch) point on the release curve L_S of S over $R' \in E_R = L_S/G_S$ and S'_i over R'_i after numeration. Since

$$
b_S = \sum_{i=1}^{b_P^{rls}} |G_S \cdot S'_i| = \sum_{i=1}^{b_P^{rls}} \frac{\#G_S}{\#G_{S'_i}} = \#G_S \cdot \sum_{i=1}^{b_P^{rls}} \frac{1}{\#G_{S'_i}},
$$

the relative local orbital property

$$
(\hat{h}(S) - h(S))_R \;= \sum_{D\ni S/R} (\hat{h}(S) - h(S)) = [D : C] \cdot \hat{h}(\hat{\mathbf{R}}) - h(\mathbf{R})
$$

follows immediately, and also the global one after summation over all infinitely weighted $\mathbf{R} \in \hat{\mathbf{C}}$:

$$
\begin{aligned}
\hat{h}(D) = h(D) + (\hat{h}(D) - h(D)) &= [D : C] \cdot \Big(h(\mathbf{C}) + ((\hat{h}(\mathbf{C}) - h(\mathbf{C})) \Big) \\
&= [D : C] \cdot \hat{h}(\mathbf{C}).
\end{aligned}
$$

We have to distinguish abelian points $\mathbf{P} \in \hat{\mathbf{X}}$, which will not be released along $\mathbf{X}'\to\hat{\mathbf{X}}$ and those $\mathbf{P}'$, which arise from releasing. The former appear in (50) by identifying $\mathbf{P}' = \mathbf{P}$.

Convention 5.17 *For an abelian point* $\mathbf{Q} = (\mathbf{C}, Q, \mathbf{D})$*,* C, D *crossing curve germs at* Q *with maximal weight product* $w(\mathbf{C}) \cdot w(\mathbf{D})$ *around, we set in any case*

$$w(\mathbf{Q}) := d_Q \cdot w(\mathbf{C}) \cdot w(\mathbf{D}),$$

where $< d_Q, e_Q >$ *is the cyclic singularity type of* Q*. If* $\mathbf{Q}$ *is, more distinguished, understood as abelian cross point on* $\mathbf{C}$*, then we set*

$$w_Q := w(\mathbf{C}),\ v_Q := w(\mathbf{D}),\ \textit{hence}\ w(\mathbf{Q}) := d_Q \cdot w_Q \cdot v_Q \tag{52}$$

and call v_Q *the* **opposite weight** *to* w_Q *(or to* $w(\mathbf{C})$*) at* $\mathbf{Q}$

Definition 5.18 *We call the abelian point* $\mathbf{Q}$ *on* $\hat{\mathbf{C}}$ *a* **general point** *of* $\hat{\mathbf{C}}$ *if and only if* $w(\mathbf{Q}) = w(\hat{\mathbf{C}})$*. The other orbital points on* $\hat{\mathbf{C}}$ *are called* **special**. *We use the notations* $\hat{\mathbf{C}}^{gen}$ *for the open orbital curve of general points and* $\hat{\mathbf{C}}^{sp}$ *for the complementary (orbital) cycle (or set) of special orbital points.*

Each abelian cross point $\mathbf{P}$ on $\hat{\mathbf{C}}$ yields the contribution $1 - \frac{1}{d_Q \cdot v_Q}$ in the middle sum of (51), and the general points of $\hat{\mathbf{C}}$ are precisely those with contribution 0. The summands 1 in the point contributions disappear, if we change to the open curve $\hat{C}^{gen} \cong C^{gen} \cong {C'}^{gen}$ and its Euler number:

$$\begin{aligned} \hat{h}(\hat{\mathbf{C}}) &= eul(\hat{C}^{gen}) + \sum_{\mathbf{P} \in \hat{\mathbf{C}}^{sp}} \sum_{\mathbf{P'} \to \mathbf{P}} \frac{1}{d_P \cdot v_P} \\ &= eul(\hat{C}^{gen}) + \sum_{\mathbf{P} \in \mathbf{C}^{sp}_{fin}} \sum_{\mathbf{P'} \to \mathbf{P}} \frac{1}{d_P \cdot v_P}, \end{aligned} \tag{53}$$

the latter because $v_P = \infty$ outside of the set $\mathbf{C}^{sp}_{fin}$ of finitely weighted special points.

□

We will write h_e for the orbital Euler height $\hat{h}$ and $\hat{\mathbf{h}}_{\mathbf{e}}$ for the corresponding orbital Euler invariant.

5.8 Released weights

Denote by $v = w(\mathbf{E}_P)$ the released weight of $\mathbf{P}$ defined as weight of the exceptional release curve $\mathbf{E}_P$ over P and b_P^{rls} is the number of exceptional curve branches of $\mathbf{C}$ at P, as above. The weight $w(\mathbf{E}_P)$ is uniquely determined by the coniform release. This follows from the self-intersection and Euler degree formulas applied to $L = L_Q \to \mathbf{E} = \mathbf{E}_P = L_Q/G_Q$, L_Q the releasing resolution curve of the cone singularity $Q \in D$ over P. Namely, the orbital degree formulas yield

$$0 > (L^2) = [L:E] \cdot h_\tau(L) = \frac{\#G_Q}{v} \cdot \frac{1}{v} \left((E^2) + \sum_i \frac{e_i}{d_i} \right)$$

$$2 - 2g(L) = eul(L) = [L:E] \cdot h_e(L) = \frac{\#G_Q}{v} \cdot \left(eul(E) - \sum_i (1 - \frac{1}{v_i d_i}) \right)$$

where the sum runs through the branches $\mathbf{P}'_i \in \mathbf{C}'$ of $(\mathbf{C}, \mathbf{P})$. It follows that

$$\frac{eul(L)}{(L^2)} = v \cdot \frac{eul(E) - \sum\limits_i (1 - \frac{1}{v_i d_i})}{(E^2) + \sum\limits_i \frac{e_i}{d_i}},$$

from where one gets v uniquely, if the numerators on both sides do not vanish. In the opposite case of an elliptic curve we work with the cusp weight $v = \infty$.

For uniform releases we have $L \cong \mathbb{P}^1$, $(L^2) = -1$, hence

$$-2 = v \cdot \frac{2 - \sum\limits_i (1 - \frac{1}{v_i d_i})}{-1 + \sum\limits_i \frac{e_i}{d_i}},$$

$$w(\mathbf{E}_P) = \begin{cases} 2(1 - \frac{e_1}{d_1} - \frac{e_2}{d_2})/(\frac{1}{v_1 d_1} + \frac{1}{v_2 d_2}), & \text{if } \mathbf{P} \text{ is abelian,} \\ 2(1 - \frac{e_1}{d_1} - \frac{e_2}{d_2} - \frac{e_3}{d_3})/(-1 + \frac{1}{v_1 d_1} + \frac{1}{v_2 d_2} + \frac{1}{v_3 d_3}), & \text{if } \mathbf{P} \text{ is non-abelian.} \end{cases}$$

6 Relative proportionality relations, explicit and general

Now we change notations to connect these numbers with the algebraically defined orbital invariants. We write D_Γ for the compactification of $\Gamma\backslash\mathbb{D}$ on the minimal surface singularity resolution X_Γ of the Baily-Borel compactification $\hat{X}_\Gamma$ of $\Gamma\backslash\mathbb{B}$. Since Γ is $\mathbb{D}$-neat we have $\Gamma\backslash\mathbb{D} = \Gamma_\mathbb{D}\backslash\mathbb{D}$ (smooth) and we have only to resolve the cusp singularities. In the Picard case the curve D_Γ is already smooth, but in the Hilbert case we have to release in general curve hypercusps at infinity. In any case we have a release diagram

$$\begin{array}{ccc} \overset{o}{X}_\Gamma & & \\ \downarrow & \searrow & \\ X'_\Gamma & \longrightarrow & \hat{X}_\Gamma \\ \uparrow & & \uparrow \\ D'_\Gamma & \longrightarrow & \hat{D}_\Gamma \\ \uparrow & \nearrow & \\ \overset{o}{D}_\Gamma & & \end{array}$$

with horizontal birational morphisms (releases) and vertical embeddings, closed in the middle part and open in the top and bottom parts. The only non-trivial weights are at infinity, especially D_Γ has weight 1. Therefore we get the algebraic orbital Euler height and orbital signature as volumes:

$$\begin{aligned} h_e(\mathbf{D}_\Gamma) &= eul(Reg\,\mathbf{D}_\Gamma) + \sum_{P\in D_\Gamma^\infty}\sum_{P'\to P}\frac{w(D_\Gamma)}{w(P')} \\ &= eul(\Gamma\backslash\mathbb{D}) + \sum_{P\in D_\Gamma^\infty}\sum_{P'\to P}\frac{1}{\infty} \\ &= eul(\Gamma\backslash\mathbb{D}) = vol_{EP}(\Gamma_\mathbb{D}). \end{aligned} \tag{54}$$

$$\begin{aligned} h_\tau(\mathbf{D}_\Gamma) &= \#Sing^1\overset{\circ}{\mathbf{D}} + (D_\Gamma'^2) + \sum_{P\in D_\Gamma^\infty}\sum_{P'\to P}\frac{e(P')}{d(P')} \\ &= (D_\Gamma'^2) + \sum_{P\in D_\Gamma^\infty}\sum_{P'\to P}\frac{0}{1} \\ &= (D_\Gamma'^2) = (\overline{\Gamma\backslash\mathbb{D}}^2) = \begin{cases} \frac{1}{2}\cdot vol_{EP}(\Gamma_\mathbb{D}), & \text{Picard case} \\ 1\cdot vol_{EP}(\Gamma_\mathbb{D}), & \text{Hilbert case} \end{cases} \end{aligned} \tag{55}$$

Comparing the identities (54) and (55) we come to

Theorem 6.1 *If Γ is a $\mathbb{D}$-neat arithmetic group of Picard or Hilbert modular type acting on $\mathbb{B} = \mathbb{B}$ or $\mathbb{H}^2$, respectively, then the orbital Euler and signature heights of $\mathbf{D}_\Gamma$ are in the following* ***relative proportionality relation****:*

$$h_e(\mathbf{D}_\Gamma) = \begin{cases} 2\cdot h_\tau(\mathbf{D}_\Gamma) & \textit{Picard case} \\ 1\cdot h_\tau(\mathbf{D}_\Gamma) & \textit{Hilbert case} \end{cases}$$

In the last paragraph we extended the heights to arbitrary Picard and Hilbert orbifaces satisfying the defining height rule $R.1$. With Remark 5.2 we orbitalize the heights of arithmetic curves $\mathbf{C} = \hat{\mathbf{C}}_\mathbf{\Gamma}$ or $\mathbf{C}'_\mathbf{\Gamma}$ to get the orbital Euler and self-intersection invariants setting

$$\mathbf{Eul}(\mathbf{C}) := \mathbf{h_e}(\mathbf{C}) = \frac{1}{w_\mathbf{C}}h_e(\mathbf{C}) \quad , \quad \mathbf{Self}(\mathbf{C})) := \mathbf{h}_\tau(\mathbf{C}) = \frac{1}{w_\mathbf{C}}h_\tau(\mathbf{C}).$$

They satisfy the defining orbital height rule $\mathbf{R.1}$, also called orbital degree formula (see also subsection 5.1). The Main Theorem of the article is the

Relative Orbital Proportionality Theorem 6.2 *If Γ is an arbitrary arithmetic group of Picard or Hilbert modular type acting on $\mathbb{B}$ or $\mathbb{H}^2$, respectively, then the orbital Euler and self-intersection of the orbital arithmetic curve $\mathbf{D}_\Gamma$ satisfy the following* ***relative proportionality relation****:*

$$\mathbf{Eul}(\mathbf{D}_\Gamma) = \begin{cases} 2\cdot \mathbf{Self}(\mathbf{D}_\Gamma) & \textit{Picard case} \\ 1\cdot \mathbf{Self}(\mathbf{D}_\Gamma) & \textit{Hilbert case} \end{cases}$$

7 Orbital Heegner Invariants and Their Modular Dependence

We denote by $\mathbf{Pic^2}$ and $\mathbf{Hilb^2}$ the categories of all Picard respectively Hilbert orbifaces, including releases, finite orbital coverings and open embeddings corresponding to ball lattices commensurable with a group Γ_K, K a quadratic number field. If we restrict ourselves to Baily-Borel compactifications and orbital finite coverings only, then we write $\widehat{\mathbf{Pic}}^{\mathbf{2}}$. By restrictions we get the categories $\widehat{\mathbf{Pic}}^{\mathbf{2,1}}$, $\widehat{\mathbf{Hilb}}^{\mathbf{2,1}}$, of orbital arithmetic curves on the corresponding surfaces. Disjointly joint we denote the arising category by $\mathbf{Shim}^{2,1}$ because the objects are supported by surface embedded Shimura varieties of (co)dimension 1. We admit as finite coverings only those, which come from a restriction $\mathbf{X_\Delta} \to \mathbf{X_\Gamma}$ with objects from $\mathbf{Shim}^2$, where Δ is a sublattice of Γ. The notations for the subcategories $\widehat{\mathbf{Shim}}^{2,1}$, $\overset{\mathbf{o}}{\mathbf{Shim}}^{2,1}$, $\mathbf{Shim}^{2,1,'}$ of $\mathbf{Shim}^{2,1}$, should be clear, also for the correspondence classes $\widehat{\mathbf{Shim}}_K^{2,1}$, $\overset{\mathbf{o}}{\mathbf{Shim}}_K^{2,1}$, $\mathbf{Shim}_K^{2,1,'}$ in $\mathbf{Shim}_K^{2,1}$, K a quadratic number field.

We look for further orbital invariants for orbital arithmetic curves.

$$0 \neq \mathbf{h}: \ \mathbf{Shim^{2,1}} \longrightarrow \mathbb{Q}$$

satisfying, by definition, the orbital degree formula

$$\mathbf{h}(\hat{\mathbf{D}}) = [\hat{\mathbf{D}} : \hat{\mathbf{C}}] \cdot \mathbf{h}(\hat{\mathbf{C}})$$

with orbital degree

$$[\hat{\mathbf{D}} : \hat{\mathbf{C}}] \ := \frac{w(\hat{\mathbf{D}})}{w(\hat{\mathbf{C}})} \cdot [\hat{C} : \hat{D}]$$

for orbital finite coverings $\hat{\mathbf{D}}/\hat{\mathbf{C}}$ of orbital arithmetic curves (in $\widehat{\mathbf{Shim}}^{2,1}$). For each level group Γ we dispose on the $\mathbb{Q}$-vector space $\mathbf{Div^{ar}}\hat{\mathbf{X}}_\mathbf{\Gamma}$ of orbital divisors generated by the (irreducible) arithmetic ones. The rational intersection product extends to the **orbital intersection product**

$$< \cdot >: \ \mathbf{Div^{ar}}\hat{\mathbf{X}} \times \mathbf{Div^{ar}}\hat{\mathbf{X}} \longrightarrow \mathbb{Q}$$

defined by

$$< \hat{\mathbf{C}} \cdot \hat{\mathbf{D}} > := \frac{< \hat{C} \cdot \hat{D} >}{w(\hat{\mathbf{C}})w(\hat{\mathbf{D}})}$$

for (irreducible) arithmetic curves and $\mathbb{Q}$-linear extension.

For finite orbital coverings $\mathbf{f}: \ \hat{\mathbf{Y}} \to \hat{\mathbf{X}}$ in $\widehat{\mathbf{Shim}}^2$ we dispose also on $\mathbb{Q}$-linear orbital direct and orbital inverse image homomorphisms

$$\mathbf{f}_\#: \ \mathbf{Div^{ar}}\hat{\mathbf{Y}} \longrightarrow \mathbf{Div^{ar}}\hat{\mathbf{X}}, \quad \mathbf{f}^\#: \ \mathbf{Div^{ar}}\hat{\mathbf{X}} \longrightarrow \mathbf{Div^{ar}}\hat{\mathbf{Y}}$$

Restricting to coverings of arithmetic orbital curves $\hat{\mathbf{D}}/\hat{\mathbf{C}}$, the former is basically defined by

$$\mathbf{f}_{\#}\hat{\mathbf{D}} := [\hat{\mathbf{D}} : \hat{\mathbf{C}}] \cdot \hat{\mathbf{C}}, \quad (\hat{C} = f(\hat{D})).$$

The orbital inverse image of $\hat{\mathbf{C}}$ is nothing else but the reduced preimage divisor $f^{-1}C$ endowed componentwise with the weights on $\hat{\mathbf{Y}}$. In the orbital style of writing we set

$$\mathbf{f}^{\#}\hat{\mathbf{C}} := \mathbf{f}^{-1}\hat{\mathbf{C}}.$$

In [H02] we proved the projection formula in the Picard case. The proof transfers without difficulties to the Hilbert case, because it needed only the general orbital language. So

$$< \mathbf{f}_{\#}\mathbf{B} \cdot \mathbf{A} > = < \mathbf{B} \cdot \mathbf{f}^{\#}\mathbf{A} >$$

holds for all arithmetic orbital divisors $\mathbf{B}$ on $\hat{\mathbf{Y}}$ or $\mathbf{B}$ on $\hat{\mathbf{X}}$, respectively. It follows by $\mathbb{Q}$-linear extension after proving it for arithmetic orbital curves.

Definition 7.1 *The N-th* **Heegner divisor** *H_N on $\hat{X} = \hat{X}(\Gamma)$ is the reduced (Weil-) divisor with irreducible components $\widehat{\Gamma\backslash\mathbb{D}}$, $\mathbb{D}$ a K-disc on $\mathbb{B}$ of norm $N \in \mathbb{N}_+$ with respect to a maximal hermitian $\mathfrak{O}_K$-lattice in K^3. The N-th* **orbital Heegner divisor** *$\mathbf{H}_N = \mathbf{H}_N(\Gamma) \in \mathbf{Div}^{\mathbf{ar}}\hat{\mathbf{X}}$ is the sum of the orbitalized components $\widehat{\Gamma\backslash\mathbb{D}} \subset \widehat{\Gamma\backslash\mathbb{B}}$ of H_N.*

For finite coverings $\mathbf{f} : \hat{\mathbf{Y}} \to \hat{\mathbf{X}}$ corresponding to Picard lattices $\Gamma' \subset \Gamma$ it holds that

$$\mathbf{f}^{\#}\mathbf{H}_N(\Gamma) = \mathbf{H}_N(\Gamma'),$$

This property is called the **orbital preimage invariance** of Heegner divisors along finite coverings.

Theorem 7.2 *The correspondences*

$$\mathbf{h}_N : \widehat{\mathbf{Shim}}^{2,1} \longrightarrow \mathbb{Q}, \quad \hat{\mathbf{C}} \mapsto < \hat{\mathbf{C}} \cdot \mathbf{H}_N >,$$

where $\hat{\mathbf{C}} \subset \hat{\mathbf{X}}(\Gamma)$ and $\mathbf{H}_N = \mathbf{H}_N(\Gamma)$ are taken on the same level Γ, are orbital invariants.

We use the neutral notation $\mathbf{h}$. We should denote it by $\hat{\mathbf{h}}$, and introduce $\overset{o}{\mathbf{h}}$ and $\mathbf{h}'$ by same values on corresponding open or released orbital surfaces. The reader should keep it in mind.

Proof . Let $\mathbf{f} : \hat{\mathbf{D}} \to \hat{\mathbf{C}}$ be a finite covering in $\widehat{\mathbf{Shim}}_K^{2,1}$ corresponding to $\Gamma' \subset \Gamma$, then

$$\begin{aligned} \mathbf{h}_N(\hat{\mathbf{D}}) &= < \hat{\mathbf{D}} \cdot \mathbf{H}_N(\Gamma') > = < \hat{\mathbf{D}} \cdot \mathbf{f}^{\#}\mathbf{H}_N(\Gamma) > = < \mathbf{f}_{\#}\hat{\mathbf{D}} \cdot \mathbf{H}_N(\Gamma) > \\ &= [\hat{\mathbf{D}} : \hat{\mathbf{C}}] \cdot < \hat{\mathbf{C}} \cdot \mathbf{H}_N(\Gamma) > = [\hat{\mathbf{D}} : \hat{\mathbf{C}}] \cdot \mathbf{h}_N(\hat{\mathbf{C}}). \end{aligned}$$

□

We look for a normalization of the three equal orbital invariants in the Proportionality Theorem 2.1 of Part 5 and a synchronization with the orbital Heegner invariants for establishing orbital power series.

Definition 7.3 *We call* $\mathbf{h}_0 : \widehat{\mathbf{Shim}_{\mathbf{K}}^{\mathbf{2,1}}} \to \mathbb{Q}$ *with*

$$\mathbf{h}_0(\hat{\mathbf{C}}) := \mathbf{Eul}(\hat{\mathbf{C}}) = (1 - sign\, D_{K/\mathbb{Q}})/2) \cdot \mathbf{Self}(\hat{\mathbf{C}}) = \mathbf{vol_{EP}}(\Gamma_{\mathbb{D}}) := \frac{1}{w(\hat{\mathbf{C}})} vol_{EP}(\Gamma_{\mathbb{D}})$$

for all orbital arithmetic curves $\hat{\mathbf{C}} = \widehat{\mathbf{\Gamma} \backslash \mathbb{D}}$*, the* **0-th orbital Heegner invariant***.*

We define the ***Heegner Series*** of $\hat{\mathbf{C}}$ by

$$\mathbf{Heeg}_{\hat{\mathbf{C}}}(\tau) := \sum_{N=0}^{\infty} \mathbf{h}_N(\hat{\mathbf{C}}) \cdot q^N, \; q = \exp(2\pi i N \tau), \; Im\, \tau > 0$$

Theorem 7.4 *The Heegner series are elliptic modular forms belonging to* $\mathcal{M}_3(D_{K/\mathbb{Q}}, \chi_K)$ *in the Picard case or* $\mathcal{M}_2(D_{K/\mathbb{Q}}, \chi_K)$ *in the Hilbert case of the corresponding quadratic number field* K *with discriminant* $D_{K/IQ}$*.*

A detailed explanation of the vector spaces $\mathcal{M}_k(m, \chi_K)$ of elliptic modular forms of weight k, level m and Nebentypus χ_K you find in the appendix.

Proof . We can refer to [H02] again. We used simply the orbital degree formula working simultaneously for each coefficient. We proved, that we find a $\mathbb{D}$-neat covering in any case. But then we get a Hirzebruch-Zagier series in the Hilbert case, or a Kudla-Cogdell series in the Picard case. These are elliptic modular forms of described type. The Heegner series we started with distinguish from the latter by the orbital property (orbital degree formula) for the coefficients only by a constant factor.

□

Definition 7.5 *An infinite series* $\{\mathbf{h}_N\}_{N=0}^{\infty}$ *of orbital invariants on an orbital category (or correspondence class only) is called* ***modular dependent****, if the corresponding series* $\sum_{N=0}^{\infty} \mathbf{h}_N(\hat{\mathbf{C}}) \cdot q^N$ *are elliptic modular forms of same type (weight, level, Nebentypus character) for all objects* $\hat{\mathbf{C}}$ *of the category. A countable set of orbital invariants is called* ***modular dependent****, if and only if there is a numeration such that the corresponding series is.*

Since the spaces of modular forms of same type are of finite dimension, it suffices to know the first coefficients of the series to know them completely, if the space is explicitly known. We proved

Theorem 7.6 *On each correspondence class* $\mathbf{Pic}_K^{2,1}$ *or* $\mathbf{Hilb}_K^{2,1}$ *are the corresponding orbital Heegner invariants* $\mathbf{h_N}$ *modular dependent. For each quadratic number field there is up to a constant factor only one Heegner series. The coefficients are rational numbers.*

8 Appendix: Relevant Elliptic Modular Forms of Nebentypus

We consider the congruence subgroups

$$\Gamma_0(m) \ := \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbb{S}l_2(\mathbb{Z});\ c \equiv 0 \ mod\ m\}$$

of the modular group $\mathbb{S}l_2(\mathbb{Z})$ acting on the upper half plane $\mathbb{H} \subset \mathbb{C}$. We also need characters $\chi = \chi_K : \mathbb{Z} \to \{\pm 1\}$ of quadratic number fields K. They factorize through residue class rings of the corresponding discriminants. A holomorphic functions $f = f(\tau)$, $\tau \in \mathbb{H}$, is called (elliptic) ***modular form*** of weight k, (scew) level m and Nebentypus χ, if and only if it satisfies the following functional equations:

$$f(\frac{a\tau + b}{c\tau + d}) = (c\tau + d)^k \chi(d)^k f(\tau) \quad \forall \ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(m),$$

and it must be regular at cusps. The space of these modular forms is denoted by $\mathcal{M}_k(m, \chi)$. This is a finite dimensional $\mathbb{C}$-vector space, which is O for $k < 0$. In [H02] we explained how to get

Example 8.1 . *Take weight $k = 3$, level $m = 4 = |D_{K/\mathbb{Q}}|$ and the Dirichlet character $\chi = \chi_K$ of the Gauß number field $K = \mathbb{Q}(i)$.*

$$\mathcal{M}_3(4, \chi) = \mathbb{C}\vartheta^6 + \mathbb{C}\vartheta^2\theta$$

with

$$\vartheta \ := \sum_{n \in \mathbb{Z}} q^{n^2} = 1 + 2\sum_{n>0} q^{n^2}, \qquad \textit{(Jacobi)}\ ,$$

$$\theta \ := \sum_{0<u\, odd} \sigma(u) q^u = q \cdot \prod_{m=1}^{\infty} (1 - q^{4m})^4 \prod_{n=1}^{\infty} (1 + 2q^n)^4, \qquad \textit{(Hecke)}.$$

In [H02] we explained how to get the "Heegner-Apollonius modular form" (2) in the Introduction from the extended orbital Apollonius cycle on the projective plane, visualized in picture 3.

Examples 8.2 . *Let D be the discriminant of a real quadratic number field K. Hecke [Heck] defined Eisenstein series in $\mathcal{M}_2(D, \chi_K)$ for prime discriminants. From [vdG], V, Appendix, we take more generally:*

$$\frac{1}{2}L(-1, \chi_K) + \sum_{N=1}^{\infty} \left(\sum_{0<d|N} \chi_K(d) d \right) q^N,$$

$$= \sum_{N=1}^{\infty} \left(\sum_{0<d|N} \chi_{D_1}(d) \chi_{D_2}(N/d) \cdot d \right) q^N,$$

with honest decompositions $D = D_1 \cdot D_2$ in two smaller discriminants.

Knowing dimensions of $\mathcal{M}_2(D, \chi_K)$ (see tables at the end of [vdG] and the first coefficient of the Heegner series for the Hilbert-Cartesius orbiplane ($K = \mathbb{Q}(\sqrt{(}2))$ of the introduction we come in the same manner as in the Picard-Apollonius orbiplane to the Heegner series (5).

On the orbiplanes we have a simple intersection theory. The intersection of an arbitrary plane curve with a quadric is nothing else but the double degree of the curve. In general for orbiplanes we left it as exercise for the reader to define the orbital degree **degree C** of arithmetic curves there, such that the following result holds.

Theorem 8.3 *For each orbital arithmetic curve* **C** *on an Picard or Hilbert orbiplane of the quadratic number field* K*, say, the Heegner series*

$$\mathbf{Heeg_C}(\tau) = \mathbf{Eul(C)} + \mathbf{degree\,C} \cdot \sum_{N=1}^{\infty} (\mathbf{degree\,H_N}) q^N \tag{56}$$

is an elliptic modular form belonging to $\mathcal{M}_2(D_{K/\mathbb{Q}}, \chi_K)$ *or* $\mathcal{M}_2(D_{K/\mathbb{Q}}, \chi_K)$*, respectively.*

Comparing the coefficients in (56) with those of the explicit arithmetic elliptic modular forms of Picard-Apollonius (2) and Hilbert-Cartesius (5) in the introduction we get a convenient counting of arithmetic curves sitting all in Heegner divisors with orbital degree multiplicities.

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