JGSP **47** (2018) 1–40

JOURNAL OF
Geometry and Symmetry in Physics
ISSN 1312-5192

INDEFINITE EISENSTEIN LATTICES: A MODERN BALL-RENDEVOUS WITH POINCARÉ, PICARD, HECKE, SHIMURA, MUMFORD, DELIGNE AND HIRZEBRUCH*

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Communicated by Ivaïlo M. Mladenov

Abstract. In [21] we have counted indefinite metrics (two-dimensional, integrally defined, over Gauss numbers) with a fixed norm (discriminant). We would like to call them also *indefinite class numbers*. In this article we change from Gauss to Eisenstein numbers. We have to work on the complex two-dimensional unit ball, an Eisenstein lattice on it and the quotient surface. It turns out that the compactified quotient is the complex plane $\mathbb{P}^2$. In the first part we present a new proof of this fact. In the second part we construct explicitly a Heegner series with the help of Legendre-symbol coefficients. They can be interpreted as "indefinite class numbers" we look for. Geometrically they appear also as number of plane curves with (normed) Eisenstein disc uniformization.

MSC:11F11, 11F30, 14C20, 14E20, 14G35, 14H45, 20H05, 30F45
Keywords: Eisenstein series, Fourier coefficients of modular forms, intersection numbers of curves, modular forms, Picard modular surfaces, Shimura curves, unit ball

Contents

*Dedicated to the memory of Professor Vasil V. Tsanov 1948-2017.

doi: 10.7546/jgsp-47-2018-1-40

1. Preface

It is historically interesting that Poincaré [26] extended Gauss' reduction theory of real quadratic forms to complex ones using linear transformations with integral Eisenstein or Gauss numbers as coefficients. In Bianchi's article [2] one can find a better understandable (German) version. It seems to be necessary (at least convenient) to work with Euclidean rings. For the same reason we are able to get our results for Picard modular surfaces over Gauss [21] and, respectively, Eisenstein numbers (this paper).

We consider two-dimensional submetrics of the three-dimensional hermitian indefinite unimodular diagonal metric. The latter is defined by the sesquilinear form $\langle\, ,\rangle$ with Gram matrix $Dg = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. The complex hermitian space $(\mathcal{C}^3, \langle\, ,\rangle)$ is denoted by $\mathbb{C}^{2,1}$. Let K be an imaginary quadratic number field and $\mathcal{O} = \mathcal{O}_K$ the ring of integers in it. In $\mathbb{C}^{2,1}$ sits the indefinite hermitian lattice $\mathcal{O}^{2,1} = (\mathcal{O}^3, Dg)$. Main objects of our study are indefinite two-dimensional hermitian sublattices $\mathcal{E}$ of $\mathcal{O}^{2,1}$. We restrict ourselves through the paper to fields K with class number one, mainly to the field $\mathbb{Q}(\sqrt{-3})$ of Eisenstein numbers. Then $\mathcal{E}$ and its orthogonal

$\mathcal{O}$-line $\mathcal{L} \subset \mathcal{O}^{2,1}$ have two respectively one $\mathcal{O}$-generator(s). The latter is uniquely determined up to an $\mathcal{O}$-unit factor: $\mathcal{L} = \mathcal{O}\mathfrak{l}$, where $\mathfrak{l}$ is a primitive $\mathcal{O}^{2,1}$-vector. The *norm* $n(\mathcal{E})$ is well-defined as norm $n(\mathfrak{l}) = \langle \mathfrak{l}, \mathfrak{l} \rangle$.

Let $V_- = \{\mathfrak{z} \in \mathbb{C}^{2,1};\ n(\mathfrak{z}) < 0\}$ be the set of complex negative norm vectors. The two-dimensional complex unit ball $\mathbb{B}$ consists of the negative lines in the projective plane

$$\mathbb{B} = \mathbb{P}V_- = V_-/\mathbb{C}^* \subset \mathbb{P}V = V/\mathbb{C}^* = \mathbb{P}^2.$$

We restrict the group action of $\mathbb{U}((2,1),\mathbb{C})$ on $\mathbb{B}$ to the arithmetic subgroups $\Gamma \subseteq \mathbb{U}((2,1),\mathcal{O})$ of finite index. The quotients $\Gamma\backslash\mathbb{B}$ are normal complex algebraic surfaces. The same is true also for their compactifications considered in this article. They are called *Picard modular surfaces*. Refinements and extensions you find in Subsection 2.1. For basic definitions and properties we refer to [18].

We continue with sublattices $\mathcal{E}, \mathcal{L} \subset \mathcal{O}^{2,1}$ as above, but admit all submetrics of V. Tensoring with $\mathbb{R}$ yield a hermitian plane E respectively a line L in V. Excluding positive definiteness, we get non-zero intersections $E_- = E \cap V_-$, $L_- = L \cap V_-$. Their $\mathbb{P}$-projection to $\mathbb{B}$ is a disc $\mathbb{E} = \mathbb{P}E_- \subset \mathbb{B}$ respectively a point $\lambda = \mathbb{P}L_- = \mathbb{P}L \in \mathbb{B}$. They come with discriminants and norms heritaged from $\mathcal{E}$ or $\mathcal{L}$, respectively.

In Section 2 (Subsections 2.1-2.7) we concentrate ourselves on the *Eisenstein Congruence Subgroup* (ECS) $\Gamma(\sqrt{-3}) \subset \mathbb{U}((2,1),\mathcal{O}_{\mathbb{Q}(\sqrt{-3})})$ and its ball quotient surface $X_{\Gamma(\sqrt{-3})} := \Gamma(\sqrt{-3})\backslash\mathbb{B}$ with (smallest) compactification $\widehat{X}_{\Gamma(\sqrt{-3})}$. Investigating a special system of partial differential equations Picard saw a close connection between the ball, the ECM and the complex plane. But he could not correctly prove it. It needed a long mathematical process for doing that. With strong results of Deligne and Mostow around 1970 the present author was able to prove that $\widehat{X}_{\Gamma(\sqrt{-3})} = \mathbb{P}^2$.[1] Here we give a new (more direct) proof without use of Deligne's result. The whole Section 2 is dedicated to the surface classification of $\widehat{X}_{\Gamma(\sqrt{-3})}$. An essential role plays the ramification locus of the (locally finite) covering $\mathbb{B} \to X_{\Gamma(\sqrt{-3})}$. It consists of all discs of discriminant -1. The image of them (branch locus) are six smooth curves. A preview with pictures (restricting to real points, naturally) is presented in Subsubsection 2.1.1. It gives a guideline of vision for the later proof steps.

In the Subsections 2.2-2.5 we collect and prove all details we need for the classification attacks in the last two Subsections of the first Section. There we have to verify the smoothness of the ECS $\widehat{X}_{\Gamma(\sqrt{-3})}$. But also rationality and smoothness of the irreducible branch curve components must be shown. Then the Chern invariants of surface and curves will be determined. Here we used L-series values

[1] First published together with historical background and motivations in [17].

(proved by means of higher number theory) and Proportionality Theorems, both in [18]. Last but not the least we need a celebrated Theorem of Miayaoka-Yau on classification of surfaces with extreme Chern numbers. At the end we have $\mathbb{P}^2$ with branch locus consisting of six embedded lines arranged as complete quadrilateral, see Fig. 1 in Subsection 2.1.

The main result of Section 3 (Subsections 8 -13) is Theorem 64. It counts the curves on $\mathbb{P}^2$ of Eisenstein norm N. If, for instance, $N \in \mathbb{N}_+$ is not divisible by 3 then this cardinality is equal to

$$h_N = \sum_{0<d|N} d^2 \cdot \left[\left(\frac{d}{3}\right) + \left(\frac{N/d}{3}\right) \right]. \tag{1}$$

Thereby $\left(\frac{t}{3}\right)$ is the quadratic rest symbol. A simple modification is necessary, if $3|N$, see Subsection 3.6 for details. The h_N's appear as coefficients of an Heegner series. Its explicit construction for our Eisenstein curves is the main goal of Section 3, one can say: of the whole article.

More precisely, we call the above plane curves also *Picard-Eisenstein Curves* (PEC). These are the (irreducible) $\mathbb{P}^2$-curves $\hat{C}$, which have along the ball quotient morphism $\mathbb{B} \twoheadrightarrow \Gamma(\sqrt{-3})\backslash\mathbb{B}$ a disc $\mathbb{D}$ of norm N as uniformizing preimage

$$\begin{array}{ccccc} \mathbb{B} & \twoheadrightarrow & \Gamma(\sqrt{-3})\backslash\mathbb{B} & \hookrightarrow & \mathbb{P}^2 \\ \uparrow & & \uparrow & & \uparrow \\ \mathbb{D} & \twoheadrightarrow & \Gamma(\sqrt{-3})\backslash\mathbb{D} & \hookrightarrow & \hat{C} \end{array}$$

Basic reference for Section 3 is [21], extended in [22]. In Subsections 3.1 and 3.2 we remind to the construction of "orbital invariants" on commensurability classes of Picard modular surfaces and curves. In this framework we have explained the notions of "orbital heights", "orbital curves","orbital intersection products" of them.[2] The work in the just cited papers extended Cogdell's result to all Picard modular surfaces. The Heegner series of the Picard-Gauss plane $\mathbb{P}^2 = \hat{X}_{\Gamma(1+i)}$ in the article of 2002 explicitly found. Explicit knowledge of Koblitz' work on modular forms, see [24]. In the mean time we discovered in the work of Erich Hecke (1887 - 1947) explicit constructions of Eisenstein series $\in M_k(n, \chi)$, see [12–14]. They were of type we look for: weight k, level n, discriminant-Nebentypus χ, see Subsection 3.5.

[2] Historical starting point was Cogdell's thesis publication [5], dedicated to neat Picard modular congruence subgroups, where the connection with elliptic modular forms of Nebentypus was found.

For the field $K = \mathbb{Q}(\sqrt{-3})$, of Eisenstein numbers with Nebentypus-character $\chi_K(n) = \left(\frac{-3}{n}\right)$ the relevant space $M_3(3, \chi_K)$ is two-dimensional. For a proof we have used a dimension-formula of Oesterlé-Cohen [6]. On the other hand Hecke's articles present two linearly independent series in $M_3(3, \chi_K)$. So, the latter space of modular forms is generated by two explicitly known Hecke-Eisenstein series E_1, E_2 (reproduced at the end of Subsection 3.5. With our combination of arithmetic-geometric methods we calculated the first two coefficients h_1 and h_2 in the general formula (1). Since the Heegner series belongs also to $M_3(3, \chi_K) = \mathbb{C}E_1 + \mathbb{C}E_2$ we can determine precisely its E_1, E_2- linear combination (see Theorem 64 again).

2. The Eisenstein Congruence Surface

2.1. Introduction

Let $\mathbb{P}^k = \mathbb{P}^k_{\mathbb{C}}$ denotes the k-dimensional complex projective space. $K = \mathbb{Q}(\sqrt{-d})$ is an imaginary quadratic number field, $d \in \mathbb{N}_+$, squarefree. $\mathcal{O}_K$ denotes the ring of integers in K, $h(K)$ the class number of K. The modules/spaces $\mathcal{O}^{n+1}$, K^{n+1}, $\mathbb{C}^{n+1}$, will be endowed with the unimodular indefinite hermitian metric $\langle . , . \rangle$ with Gram diagonal matrix

$$\begin{pmatrix} +1 & 0 & . & . & . & 0 & 0 \\ 0 & +1 & 0 & . & . & 0 & 0 \\ & & & \cdots & & & \\ & & & \cdots & & & \\ & & & \cdots & & & \\ 0 & & . & . & . & 0 \; +1 & 0 \\ 0 & & . & . & . & . \; 0 & -1 \end{pmatrix}$$

w.r.t. to the canonical basis. The corresponding hermitian $\mathcal{O}$-modules/spaces are denoted by $\mathcal{O}^{n,1}$, $K^{n,1}$ or $\mathbb{C}^{n,1}$, respectively.

Let generally V be an $(n+1)$-dimensional hermitian $\mathbb{C}$-vector space of signature $(n, 1)$. Looking at norms w.r.t. hermitian sesquilinear form $h = \langle ., . \rangle$ we will speak about negative, positive or cusp vectors v, if the norm $n(v) = \langle v, v \rangle$ is negative, positive, or zero, respectively. We introduce the notations

$$V_- = \{v \in V;\ n(v) < 0\}, V_+ = \{v \in V;\ n(v) > 0\}, V_0 = \{v \in V;\ n(v) = 0\}.$$

The complex lines in V through the 0-point will be identified with the points of the n-dimensional projective space $\mathbb{P}^n$. The negative vectors project along $V \to$

$\mathbb{P}^n = \mathbb{P}V$ (keep in mind that you have to exclude the O-vector in V) onto the n-dimensional (complex unit) ball $\mathbb{B}^n$. The projection maps V_0 onto the boundary $\partial\mathbb{B}$ of the ball. We will always assume, that V is generated by a hermitian K-space V_K with hermitian product h_K, K as above. Notice that $V = V_{\mathbb{C}} = V_K \otimes \mathbb{R}$, $h_K : V_K \times V_K \to K$.

From now on we concentrate ourselves to the second dimension. We call $\mathbb{B} := \mathbb{B}^2$ the (complex two-dimensional) unit ball. With $\mathbb{D}$ we denote generally a complete linear subdisc of $\mathbb{B}$. It is defined as non-void intersection of complex projective line L with our fixed ball $\mathbb{B}$. Consider the projective projections

$$\begin{array}{ccc} V_- & \hookrightarrow & V \\ \downarrow & & \downarrow \\ \mathbb{B} & \hookrightarrow & \mathbb{P}^2 \end{array} \tag{2}$$

again (where V is three-dimensional complex vector space. A (complete linear) subdisc $\mathbb{D}$ of our fixed ball $\mathbb{B}$ is a non-void intersection of a projective line L with $\mathbb{B}$. Look at the lower rectangle of the following diagram

$$\begin{array}{rccc} & E & \hookrightarrow & V \\ & \downarrow & & \downarrow{\scriptstyle p} \\ & L & \hookrightarrow & \mathbb{P}^2 \\ & \uparrow & & \uparrow \\ \mathbb{B} \cap L = & \mathbb{D} & \hookrightarrow & \mathbb{B} \end{array} \tag{3}$$

In the upper part E denotes a subplane of V projecting projectively onto the line L along p.

Definition 1. *$\mathbb{D}$ is called a K-disc, iff the line embedding in the above diagram is defined over K. It's the same to say, that the L covering the plane E is a K-defined subplane of V. Equivalently*: *one has two different K-points on L* (*or on* $\mathbb{D}$).

The action of unitary group $\mathbb{U}((2,1),\mathbb{C}) \subset \mathbb{G}L_3(\mathbb{C})$ on $V \cong \mathbb{C}_3$ is compatible with the hermitian structure. Therefore it induces an action on V_-. Dividing out the ineffectively acting central diagonal subgroup, the unitary action goes down to that of $\mathbb{P}\mathbb{U}(2,1)$ on our ball. The corresponding actions are called fractional transformations.

Now fix the imaginary quadratic number field K, also $\mathcal{O} = \mathcal{O}_K$.

Definition 2. *The arithmetic group* $\Gamma^{2,1} = \mathbb{U}((2,1),\mathcal{O})$ *is called the* full Picard modular group *of the field* K. *Any subgroup* Γ *of* $\mathbb{U}((2,1),K)$*, commensurable with* $\Gamma^{2,1}$*, will be called* Picard modular group (*or* Picard modular lattice).

The well-known (e.g. from [18] and/or references there) are the following

Facts 3. *The quotient surface* $\Gamma\backslash\mathbb{B}$ *is an* (*open, t.m. non-compact*) *normal complex algebraic surface. The same is true for its Baily-Borel compactification* (*BB*) $\widehat{\Gamma\backslash\mathbb{B}}$*. Both types are called Picard modular surfaces. Adding finitely many* (*normal*) *points one gets the BB compactification from the open model. These points are called cusp points or cusp singularities.*

With the notations

$$\partial_K\mathbb{B} = \partial\mathbb{B} \cap K^2, \qquad \widehat{\mathbb{B}} = \mathbb{B} \cup \partial_K\mathbb{B}$$

we get the inclusion diagram with vertical surjections

$$\begin{array}{ccc} \mathbb{B} & \hookrightarrow & \widehat{\mathbb{B}} \\ \big\downarrow & & \big\downarrow \\ \Gamma\backslash\mathbb{B} & \hookrightarrow & \widehat{\Gamma\backslash\mathbb{B}} \end{array}$$

Most important Picard modular subgroups of Γ (as above) are the congruence subgroups of $\mathcal{O}$-ideals $\mathfrak{a}$. They are defined as

$$\Gamma(\mathfrak{a}) = \{\gamma \in \Gamma;\ \gamma \equiv E \mod \mathfrak{a}\}$$

where E is the unit matrix of order three.

We would like to define arithmetic curves on Picard modular surfaces. For this purpose consider a K-disc $\mathbb{D}$ in $\mathbb{B}$ (in the sense of Definition 1) together with a Picard modular lattice Γ. The subgroup of all its elements γ acting on $\mathbb{D}$ itself is denoted by $N_\Gamma(\mathbb{D})$. It is called the *normalizer group* of Γ and $\mathbb{D}$. Dividing out the ineffective (on $\mathbb{D}$ acting) kernel $Z_\Gamma(\mathbb{D})$ we get the effective on $\mathbb{D}$ acting Fuchsian group $\Gamma_\mathbb{D} := N_\Gamma(\mathbb{D})/Z_\Gamma(\mathbb{D})$. The quotient curve $\Gamma\backslash\mathbb{D}$ is an (sometimes open, sometimes compact) algebraic curve sitting on $\Gamma\backslash\mathbb{B}$. In general, there exist curve singularities on it. Its closure on any compactification $\overline{\Gamma\backslash\mathbb{B}}$ of $\Gamma\backslash\mathbb{B}$ is denoted by $\overline{\Gamma\backslash\mathbb{D}}$. The resolution of singularities of this curve is nothing else but the BB-compactified quotient curve $\widehat{\Gamma_\mathbb{D}\backslash\mathbb{D}}$. Altogether we illustrate the situation in the following commutative diagram of embeddings and surjections

$$
\begin{array}{ccccccc}
\widehat{\mathbb{D}} & \hookleftarrow & \mathbb{D} & \hookrightarrow & \mathbb{B} & \hookrightarrow & \widehat{\mathbb{B}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Gamma_{\mathbb{D}}\backslash\widehat{\mathbb{D}} & \hookleftarrow & \Gamma\backslash\mathbb{D} & \hookrightarrow & \Gamma\backslash\mathbb{B} & \hookrightarrow & \widehat{\Gamma\backslash\mathbb{B}} \\
& \searrow^{\text{desing}} & \downarrow^{\text{closure}} & \searrow & \downarrow & & \uparrow^{\text{closure}} \\
& & \overline{\Gamma\backslash\mathbb{D}} & \twoheadrightarrow & \overline{\Gamma\backslash\mathbb{B}} & & \widehat{\Gamma_{\mathbb{D}}\backslash\mathbb{D}}
\end{array}
\qquad (4)
$$

where $\widehat{\mathbb{D}} := \mathbb{D} \cup \partial_K \mathbb{D}$ with the set $\partial_K\mathbb{D} = \partial\mathbb{D} \cap_K \mathbb{B}$ of K-rational boundary points of our disc.

By the way, the K-boundary points of $\mathbb{D}$ or $\mathbb{B}$ are called cusps of these objects. Their image points along the left or right vertical arrows of the above diagram are called *cusp points* of the curve or surface, respectively.

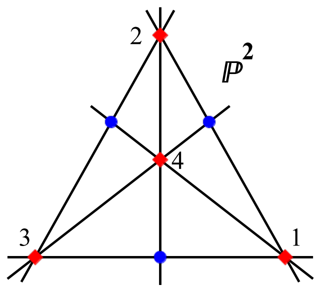
Definition 4. *Let $\mathbb{D}$ be a K-disc, Γ a Picard modular group of K and $\Gamma\backslash\mathbb{B}$ the corresponding Picard modular surface. The image curve $\Gamma\backslash\mathbb{D}$ on the surface will be called a K-*arithmetic curve *or* Picard modular curve (*PM-curve*). *We use the same notation for the compactifications of this curve on the surfaces appearing in* (4).

2.1.1. Preview: Picard Modular Surfaces of Eisenstein Numbers

We fix now $K = \mathbb{Q}(\sqrt{-3})$, known as field of Eisenstein numbers. The ring of (integral) Eisenstein numbers is the unique factorization domain $\mathcal{O} = \mathcal{O}_K = \mathbb{Z} + \mathbb{Z}\omega$, where $\omega = \mathrm{e}^{\frac{2\pi \mathrm{i}}{3}}$ is one of the two primitive third unit roots. Main objects of this paper will be the congruence subgroup $\Gamma(\sqrt{-3}) \ := \Gamma^{2,1}(1-\omega)$ of the principal $\mathcal{O}$-ideal $(1-\omega)$ generated by the $\mathcal{O}$-prime element $1-\omega$ of norm three.

Theorem 5 ([17]). *The BB compactification $\widehat{\Gamma(\sqrt{-3})\backslash}\mathbb{B}$ is the projective complex plane $\mathbb{P}^2$. There are precisely four cusp points on it. The six lines through pairs of them is the* (*compactified*) *branch locus of the quotient morphism $\mathbb{B} \to \Gamma(\sqrt{-3})\backslash\mathbb{B}$.*

We visualize the (compactified) branch curve in Fig. 1 below (real part). The configuration is known as *complete quadrilateral* on the complex plane $\mathbb{P}^2$. The ramification locus on $\mathbb{B}$ consists of infinitely many discs: the $\Gamma(\sqrt{-3})$ - orbit of the six ramification discs drawn in Fig. 2 below (real cut). Representative $\Gamma(\sqrt{-3})$-cusps, covering (all four) plane cusp points along the cusp-extended quotient morphism (see Diagram 4), are marked by little red diamonds in the pictures.



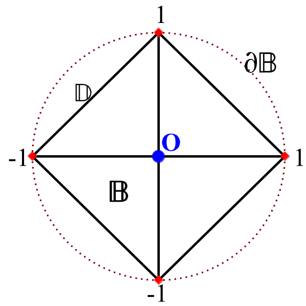


Figure 1. Branch locus Quadrilateral.

Figure 2. Covering ramification discs.

We fix a numeration (1,2,3,4) of the cusp points. It allows us to denote each branch line by the two cusp points lying on it. For instance $L_{1,2} = L_{2,1}$ is the line through the points "1" and "2".

Corollary 6. *The branch/ramfication index at each of the six lines/discs is equal to three.*

Here the numbers ± 1 are nothing else but the x-coordinates on the horizontal disc or the y-coordinates on the vertical one, respectively. More precisely, we marked the cusps by (affine) coordinates

$$\kappa_1 = (1,0), \qquad \kappa_2 = (0,1), \qquad \kappa_3 = (-1,0), \qquad \kappa_4 = (0,-1). \tag{5}$$

The disc joining the two cusps κ_i and κ_j, $1 \le i < j \le 4$, will be denoted by $\mathbb{D}_{i,j}$. Blue points represent ($\mathbb{R}$-visible) intersection points of two ramification discs. Also their projections along the quotient morphism $\mathbb{B} \to \Gamma(\sqrt{-3})\backslash\mathbb{B} = \mathbb{P}^2$ are marked by the same color. On $\mathbb{P}^2$ (all) three of them are $\mathbb{R}$-visible, drawn in Fig. 1.

Remark 7. *In this paper we will give a new proof of the statements of this Subsection. It is - other than in* [17] *- essentially supported by the branch locus. Emmy Noether emphasized a century before today the important role of ramifications. Our aim is to improve it geometrically in dimension two.*

The proof will be finished at the end of Section 2.7.

Remark 8. *With*

$$G=\begin{pmatrix}\frac{\sqrt{-3}}{2} & 0 & -\frac{\sqrt{-3}}{2}\\ 0 & 1 & 0\\ 1 & 0 & 1\end{pmatrix},\qquad N=\begin{pmatrix}0 & 0 & -\frac{1}{\sqrt{-3}}\\ 0 & 1 & 0\\ \frac{1}{\sqrt{-3}} & 0 & 0\end{pmatrix}$$

the hermitian spaces $\mathbb{C}^{2,1}=(\mathbb{C}^3,Dg)$ *and* $(\mathbb{C}^3,N)$ *are isometric. Namely the Gram matrices* Dg *and* N *are conjugated to each other*: ${}^t\overline{G}\cdot N\cdot G=Dg$. *It defines the isometry*

$$\Theta: V=\mathbb{C}^{2,1}\xrightarrow{\sim}(\mathbb{C}^3,N)=V',\qquad \mathfrak{c}\mapsto G\mathfrak{c}.$$

It induces an isomorphism of unitary groups

$$\mathbb{U}((2,1),\mathbb{C})=\mathbb{U}(Dg,\mathbb{C})\xrightarrow{\sim}\mathbb{U}(N,\mathbb{C}),\qquad A\mapsto G^{-1}AG.$$

It restricts to the unitary subgroups with coefficients in $K=\mathbb{Q}(\sqrt{-3})$. *The image of the Picard modular Eisenstein lattice* $\Gamma(\sqrt{-3})$ *lands in the commensurability class of* $\Delta=\mathbb{U}(N,\mathcal{O})$, *say* $\Gamma'=G^{-1}\Gamma(\sqrt{-3})G$.

Cogdell established the Heegner series for neat principal congruence subgroups $\Delta(\alpha),\alpha\in\mathcal{O}_K$. *How Heegner's construction extends to all members of the commensurability class of* Δ *is explained in* [21, 22]. *We pull the pair* (V',Γ') *back along the isometry* Θ *to* $(V,\Gamma(\sqrt{-3}))$. *Along this way we get the same Heegner series for* $\Gamma(\sqrt{-3})$ *and* G'. *We get also the same quotient surface* $\mathbb{P}^2$ *for both arithmetic groups. Also the plane curve interpretations are the same.*

2.2. Unimodular Sublattices

We fix an imaginary quadratic number field K, for simplicity of class number one. Moreover, the ring $\mathcal{O}=\mathcal{O}_K$ of K-integers is assumed to be a unique factorization domain (Gauss and Eisenstein numbers belong to this class). For any hermitian $\mathcal{O}$-sublattice Λ of $\mathcal{O}^{2,1}$ of $\mathcal{O}$-rank $1,2$ or 3 we define the dual lattice as $\mathcal{O}$-module

$$\Lambda^{\#}=\{\mathfrak{x}\in\Lambda\otimes K\ ;\ \langle\mathfrak{x},\mathfrak{l}\rangle\ \in\mathcal{O},\text{for all }\mathfrak{l}\in\Lambda\}.$$

Notice that $\Lambda\subseteq\Lambda^{\#}$ with equality if and only if Λ is unimodular. Equivalently: the discriminant $\mathrm{dcr}(\Lambda)$ is equal to ±1. Thereby the discriminant of Λ is defined to be the determinant of the Gram-Matrix of an $\mathcal{O}$-basis of Λ with respect to the hermitian form. It is defined up to multiplication with a norm unit of $\mathcal{O}$.

Two subsets M,N of $K^{2,1}$ are said to be *orthogonal*, iff $\langle m,n\rangle=0$ for all $m\in M,n\in N$. We write $M\perp N$ in this case. Orthogonal direct sums of lattices are written as $M \boxdot N$. The orthogonal complement of $M \subseteq \mathcal{O}^{2,1}$ in $\mathcal{O}^{2,1}$ is the sublattice

$$M^{\perp} = \{\mathfrak{n} \in \mathcal{O}^{2,1}\ ;\ \mathfrak{n} \perp M\}.$$

Obviously, the discriminants of orthogonal lattices $M, N \subseteq \mathcal{O}^{2,1}$ behave multiplicatively

$$\mathrm{dcr}(M \boxdot N) = \mathrm{dcr}(M) \cdot \mathrm{dcr}(N). \tag{6}$$

Two sublattices M, N of Λ are called *orthogonal complementary* (in $\mathcal{O}^{2,1}$), iff $M \cap N = \{\mathfrak{o}\}$, $M^{\perp} = N$ and $N^{\perp} = M$.

Fact 9. **([20, Proposition 6.1]).** *If M and N are orthogonal complementary sublattices of $\mathcal{O}^{2,1}$, then $M^{\#}/M \cong N^{\#}/N$ as $\mathcal{O}$-modules.*

Corollary 10. *Under the above conditions it holds that*: *M is unimodular if and only if N is unimodular.*

Corollary 11. *Let M be an indefinite* rank-2 *sublattice of $\mathcal{O}^{2,1}$ and G its orthogonal complement in $\mathcal{O}^{2,1}$, say $G = \mathcal{O}\mathfrak{c}$. Then $-\mathrm{dcr}(M) = n(\mathfrak{c}) > 0$.*

Hint. Express $M^{\#}\!: M$ in terms of the discriminant of M, do the same with $G^{\#} : G$ and compare.

□

Examples 12. *A central role will play the following six norm one vectors $\mathfrak{d}_{i,j}$ together with their orthogonal planes $E_{i,j}$, $1 \leq i < j \leq 4$, (of discriminant -1)*

$$\begin{array}{lll} E_{1,2} \perp \mathfrak{d}_{1,2} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, & E_{1,3} \perp \mathfrak{d}_{1,3} = t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, & E_{1,4} \perp \mathfrak{d}_{1,4} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \\ E_{2,3} \perp \mathfrak{d}_{2,3} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, & E_{2,4} \perp \mathfrak{d}_{2,4} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, & E_{3,4} \perp \mathfrak{d}_{3,4} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}. \end{array} \tag{7}$$

The unimodular plane $E_{i,j}$ is generated by the i-th and j-th column C_i, C_j, respectively, of the following "cusp matrix" (all 4 columns are cusp vectors)

$$C = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix}. \tag{8}$$

The projective projections of the C_i (see Diagram 3, Section 2.1 are the four cusps $\kappa_i = \mathbb{P}C_i$, $i = 1 \dots 4$. We visualized them in Fig. 2 at the end of Section 2.1. Consequently, the $\mathbb{P}$-projections of the planes $E_{i,j}$ are six projective lines $L_{i,j}$ whose intersections with the ball $\mathbb{B}$ are the discs $\mathbb{D}_{i,j}$ drawn in Fig. 2.

2.3. Counting Special Points

2.3.1. Cusp Points on $\widehat{\Gamma\backslash\mathbb{B}}$

We remind the following

Theorem 13 (Feustel [9], elegant proof in [29]). *Let $\Gamma_K^{2,1}$ be the full Picard modular group of an arbitrary imaginary quadratic number field K. Then the quotient surface $\widehat{\Gamma_K^{2,1}\backslash\mathbb{B}}$ has precisely $h(K)$ cusp points.*

We concentrate us to the field of Eisenstein numbers again, omitting some indices. E.g. write simply Γ for the full Picard modular group $\mathbb{U}((2,1),\mathcal{O})$, where $\mathcal{O} = \mathbb{Z}[\omega]$ is the ring of integral Eisenstein numbers.

Lemma 14. *The factor group $\Gamma/\Gamma(\sqrt{-3})$ is isomorphic to the (doubled) symmetric group $\pm S_4$. It is geometrically represented as orthogonal group $\mathbb{O}((2,1),\mathbb{F}_3)$, where $\mathbb{F} = \mathbb{F}_3$ denotes the finite field $\mathcal{O}/(1-\omega)$ consisting of three elements $0, +1, -1$, say.*

Proof: The action of Γ on $\mathcal{O}^{2,1}$ goes modulo $(1-\omega)$ down to an orthogonal action on $\mathbb{F}^{2,1}$ with ineffective kernel $\Gamma(\sqrt{-3})$. So we have an embedding

$$\Gamma/\Gamma(\sqrt{-3}) \hookrightarrow \mathbb{U}((2,1),\mathbb{F}) = \mathbb{O}((2,1),\mathbb{F}) \cong \pm S_4. \tag{9}$$

It is well-known that the projective group $\mathbb{P}\mathbb{O}((2,1),\mathbb{F})$ is isomorphic to the symmetric group S_4, see e.g. [7]. It appears as permutation group of the $\mathbb{F}$-points

$$(1:0:1), \qquad (0:1:1), \qquad (-1:0:1), \qquad (0:-1:1) \tag{10}$$

which are images along the (projectivized) residue map $\mathcal{O}^3 \to \mathbb{F}^3$ of the cusp vectors sitting in the cusp matrix C in (8). In affine coordinates the corresponding $\Gamma(\sqrt{-3})$-cusp points are listed in (5) and drawn in Fig. 2. The Γ-elements

$$\Sigma_{12} = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ -2 & -2 & -3 \end{pmatrix}, \quad \Sigma_{13} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \Sigma_{14} = \begin{pmatrix} -1 & 2 & -2 \\ 2 & -1 & 2 \\ 2 & -2 & 3 \end{pmatrix}$$

act via residue map on the $\mathbb{F}$-cusp vectors $C_i \mod 3$ as S_4-transpositions (12), $(1,3), (1,4)$ (up to sign). These three elements generate S_4. Therefore $\Gamma/\Gamma(\sqrt{-3}))$ $\cong \pm S_4$, hence, the inclusion (9) is the identical map. The Lemma is proved. ■

Corollary 15. *The Eisenstein congruence surface* $\widehat{\Gamma(\sqrt{-3})\backslash\mathbb{B}}$ *has precisely four cusp points. Lifted representants on* $\partial\mathbb{B}$ *are the* (*red*) $\mathbb{B}$*- boundary points drawn in* Fig. 1.

Proof: From Feustel's Theorem 13 we know that the cusp orbit of the full Picard modular group Γ consists of only one element, take $\Gamma\kappa_1$. The preimage of this cusp point on the congruence surface are the (projectivized) orbits

$$\Gamma(\sqrt{-3})\begin{pmatrix}1\\0\\1\end{pmatrix}, \quad \Gamma(\sqrt{-3})\begin{pmatrix}0\\1\\1\end{pmatrix}, \quad \Gamma(\sqrt{-3})\begin{pmatrix}-1\\0\\1\end{pmatrix}, \quad \Gamma(\sqrt{-3})\begin{pmatrix}0\\-1\\1\end{pmatrix}$$

see (10) as well as Figs. 1 and 2. ■

2.3.2. $\mathbb{B}$-points of Maximal Negative Norm -1

Each K-line $L = K\mathfrak{a} \in K^3$ can be generated by a primitive vector $\mathfrak{p} \in \mathcal{O}^3$, which is unique up to multiplication with a 6-th unit root.

Definition 16. *With these notations, the* norm $N = n(L)$ *is uniquely defined as* $n(\mathfrak{p}) \in \mathbb{Z}$. $\mathfrak{p}$ *is called a* norm N vector. *If the norm is negative, then* $p = \mathbb{P}\mathfrak{p} = L$ *is a ball point. We endow* p *and also its image point* P *on the Eisenstein congruence surface with the same* norm $n(P) = n(p) = n(\mathfrak{p}) = N$. *We speak then also about* norm N points p *or* P.

It is clear that only K-points with integral negative norms exist on the ball $\mathbb{B}$ and on the Picard modular quotient surfaces.

Proposition 17. *The* -1 *vectors in* $\mathcal{O}^{2,1}$ *fill precisely one* Γ*-orbit, namely* $\Gamma\begin{pmatrix}0\\0\\1\end{pmatrix}$. *This orbit splits into three* $\Gamma(\sqrt{-3})$*-orbits.*

Proof: Otremba proved in [25] that, especially for Eisenstein numbers, there is only one isometry class of positive definite unimodular hermitian $\mathcal{O}$-lattices of rank two, see also Hashimoto [11]. Take the simplest one E generated by the first two of the canonical $\mathcal{O}^3$-basis vectors

$$\mathfrak{e}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \mathfrak{e}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad \mathfrak{e}_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

If $\mathfrak{c}$ denotes an arbitrary norm -1 vector in $\mathcal{O}^{2,1}$, then its orthogonal complement there, denoted by F, is a rank 2 sublattice of $\mathcal{O}^{2,1}$ with discriminant $+1$ by Corollary 11. According to Otremba/Hashimoto there are lattice isomorphisms

$$E \cong F, \qquad \mathcal{O}\mathfrak{e}_3 \cong \mathcal{O}\mathfrak{c}, \qquad \mathcal{O}^{2,1} = E \boxdot \mathcal{O}\mathfrak{e}_3 \cong F \boxdot \mathcal{O}\mathfrak{c}.$$

So we extended the second isometry to a rank three one $\gamma \in \Gamma$, hence $\mathfrak{c} \in \Gamma\mathfrak{e}_3$.
For the second statement we must check the orbit $\Gamma\mathfrak{e}_3$ modulo $\Gamma(\sqrt{-3})$. We deal with the action of the factor group S_4 on the residue space $\mathbb{F}^{2,1}$. The Klein's four group $K4 \subset S_4$ fixes $\mathfrak{e}_3$. The factor group $S_3 \cong S_4/K4$ moves effectively this vector. Geometrically, on the Eisenstien congruence surface, there are three moved points, visualized in Fig. 1 by blue bullets. Pulling them back to the ball we get the explicit splitting

$$\Gamma\mathfrak{e}_3 = \pm\Gamma(\sqrt{-3})\begin{pmatrix}0\\0\\1\end{pmatrix} \sqcup \pm\Gamma(\sqrt{-3})\begin{pmatrix}1\\1\\1-\omega\end{pmatrix} \sqcup \pm\Gamma(\sqrt{-3})\begin{pmatrix}1\\-1\\1-\omega\end{pmatrix}.$$

Namely, the three -1 vectors inside are not pairwise $\Gamma(\sqrt{-3})$-equivalent. ■

2.3.3. Norms of K-Discs and Their Quotient Curves

We denote by $\mathcal{E}$ now an indefinite rank-two sublattice of $\mathcal{O}^{2,1}$. Its orthogonal complement in $\mathcal{O}^{2,1}$ is an $\mathcal{O}$-line $\mathcal{L}$. We know that $n(\mathcal{L}) = -\mathrm{dcr}(\mathcal{E}) > 0$ by Corollary 11. The complex plane $E = \mathbb{R} \otimes \mathcal{E}$ defines the K-disc $\mathbb{D} = \mathbb{P}E \cap \mathbb{B}$ in the ball. Similarly, we set $L = \mathbb{R} \otimes \mathcal{L}$.

Definition 18. *With the above notations we call* $n(E) := n(\mathcal{E}) =: n(\mathbb{D})$ *the* norm of the K-disc $\mathbb{D}$. *Also the image curve* $\Gamma(\sqrt{-3})\backslash\mathbb{D}$ (*as well any closure/compactification*) *on the Eisenstein Congruence Surface will be endowed with the same norm.*

Observe, that conversely a given K-disc $\mathbb{D}$ determines uniquely the line L as projective (algebraic) closure of the disc in the projective plane $\mathbb{P}^2 \supset \mathbb{B}$. Furthermore this line is uniquely lifted to the plane $E \subset V$, see Diagram 3 in Section 2.1. Moreover, we get with $\mathcal{E} = E \cap \mathcal{O}^{2,1}$ the corresponding indefinite rank 2 sublattice of $\mathcal{O}^{2,1}$. Its absolute discriminant is the norm of $\mathbb{D}$ (see Corollary 11). Moreover, given an arithmetic curve C on the Eisenstein Congruence Surface, say $C = \Gamma(\sqrt{-3})\backslash\mathbb{D}$, then the covering K-disc $\mathbb{D}$ is determined up to $\Gamma(\sqrt{-3})$-equivalence. But the disc-norm is stable under $\Gamma(\sqrt{-3})$-transformations. So also the *curve norm* $n(C)$ is well-defined.

Of special interest are the discs and their quotient curves of norm $+1$. Parallel to Proposition 17 we have

Proposition 19. *The $+1$ vectors in $\mathcal{O}^{2,1}$ fill precisely one Γ-orbit. (For simplicity you can take $\Gamma\mathfrak{e}_1$). This orbit splits into six $\Gamma(\sqrt{-3})$-orbits, generated e.g. by the following six norm 1 vectors (arranged in three orthogonally intersecting pairs)*

$$\begin{pmatrix}1\\1\\1\end{pmatrix}\perp\begin{pmatrix}\omega\\\omega^2\\-1\end{pmatrix},\quad\begin{pmatrix}0\\1\\0\end{pmatrix}\perp\begin{pmatrix}1\\0\\0\end{pmatrix},\quad\begin{pmatrix}1\\-1\\1\end{pmatrix}\perp\begin{pmatrix}-1\\\omega\\\omega^2\end{pmatrix}. \tag{11}$$

Proof: Obviously, they are not pairwise $(1-\omega)$-congruent to each other. Therefore they generate different $\Gamma(\sqrt{-3})$-orbits of norm one vectors. The symmetric group S_4 acts on their residue classes modulo $(1-\omega)$ with ineffective kernel $K4$. The residue vectors coincide with the six $\mathbb{F}$-vectors $\mathfrak{d}_{i,j}\in\mathbb{F}^{2,1}$, $1\le i<j\le 4$ listed in (7). There are no more one-vectors in $\mathbb{F}^{2,1}$. Therefore the six ones in (11) form a complete set of representatives of norm one vectors modulo $\Gamma(\sqrt{-3})$. ■

Remark 20. *Each of the three ortho-pairs indicated in (11) generate a positive definite subplane in $\mathcal{O}^{2,1}$ of minimal discriminant $+1$. They have orthovectors of norm -1, namely*

$$\begin{pmatrix}\omega\\-\omega^2\\\omega-\omega^2\end{pmatrix},\quad\begin{pmatrix}0\\0\\1\end{pmatrix},\quad\begin{pmatrix}1\\\omega\\1-\omega\end{pmatrix}. \tag{12}$$

Visualisation. The six discs $\mathbb{D}_{i,j}$ have been defined at the end of Section 2.2 by their norm one vectors in (11). They are drawn in Fig. 2. Their quotient curves on the congruence surface appear in Fig. 1. The three -1 points on $\mathbb{B}$, projected from the vectors in (11, are intersections of the disc pairs with orthogonal vectors listed in (11). Only one of them is visible (real coordinates). But along the quotient map $\mathbb{B}\to\Gamma(\sqrt{-3})\backslash\mathbb{B}$ all three points will be visible (as blue bullets) on the quotient surface. By Proposition 17 these are all -1 points on the congruence surface. Each of them is an intersection point of two norm 1 curves.

2.3.4. Stabilizing Subgroups

Lemma 21. *The stabilizer group of e_3 in $\Gamma(\sqrt{-3})$ is the diagonal group*

$$\left\{\begin{pmatrix}\omega^i&0&0\\0&\omega^j&0\\0&0&\omega^k\end{pmatrix};\ 1\le i,j,k\le 3\right\}.$$

The projective stabilizer of the (blue) -1 point $O \in \mathbb{B}$, is represented by

$$\left\langle \begin{pmatrix} \omega & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle \times \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle = \left\{ \begin{pmatrix} \omega^i & 0 & 0 \\ 0 & \omega^j & 0 \\ 0 & 0 & 1 \end{pmatrix} ; 1 \leq i, j \leq 3 \right\}. \tag{13}$$

The projective stabilizer of any -1 point on $\mathbb{B}$ is isomorphic to the bicyclic group $K9 := Z_3 \times Z_3$ of order nine.

Proof: Obviously, the Γ-stabilizer of $\mathfrak{e}_3$ is $\begin{pmatrix} U & \mathfrak{o} \\ {}^t\mathfrak{o} & \langle -\omega \rangle \end{pmatrix}$ with $U = \mathbb{U}(2, \mathcal{O})$ and $\mathfrak{o}$ the two-dimensional zero vector. Intersection with the congruence subgroup yield the diagonal group in Lemma 21 and finally the projective $K9$-representants in (12). ■

Definition 22. *For any Picard modular group Δ we call $\delta \in \Delta$ a* Δ-elliptic element *iff it has finite order and three different eigenvalues. The notation will also used for the projectivized elements. The point $Q \in \mathbb{B}$ is called a* Δ-elliptic point *iff the stabilizing* (*isotropy, stationary*) *group* $\mathrm{Stab}_\Delta(Q)$ *contains a Δ-elliptic element.*

Examples 23. *The central point $O \in \mathbb{B}$ is a $\Gamma(\sqrt{-3})$-elliptic point because its stationary group contains the Gw-elliptic element* $\mathrm{diag}(\omega, \omega^2, 1)$. *The elliptic property can be easily transported via* Proposition 17 *to all* (*blue*) -1 *points of* $\mathbb{B}$.

Remark 24. *Any elliptic $\delta \in \Delta$ has precisely one fixed point $Q \in \mathbb{B}$.*

Proof: It is easy to see that the eigenbasis of δ consists of an orthogonal basis of V. This is only possible with two positively normed vectors and a negative one, say $q \in V_-$. Then $Q = \mathbb{P}q$ is the point we looked for. ■

Definition 25. *With the notations of* Definition 22 *we call δ a Δ-reflection, iff it has finite order and precisely two different eigenvalues, where the eigenline of the simple one belongs to V_+.*

Remark 26. *Eigenvectors of different eigenvalues are orthogonal to each other. Therefore the eigenplane $E \subset \mathbb{C}^{2,1}$ of the double eigenvalue is indefinite being orthogonal to a positively normed vector. It follows that E projects and restricts to a disc $\mathbb{D} = \mathbb{B} \cap \mathbb{P}E$ on our ball.*

Definition 27. *Such $\mathbb{D}$ as above is called a Δ-reflection disc* (*precisely*: *of the reflection δ*). *The centralizer* (*group*) *of $\mathbb{D}$ is the ineffectively on $\mathbb{D}$ acting subgroup*

$$Z_\Delta(\mathbb{D}) = \{\gamma \in \Delta;\ \gamma|_\mathbb{D} = \mathrm{id}|_\mathbb{D}\}. \tag{14}$$

Example 28. *The element* $\mathrm{diag}(\omega, 1, 1)$ *acts identical on the horizontal disc* $\mathbb{D}_{2,4}$, *and* $\mathrm{diag}(1, \omega, 1)$ *acts inefficiently on the vertical coordinate disc* $\mathbb{D}_{1,3}$.

Compare the notations at the end of Section 2.2 and the Visualisation 2.

Lemma 29. *All* -1 *points* Q *on* $\mathbb{B}$ *are intersection points of two* $\Gamma(\sqrt{-3})$*-reflection discs* $\mathbb{D}, \mathbb{D}'$. *The stationary groups are isomorphic to* $K9 = Z_3 \times Z_3$, *with centralizer groups* $Z_{\Gamma(\sqrt{-3})}(\mathbb{D}_i) \cong Z_3$ *of the* $\mathbb{D}$ *and* $\mathbb{D}'$ *as generators.*

Proof: The situation around O described in Lemma 21 is transported with the help of Γ to each -1 point Q (see Proposition 17). ■

2.4. All Elements of Finite Order

Let K be an imaginary quadratic number field and $\gamma \in \mathbb{G}L_3(K)$ an element of finite order n. The characteristic polynomial of γ over K has degree three. All zeros of it are n-th unit roots $\zeta_n \in \mathbb{C}$. Over Q the roots have degree not greater than six. Denote by $\chi(x) \in \mathbb{Q}[x]$ a polynomial with zero ζ_n of degree ≤ 6. We want to know, which unit roots can occur for $\gamma \in \Gamma(\sqrt{-3})$. We concentrate us first to prime numbers $n = p$ and to the field K of Eisenstein numbers. The minimal polynomial over Q with zero ζ_p has degree $p - 1$. It divides $\chi(x)$, therefore $p - 1 \leq 6$. The only possibilities are $p = 2, 3, 5, 7$. The latter two primes can be excluded, because ζ_5, ζ_7 are not zeros of a polynomial of degree ≤ 3 over K.
If $n = p^k$ is a prime power, then the degree of the prime polynomial of ζ_{p^k} over $\mathbb{Q}$ is equal to $(p - 1)p^{k-1}$. Over $\mathbb{Q}(\sqrt{-3})$ survive only the honest powers $2^2, 3^2$. In congruence subgroups we can further restrict the possible orders of finite elements. In [20, Lemma 7.1], we proved the elementary

Lemma 30. *Let* $\mathfrak{a}$ *be an ideal in* $\mathcal{O}$, $\gamma \in \Gamma(\mathfrak{a})$ *of finite order* n *with* n*-th unit root* ζ *as eigenvalue. Then* $\mathfrak{a}$ *divides the principal ideal* $(1 - \zeta)$ *in* $\mathcal{O}_L$, *where* $L = K(\zeta)$.

For the Eisentein congruence subgroup $\Gamma(\sqrt{-3})$ we get for $\mathfrak{a} = (1-\omega)$ the relation $(1-\omega) \,|\, (1-\zeta) \in \mathcal{O}_{K(\zeta)}$. The only possible prime order for ζ is $p = 3$. The prime powers $2^2, 3^2$ are excluded, but also $p = 2$. We notice the following

Corollary 31. *The only elements of finite order in* $\Gamma(\sqrt{-3})$ *have order three. Each of them is conjugated to one of the diagonal elements* $\mathrm{diag}(\omega^i, \omega^j, \omega^k)$.

We want to determine all reflections in $\Gamma(\sqrt{-3})$. Such a reflection σ must be of order 3 by Corollary 31 with a double and a single eigenvalue. Without loss of generality, we assume that 1 is the double eigenvalue and ω the other one (if not, take the inverse or/and multiply σ with ω or ω^2). The eigenplane is denoted by E. Its orthogonal complementary line in V is denoted by G. The integral part $\mathcal{G} = \mathcal{O}^3 \cap G$ is spanned by a primitive (eigen) vector $\mathfrak{g}$ with positive norm and sits on the line G orthogonal to E. We assert that its norm $n(\mathfrak{g}) = n(G)$ is equal to three.

2.5. Elliptic Elements

We want to prove that the Eisenstein quotient surface $\Gamma(\sqrt{-3})\backslash\mathbb{B}$ is smooth and that the $\Gamma(\sqrt{-3})$-reflection discs fill completely the ramification locus of the (locally finite) quotient morphism $\mathbb{B} \to \Gamma(\sqrt{-3})\backslash\mathbb{B}$. For this purpose we must find all elliptic elements, points, of our Picard Eisenstein congruence lattice $\Gamma(\sqrt{-3})$ or on the surface $\Gamma(\sqrt{-3})\backslash\mathbb{B}$, respectively.

Characterisation 32 (of elliptic elements). *Let Δ be a Picard modular group of the two-ball. An element $\delta \in \Delta$ is* elliptic *iff it has an isolated fixed point P on $\mathbb{B}$. This means that in a small open neighbourhood of P there is no other fixed point of δ. A point $P \in \mathbb{B}$ is a Δ*-elliptic point *iff the stationary group* $\mathrm{Stab}_\Delta(P) = \{\gamma \in \Delta\,;\, \gamma(P) = P\}$ *contains an elliptic group element* (cf. Shimura [28, Chapter I]).

Remarks 33. *For each ball point P the stationary group $\mathrm{Stab}_\Delta(P)$ is finite because Δ acts proper discontinuously on $\mathbb{B}$. An elliptic element $\delta \in \Delta$ has finite order, say n. Being not a reflection it has three different eigenvalues. Each of them is an n-th unit root. Any triple of eigenvectors of the different eigenvalues is an orthobasis of $\mathbb{C}^{2,1}$. Precisely one of these basis vectors, let us denote it by $\mathfrak{c}$, has negative norm. Its projection P on $\mathbb{B}$ is the only ball fixed point of δ.*

Proposition 34. *The set of $\Gamma(\sqrt{-3})$-elliptic ball points coincides with the set of -1 points on the Eisenstein Congruence surface. It is the same to say: with the set of intersection points of two $\Gamma(\sqrt{-3})$-reflection discs on $\mathbb{B}$.*

Proof: The inclusion $\subseteq$ has already been proved with Lemma 29, namely the -1 point O contains the elliptic element $\delta = \mathrm{diag}(\omega, \omega^2, 1)$. For any -1 point P one finds a $\gamma \in \Gamma(\sqrt{-3})$ transporting O to P. Then $\gamma\delta\gamma^{-1}$ stabilizes P, has the same eigenvalues as δ, is therefore elliptic. The inverse inclusion has been proved in [17, Ch.I, 1.4.5], over two pages, elementary but a little bit tricky. ■

2.6. Smoothness of the Eisenstein Congruence Surface (ECS)

2.6.1. Smoothness of the Open Surface $\Gamma(\sqrt{-3})\backslash\mathbb{B}$

Corollary 35. *The projective stabiliser group* $\mathrm{Stab}_{\Gamma(\sqrt{-3})}(p)$ *of each* $\Gamma(\sqrt{-3})$*-elliptic point* $p \in \mathbb{B}$ *is isomorphic to* $Z_3 \times Z_3$*. It is generated by two* $\Gamma(\sqrt{-3})$*-reflections. Therefore the image point* P *on* $\Gamma(\sqrt{-3})\backslash\mathbb{B}$ *is smooth. Hence the whole* (*open*) *quotient surface is smooth.*

Proof: We saw, that p must be a ballpoint of norm -1 and that it is the intersection of precisely two reflection disc (Proposition 34). Because of the Γ-equivalence of all -1-points we can assume that $p = O$. We visualized the local situation around O in Fig. 2. The reflections $\begin{pmatrix} \omega & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & 1 \end{pmatrix}$ generate obviously $K9 = Z_3^2 \cong \mathrm{Stab}_{\Gamma(\sqrt{-3})}(O)$. The image point of O, and also of p on the Congruence Surface is locally isomorphic to its image of O on $\mathbb{C}^2/\mathrm{Stab}_{\Gamma(\sqrt{-3})}(O)$. The latter image point is a smooth one by the following old result

Proposition 36 (Chevalley [4]). *Let G be a finite subgroup of* $\mathbb{G}L_n(\mathbb{C})$*,* $p : \mathbb{C}^n \to \mathbb{C}^n/G$ *the quotient map. Then* $p(O)$ *is a regular point if and only if G is generated by reflections.*

■

We remember the definitions for arbitrary $\mathbb{B}$-lattices Δ and Δ-discs $\mathbb{D}$ to its *normalizer* and *centralizer* group (see the text before Diagram (4) in Section 2.1)

$$N_\Delta(\mathbb{D}) = \{\alpha \in \Delta;\, \alpha(\mathbb{D}) = \mathbb{D}\}, \qquad Z_\Delta(\mathbb{D}) = \{\alpha \in \Delta;\, \alpha|_{\mathbb{D}} = \mathrm{id}\,|_{\mathbb{D}}\}. \quad (15)$$

The effectively on $\mathbb{D}$ acting group is: $\Delta_{\mathbb{D}} = N_\Delta(\mathbb{D})/Z_\Delta(\mathbb{D})$.

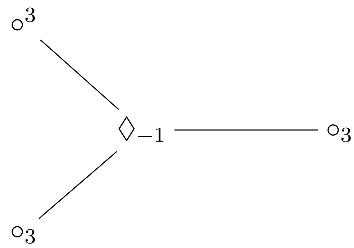
Proposition 37. *The image curve of any* $\Gamma(\sqrt{-3})$*-reflection disc* $\mathbb{D}$ *on the* (*open*) *Congruence Surface* $\Gamma(\sqrt{-3})\backslash\mathbb{B}$ *is smooth.*

Proof: Let $\sigma \in \Gamma' := \Gamma(\sqrt{-3})$ be a reflection and denote the corresponding reflection disc by $\mathbb{D} = \mathbb{D}_\sigma$. Knowing that the quotient curve $\Gamma'_{\mathbb{D}}\backslash\mathbb{D}$ is smooth we see that also $\Gamma'\backslash\mathbb{D} \subset \Gamma'\backslash\mathbb{B}$ has no singularities iff the Γ'-equivalence for points on $\mathbb{D}$ is not stronger than the $N_{\Gamma'}(\mathbb{D})$-equivalence. But this generally known for reflection discs (see [18, Lemma 4.5.2]). ■

2.6.2. Smoothness at the Cusps

We want not explain here the necessary notions and calculations for understanding things around cusps. We will shortly mention the things we used. The interested reader should consult [17] (for the ECS).

- Neat subgroup, toroidal compactifications, cusp bundle over elliptic curves, disc bundles over elliptic curves, their plugging in, smooth crossing fibres, cyclic quotients of elliptic cusp bundles, curve compactifications $\overline{\Delta\backslash\mathbb{B}}$ around cusps.
- explicit calculations [EPD], yield the second graph type in [16, Proposition 4.2])



(16)

This is a "dual graph" of four (irreducible) curves. The circles stand for reflection curves with (positive) indices announcing the branch order. The central diamond represents a compactifying curve with (negative) index, which means the selfintersection. The lines are interpreted as intersection points of the curves joint by them. So the cusp curve has three intersecting reflection curves. We know moreover that our ECS-cusp curves are rational.

Proposition 38. *All* (*four*) *cusp points of the ECS are non-singular. They are also regular points of each of* (*the six*) *reflection curves.*

Altogether with Propositions 35 and 37 we get

Theorem 39. *The BB-compactified Eisenstein Congruence Surface* $\widehat{\Gamma(\sqrt{-3})\backslash\mathbb{B}}$ *is smooth. All* (*six*) (*compact*) *reflection curves* $\widehat{\Gamma(\sqrt{-3})\backslash\mathbb{D}}$ *on it are smooth.*

2.6.3. Curve Classification

Let $\mathbb{D}$ be a $\Gamma(\sqrt{-3})$-reflection disc on $\mathbb{B}$ and C its (open) quotient curve $\Gamma(\sqrt{-3})\backslash\mathbb{D} \subset \Gamma(\sqrt{-3})\backslash\mathbb{B}$. The closure of the BB- compactified or curve compactified surface is denoted by $\hat{C}$ or $\overline{C}$, respectively.

Exercise. Determine the Euler-Poincaré volume of a fundamental domain $\mathcal{F}$ of the one-dimensional Eisenstein $\mathbb{D}$-lattice $\mathbb{PU}((1,1),\mathcal{O}) = \Gamma_{\mathbb{D}}$ and $\mathcal{F}'$ of $\Gamma'_{\mathbb{D}} = \Gamma(\sqrt{-3})_{\mathbb{D}}$.

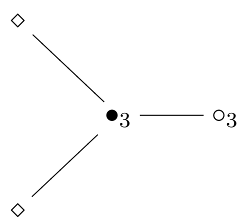
Answer (Feustel [8])

$$\int_{\mathcal{F}'} \gamma_1 = -2/3 \text{ with Euler form } \gamma_1 \text{ on } \mathbb{D}. \tag{17}$$

Hint. Use a transfer of $\mathbb{U}((1,1),\mathcal{O}$ to a $\mathbb{S}L_2(\mathbb{Z})$-commensurable group acting on the upper half plane $\mathbb{H}$ along a suitable biholomorphic fractional map $\mathbb{D} \xrightarrow{\sim} \mathbb{H}$. Relate it with the volume of a $\mathbb{S}L_2$-fundamental domain on $\mathbb{H}$, which is equal to $-\frac{1}{6}$. For more details see [17, I.1.5.]

Next we want to calculate Euler number $e(\overline{C})$ and signature $\sigma(\overline{C})$ of the $\overline{\Gamma(\sqrt{-3})\backslash\mathbb{B}}$ - embedded compact reflection curve $\overline{C}$. A good geometric orientation is concentrated in the following

$\overline{C}$-Graph ([18, Figure in Example 4.7.6])



(18)

It is a "dual graph" again. The small circle and the diamonds stand for curves crossing $\overline{C}$ represented by the central bullet. More precisely, the diamonds stand for cusp curves, the circle: for the reflection curve crossing the given one. The attached numbers 3 indicate the branch order (ramification index).

We encaged Euler number and selfintersection defining (one-dimensional) *orbital heights*: h_e the *Euler-height* and h_τ the *signature-height*. We only reduce the definition here to the cases, when at most two components of the irreducible branch curves are intersecting in any point of the curve compactified model of a Picard modular surface. I refer to the more general Definition 4.7.3 in my book [18]. Surface singularity data e_i, d_j can be simplified (see below) because our surface (ECS) is smooth. In [18] we wrote "e_f","τ_f" instead of $h_e, \dfrac{h_\tau}{3}$.

Definition 40. *The one-dimensional* orbital Euler hight *and* orbital signatur hight *are defined as*

$$\begin{aligned} h_e(C) &= e(\overline{C}) - \sum_i (1 - \frac{1}{v_i d_i}) - \#\overline{C}_\infty \\ h_\tau(C) &= \frac{1}{v}\left[(\overline{C}^2) + \sum_i \frac{e_i}{d_i} + \sum_j \frac{e_j}{d_j} \right] \end{aligned} \tag{19}$$

where $e(\overline{C}), (\overline{C}^2)$ are respectively the Euler number and the selfintersection index of the reflection curve C. Singularity data are trivial: $e_i = 0, d_j = 1$. The branch order of C is denoted by v, while v_i is the branch order of the i-th curve crossing C.

The number of rational compactification curves crossing $\overline{C}$ is denoted by $\#\overline{C}_\infty$. Here h_e is the algebraic expression for the Euler volume $\int_{\mathcal{F}'} \gamma_1$, and $h_\tau = \frac{1}{2} \cdot h_e$. The relations can be found in [18, 4.7.7]. The latter one is known as *proportionality*.

Now we can calculate with the help of (17), (18) and (19)

$$\begin{aligned} -\frac{2}{3} &= h_e(C) = e(\overline{C}) - (1 - \frac{1}{3 \cdot 1}) - 2 \\ -\frac{1}{3} &= h_\tau(C) = \frac{1}{3} \cdot \left[(\overline{C}^2) + 0 + 0 \right]. \end{aligned}$$

It follows that

Lemma 41. *Any of the six reflection curves $\overline{C}$ on the ECS have Euler number 2. So these are smooth rational curves. Moreover their selfintersection is equal to -1.*

2.7. Surface Classification

For brevity, e.g. at indices, we use following notations

$$\Gamma = \mathbb{U}((2,1),\mathcal{O}), \qquad \Gamma' = \Gamma(\sqrt{-3}), \qquad X = \Gamma\backslash\mathbb{B}, \qquad X' = \Gamma'\backslash\mathbb{B}.$$

We defined in [18] two-dimensional *orbital heights* H_e, H_τ for any Picard modular surface. Let us pick out the easy variant of smooth curve compactified Picard modular surfaces and (smooth) branch curve on it without triple points. The irreducible compactification curves are assumed to be rational.

Definition 42. *The* orbital Euler hight *respectively* signature hight *look like this*

$$\begin{aligned} H_e(X') &= E(\overline{X'}) - 2(\mathbf{1}-\mathbf{v}^{-\mathbf{1}}) \cdot D \cdot {}^{\mathbf{t}}\mathbf{v}^{-\mathbf{1}} - \frac{1}{2}(\mathbf{1}-\mathbf{v}^{-\mathbf{1}}) \cdot S \cdot {}^{\mathbf{t}}(\mathbf{1}-\mathbf{v}^{-\mathbf{1}}) - t \\ H_\tau(X') &= \Sigma(\overline{X'}) - \frac{1}{3}(\mathbf{v}-\mathbf{v}^{-\mathbf{1}}) \cdot D \cdot {}^{\mathbf{t}}\mathbf{v}^{-\mathbf{1}} - \frac{1}{3}(T^2). \end{aligned} \tag{20}$$

Thereby T denotes the compactification divisor. Its number of components is denoted by t. D is the (diagonal) selfintersection matrix of branch curve components. S denotes the proper intersection matrix of the same curves, that means with self intersections substituted by 0.

In the case of the Eisenstein Congruence Surface we have $t = 4$ by Corollary 15, and

$$D = \mathrm{diag}(-1, -1, -1, -1, -1, -1)$$

by Lemma 41. With suitable numeration one gets for the six reflection curves on $\overline{X'}$

$$S = \begin{pmatrix} 0&0&0&0&0&1 \\ 0&0&0&0&1&0 \\ 0&0&0&1&0&0 \\ 0&0&1&0&0&0 \\ 0&1&0&0&0&0 \\ 1&0&0&0&0&0 \end{pmatrix}$$

because each component has only one intersecting partner, see Diagram 18 with Proposition 19 in the background.

The right hand sides in (20) can now be calculated

$$\begin{aligned} &E(\overline{X'}) - \sum_{i=1}^{6}\Big((1-\frac{1}{3}) \cdot (-1) \cdot 1/3\Big) - 2 \cdot 4, \quad -\frac{1}{2}\sum_{i=1}^{6}\Big((1-\frac{1}{3}) \cdot 1 \cdot (1-\frac{1}{3})\Big) \\ &\Sigma(\overline{X'}) - \frac{1}{3}\sum_{i=1}^{6}\Big(\frac{8}{3} \cdot (-1) \cdot \frac{1}{3}\Big) - \frac{1}{3} \cdot 4 \cdot (-1). \end{aligned} \tag{21}$$

The left hand sides of (20) are volumes of fundamental domains again

$$\int_{\mathcal{F}(\Gamma')} \gamma_2 \text{ with Euler form } \gamma_2 \text{ on } \mathbb{B}, \qquad \int_{\mathcal{F}(\Gamma')} \tau_2 \text{ with signature form } \tau_2 \text{ on } \mathbb{B}.$$

For details we refer to [18] or [1]. There one can also the important Proportionality Theorem (Proposition 2) in [18, 4.9.1]. Originally it comes from Hirzebruch's comparison of Chern forms of invariant metrics on symmetric domains, well-understandable summarized in [1]. Here we need the Proportionality Relation

$$\gamma_2 = 3\tau_2 \text{ consequently } H_e(X') = 3H_\tau(X').$$

Remarks 43. *The conclusion needs higher dimensional modern Riemann-Roch Theory in the sense of Grothendieck-Hirzebruch, see* [18] *with background in Hirzebruch's book* [15].

The Euler volumes of full Picard modular groups have been determined in the second half of the 1970's. *They appear as L-series values, multiplied with an elementary number. A proof has been reproduced in* [18, Ch.V, 5A]. *Later, by means of functional equation they could be expressed in terms of (higher) Bernoulli numbers, see last formula in* [19]. *Especially, it was found*

$$H_e(X') = \int_{\mathcal{F}(\Gamma')} \gamma_2 = 1/3, \qquad H_\tau(X') = \int_{\mathcal{F}(\Gamma')} \tau_2 = 1/9$$

see e.g. [18, Proof of Lemma 5.2.2]. *We plug it into the left hand side of* (20), *calculate* (21) *to get*

$$\frac{1}{3} = E(\overline{X'}) - \frac{20}{3}, \qquad \frac{1}{9} = \Sigma(\overline{X'}) + \frac{28}{9}.$$

It follows that $E(\overline{X'}) = 7$ and $\Sigma(\overline{X'}) = -3$. Each of the four cusp lines on $\overline{X'}$ have selfintersection -1. So, the birational morphism $\overline{X'} \to \widehat{X'}$ is the simultaneous blowing down of four exceptional curves to four (regular) cusp points. It induces classically the following change of Chern numbers

$$E(\widehat{X'}) = E(\overline{X'}) - 4 = 3, \qquad \Sigma(\widehat{X'}) = \Sigma(\overline{X'}) + 4 = 1.$$

Proposition 44. *The BB-compactification* $\widehat{X'} = \widehat{\Gamma(\sqrt{-3})\backslash\mathbb{B}}$ *has the following Chern invariants*

$$\text{Euler number } E(\widehat{X'}) = 3, \qquad \text{Signature } \Sigma(\widehat{X'}) = 1.$$

One derives from them two other Chern invariants, arithmetic genus and Selfintersection of the canonical class

$$\chi = \frac{1}{4}(E + \Sigma) = 1, \qquad (K^2) = 12\chi - E = 9.$$

Now we must remember to the following highlight theorem of the 1980's

Theorem 45 (Miyaoka -Yau). *A smooth compact complex surface of general type is a ball quotient $\Delta\backslash\mathbb{B}$ (for a suitable cocompact ball lattice Δ) if and only if their Chern numbers (K^2) and E (Euler number) satisfy the relation*

$$(K^2) = 3E. \tag{22}$$

In this case there is no rational curve on the surface.

A simple resumée of Miyaoka's and Yau's approach to the above theorem can be found in [1, Anhang B.2, F]).

From classification theory of complex surfaces of Kodaira dimension < 2 one deduces (after careful check of the texts in [3] or [27] around Chern invariants)

Proposition 46. *The only smooth compact complex algebraic surface, not of general type, satisfying condition* (22) *is the projective plane $\mathbb{P}^2$.*

Corollary 47. *If there is a rational curve on the smooth compact complex surface satisfying Chern number relation* (22)*, then it cannot be of general Type. It has to be isomorphic to the projective plane $\mathbb{P}^2$.*

Main Theorem 48. *The Eisenstein Congruence Surface $\widehat{\Gamma(\sqrt{-3})\backslash\mathbb{B}}$ is the projective plane $\mathbb{P}^2$. The four cusp points K_i, $i = 1..4$, are not collinear. The compactified branch locus of the quotient morphism $\Gamma(\sqrt{-3})\backslash\mathbb{B} \to \Gamma\backslash\mathbb{B}$ is the complete quadrilateral on $\mathbb{P}^2$, visualized in* Fig. 2*. Each component is a line through two of the cusp points.*

Proof: We know by Proposition 44, that for our ECS the Chern relation (22) is satisfied. By Corollary 47 and Theorem 45 it cannot be a surface of general type. But then it follows from Proposition 46 that the BB-compactified Eisenstein Congruence Surface is the projective plane $\mathbb{P}^2$.

The six reflection discs visualized in Fig. 1 cover our the branch curves along the ball quotient morphism. On any of the above discs $\mathbb{D}$ ly, up $\Gamma(\sqrt{-3})$-equivalence, precisely two cusps. Hence the compactified image curve $\overline{C}$ goes through precisely two cusp curves of $\overline{X'}$. We see in Lemma 41 that the selfintersection index $(\overline{C}^2)$ is equal to -1. Each blowing down curve crossing $\overline{C}$ increases the selfintersection by 1, therefore $(\hat{C}^2) = (\overline{C}^2) + 2 = 1$. But $\hat{C}$ is a smooth rational curve on $\mathbb{P}^2$. There are only two possibilities: It is a quadric or an embedded line on the plane. The first case can be excluded because selfintersection of a quadric is equal to 4. The Main Theorem is proved. ∎

3. Heegner Series of Picard Modular Surfaces

3.1. Orbital Invariants

3.1.1. Orbital Surface Heights

Let $\mathcal{F}$ denotes the category of open Picard modular surfaces, but with only finite morphisms. More precisely, we pick out only the finite morphisms of $\mathcal{F}$ supported by inclusions $\Gamma' \subseteq \Gamma$, that means the quotient morphisms $\Gamma'\backslash\mathbb{B} \twoheadrightarrow \Gamma\backslash\mathbb{B}$. For any imaginary quadratic number field K we let $\mathcal{F}_K$ be the complete subcategory of all Picard modular surfaces over fixed K. Conversely, we can consider $\mathcal{F}$ as disjoint union of all $\mathcal{F}_K$.

We can also built the categories $\widehat{\mathcal{F}}, \overline{\mathcal{F}}$ of BB-compactified or curve-compactified objects and morphisms from $\mathcal{F}$. If we correspond to each object/morphism its compactification we get natural isomorphisms
$\mathcal{F} \xrightarrow{\sim} \overline{\mathcal{F}} \xrightarrow{\sim} \widehat{\mathcal{F}}$.

Definition 49. *An* orbital invariant *on $\mathcal{F}$ in a $\mathbb{Q}$-Algebra R is a degree compatible map $H: \mathcal{F} \longrightarrow R$, i.e., it holds that*

$$H(Y) = [Y : X] \cdot H(X) = \deg(F) \cdot H(X)$$

for all (finite) morphisms $F: \Gamma'\backslash\mathbb{B} = Y \twoheadrightarrow X = \Gamma\backslash\mathbb{B}$ of $\mathcal{F}$.

In other words the condition describes a contravariant numerical functor $\deg$: $\mathcal{F} \to R$, t.m. with commutative diagrams

$$\begin{array}{ccccc} Y & \longmapsto & H(Y) & = & H(\mathbb{P}\Gamma') \\ {\scriptstyle F}\big\downarrow & & \bullet \big\uparrow {\scriptstyle \deg F = [\mathbb{P}\Gamma : \mathbb{P}\Gamma']} & & \big\uparrow \\ X & \longmapsto & H(X) & = & H(\mathbb{P}\Gamma) \end{array} \tag{23}$$

where $\bullet$ is nothing but the multiplication map (with $\deg F$ - factor). Sometimes it is convenient to express the factor in terms of index of the starting groups. If $Z(\Gamma)$ denotes the (finite cyclic) center of Γ and w its order, then we have the obvious relation

$$[\mathbb{P}\Gamma : \mathbb{P}\Gamma'] = [\Gamma/Z(\Gamma) : \Gamma'/Z(\Gamma')] = [\Gamma : \Gamma']\cdot[Z(\Gamma') : Z(\Gamma)] = [\Gamma : \Gamma']\cdot\frac{w'}{w}\cdot \tag{24}$$

Important is the multiplicativity sitting in the categorial definition

$$\deg(F \circ F') = \deg(F) \cdot \deg(F')$$

for all $F' : Z \to Y$ in $\mathcal{F}$.

It is easy to transfer the notion to the (naturally isomorphic) compactified categories $\widehat{\mathcal{F}}$ and $\overline{\mathcal{F}}$. This should be done in mind by the reader. We denote e.g. simultaneously orbital heights by H on each of our open or compactified categories. Distinctions $H, \widehat{H}, \overline{H}$ are not necessary. We also use sometimes $H(\Gamma) = H(\mathbb{P}\Gamma) = H(\Gamma\backslash\mathbb{B})$ as in the above diagram.

The background are Haar measures. Denote by G the Lie group defining the ball $\mathbb{B}$ and acting on it, say $G = \mathbb{SU}((2,1),\mathbb{C})$. Take a G-invariant metric μ on $\mathbb{B}$ (known as *Bergmann metric* in Differential Geometry) and let $\mathrm{d}\mu$ be the corresponding volume form on $\mathbb{B}$. The volumes $\mu(\Gamma) := \int_{\mathcal{F}(\Gamma)} \mathrm{d}\mu$ of fundamental domains of ball lattices Γ are finite (by definition of lattice), especially for Picard modular groups. Each such volume form defines an orbital invariant on $\widehat{\mathcal{F}}$ in $\mathbb{R}$. We can choose Chern forms, especially the Euler or signature form, see Subsection 2.7, Definition 42.

Example 50. *The orbital Euler height $H_e(X') = \int_{\mathcal{F}(\Gamma')} \gamma_2$ and also the orbital Signature height $H_\tau(X') = \int_{\mathcal{F}(\Gamma')} \tau_2$ have been essentially used for questions around surface classifications. They can be expressed in terms of Riemann-Roch and singularity theory. As important example we presented the heights of the Eisenstein Congruence Surface in this manner, see* Remarks 43.

3.1.2. Picard Modular Curves

Let Γ be again a Picard modular surface, say over K. Consider a K-disc $\mathbb{D} \subset \mathbb{B}$. The quotient curve $D_\Gamma := \Gamma\backslash\mathbb{D} \subset \Gamma\backslash\mathbb{B}$ is called a *Picard Modular curve* (PMC). All of them form the *category $\mathcal{D}$ of Picard modular curves*. The morphisms are the finite coverings $\Gamma'\backslash\mathbb{D} \twoheadrightarrow \Gamma\backslash\mathbb{D}$ induced by a surface covering $\Gamma'\backslash\mathbb{B} \twoheadrightarrow \Gamma\backslash\mathbb{B}$. We illustrate the situation in the following diagram

$$\begin{array}{ccccc} D_{\Gamma'} & = & \Gamma'\backslash\mathbb{D} & \hookrightarrow & \Gamma'\backslash\mathbb{B} \\ \downarrow f & & \downarrow F|\mathbb{D}_{\Gamma'} & & \downarrow F \\ D_\Gamma & = & \Gamma\backslash\mathbb{D} & \hookrightarrow & \Gamma\backslash\mathbb{B} \end{array} \tag{25}$$

The compactified extensions $\widehat{D_{\Gamma'}} \to \widehat{D_\Gamma}$ or $\overline{D_{\Gamma'}} \to \overline{D_\Gamma}$ restricting $\widehat{F}$ or $\overline{F}$, respectively, form the categories $\widehat{\mathcal{D}}$ or $\overline{\mathcal{D}}$.

Definition 51. *$p \in \mathbb{D}$ is called a Γ-cross point (of $\mathbb{D}$) iff there exists an element $\gamma \in \Gamma \setminus N_\Gamma(\mathbb{D})$ such that and $p \in \mathbb{D} \cap \gamma\mathbb{D}$.*

In other words: γ moves $\mathbb{D}$ but not p. Or: The image point P of p along $\mathbb{D} \to D_\Gamma = \Gamma\backslash\mathbb{D}$ is a curve singularity of D_Γ. The Γ-equivalence for points on $\mathbb{D}$ is stronger than the $N_\Gamma(\mathbb{D})$-equivalence.

Definition 52. *A sublattice Γ' (of finite index) of Γ is called $\mathbb{D}$-neat if it is neat and has no Γ'-cross point.*

In other words: for any $\gamma' \in \Gamma'$ it holds that $\gamma'\mathbb{D} = \mathbb{D}$ or $\mathbb{D} \cap \gamma'\mathbb{D} = \emptyset$. For a discussion we refer the reader to the original Definition 4.4.7 in [18] and the text around. There one finds the following

Facts 53.
- *For any K-disc $\mathbb{D} \subset \mathbb{B}$ and Picard modular group Γ (over K) there exists a $\mathbb{D}$-neat sublattice Γ' of Γ.*
- *Any Picard curve $D_\Gamma = \Gamma\backslash\mathbb{D}$ (as above) has only finitely many curve singularities.*

We set shortly $v = |Z_\Gamma(\mathbb{D})|$. It's nothing else but the branch order of the covering (reflection order, ramification index) of the covering F in Diagram (25) at the quotient curve D_Γ. The degree of the curve covering $D_{\Gamma'}/D_\Gamma$ can be expressed as

$$[D_{\Gamma'} : D_\Gamma] = [\Gamma_\mathbb{D} : \Gamma'_\mathbb{D}] = \frac{[N_\Gamma(\mathbb{D}) : N_{\Gamma'}(\mathbb{D})]}{[Z_\Gamma(\mathbb{D}) : Z_{\Gamma'}(\mathbb{D})]} = \frac{[N_\Gamma(\mathbb{D}) : N_{\Gamma'}(\mathbb{D})]}{v : v'}$$

with obvious notation $v' = |Z_{\Gamma'}(\mathbb{D})|$.

3.2. Orbital Curve Categories

For Riemann-Roch calculations around curves D_Γ we need the embedding of the compact curve $\overline{D}_\Gamma$ in an open neighbourhood $U \subseteq \overline{\Gamma\backslash\mathbb{B}}$. Namely, for orbital invariants we need data of surface and curve singularities living on $\overline{D}_\Gamma$, moreover the branch order of the curve with respect to $\mathbb{D}$ and Γ. Instead of smaller neighbourhoods - as used in earlier papers, e.g. in [21] - it suffices to take only the big one $\overline{U} = \overline{\Gamma\backslash\mathbb{B}}$. So we thicken our open curves to $\mathbf{D}_\Gamma$, which is nothing else but the curve together with its embedding

$$\mathbf{D}_\Gamma : \qquad D_\Gamma \hookrightarrow \Gamma\backslash\mathbb{B}.$$

Definition 54. *The thickened curve $\mathbf{D}_\Gamma$ is called an* orbital Picard modular curve *(on $\Gamma\backslash\mathbb{B}$).*

An orbital covering *is an embedding pair $\mathbf{f} = (f, F)$ as in* Diagram (25). *More simply, it is an embedding diagram*

$$\begin{array}{ccc} D_{\Gamma'} & \hookrightarrow & \Gamma'\backslash\mathbb{B} \\ {\scriptstyle f}\downarrow & \mathbf{f} & \downarrow{\scriptstyle F} \\ D_{\Gamma} & \hookrightarrow & \Gamma\backslash\mathbb{B} \end{array} \tag{26}$$

In this way we built the *category* $\mathfrak{D}$ *of open orbital Picard curves* together with orbital coverings as defined above as (only) morphisms. Via compactifications we get in obvious manner the orbital categories $\widehat{\mathfrak{D}}$ and $\overline{\mathfrak{D}}$. If we fix the imaginary quadratic field K, then we get complete subcategories $\mathfrak{D}_K, \widehat{\mathfrak{D}}_K, \overline{\mathfrak{D}}_K$ of orbital Picard modular curves over K.

The diagram

$$\begin{array}{ccc} D_{\Gamma''} & \hookrightarrow & \Gamma''\backslash\mathbb{B} \\ {\scriptstyle g}\downarrow & \mathbf{g} & \downarrow{\scriptstyle G} \\ D_{\Gamma'} & \hookrightarrow & \Gamma'\backslash\mathbb{B} \\ {\scriptstyle f}\downarrow & \mathbf{f} & \downarrow{\scriptstyle F} \\ D_{\Gamma} & \hookrightarrow & \Gamma\backslash\mathbb{B} \end{array} \tag{27}$$

describes the multiplicativity of orbital morphisms in full details

$$\mathbf{f} \circ \mathbf{g} = (f, F) \circ (g, G) = (f \circ g, F \circ G) = \mathbf{fg}.$$

3.2.1. Orbital Invariants for Orbital Curves

Definition 55. *A* (*rational*) orbital invariant

$$\mathbf{h} : \mathfrak{D} \longrightarrow \mathbb{Q}$$

is a multiplicative numerical functor on $\mathfrak{D}$ *satisfying*

$$\mathbf{h}(\mathbf{D}) = [\mathbf{D} : \mathbf{C}] \cdot \mathbf{h}(\mathbf{C}) \qquad \text{where} \qquad [\mathbf{D} : \mathbf{C}] = \frac{[D : C]}{w : v} = \frac{[\Gamma_{\mathbb{D}} : \Gamma'_{\mathbb{D}}]}{w : v}$$

for orbital curve coverings $\mathbf{D} \twoheadrightarrow \mathbf{C}$ *in* $\mathfrak{D}$ *with branch orders* $v = v_{\mathbf{C}}$ *and* $w = v_{\mathbf{D}}$. *The numbers* v, w *are also called* (orbital) weights.

The single value $\mathbf{h}(\mathbf{C})$ *is called the* orbital hight (*w.r.t.* $\mathbf{h}$) *of the orbital curve* $\mathbf{C} = \mathbf{D}_{\boldsymbol{\Gamma}}$.

Example 56. *Let us take a curve-degree compatible numerical hight functor $h : \mathcal{D} \to \mathbb{Q}$. This means that for all covering $f : D_{\Gamma'} \twoheadrightarrow D_\Gamma$ in $\mathcal{D}$ it holds that*

$$h(D_{\Gamma'}) = [D_{\Gamma'} : D_\Gamma] \cdot h(D_\Gamma).$$

We orbitalize it to the obital invariant $\mathbf{h} : \mathfrak{D} \to \mathbb{Q}$ by setting

$$\mathbf{h}(\mathbf{D}) = \frac{1}{w} h(D).$$

It is easy to check that (with notations of Definition 55)

$$\mathbf{h}(\mathbf{D}) = \frac{1}{w} h(D) = \frac{[D : C] \cdot h(C)}{w} = \frac{[D : C] \cdot v \cdot \mathbf{h}(\mathbf{C})}{w} = [\mathbf{D} : \mathbf{C}] \cdot \mathbf{h}(\mathbf{C}).$$

So we orbitalize the Euler and signature heights in Definition 40 to *orbital Euler* respective *signature invariant* on $\mathfrak{D}$. The second formula there shows that the branch orders are very necessary. This is the reason for the change from simple heights to (fat) orbital invariants. The branch orders will be most important in the next section preparing Heegner series.

3.3. Orbital Intersection Products

It becomes convenient to work in the category $\widehat{\mathfrak{D}}$ of BB-compactified orbital Picard curves. We restrict ourselves to a fixed imaginary quadratic number field K. So we work only with Picard modular groups Γ commensurable with the full one $\mathbb{U}((2,1), \mathcal{O}_K)$ and the compactified quotient surfaces $\widehat{X_\Gamma} = \widehat{\Gamma\backslash\mathbb{B}}$. Each K-disc $\mathbb{D}$ defines an (embedded) Orbital Picard Modular Curve $\widehat{\mathbf{D}}_\Gamma : \widehat{D_\Gamma} \hookrightarrow \widehat{X_\Gamma}$ in $\widehat{\mathfrak{D}}_K$, cf. Definition 54.

We orbitalize first the well-known divisor groups

$$\text{Div } S = \bigoplus_{C \subset S} \mathbb{Q} \cdot C$$

on compact algebraic surfaces S (where C runs through all irreducible compact algebraic curves on S). There is a nice intersection product on complex compact normal algebraic surfaces for the generating curves C. The definition goes back to Mumford, (later Fulton). It can be $\mathbb{Q}$-linearly extended to a $\mathbb{Q}$-bilinear form

$$(\,\cdot\,) : \text{ Div } S \times \text{Div } S \longrightarrow \mathbb{Q}.$$

For orbitalization we fix Γ and define on $\widehat{X} = \widehat{X_\Gamma}$ as above the formal $\mathbb{Q}$-vector space

$$\mathbf{Div}\, \widehat{X} = \bigoplus_{\widehat{D_\Gamma} \subset \widehat{X_\Gamma}} \mathbb{Q} \cdot \widehat{\mathbf{D}}_\Gamma$$

generated by the OPM curves $\widehat{\mathbf{D}}_\Gamma$ on $\widehat{X_\Gamma}$, see Definition 54 again.

For two OPM curves $\widehat{\mathbf{C}}, \widehat{\mathbf{D}}$ of orbital weight w respectively v on $\hat{X}$ we define the *orbital intersection*

$$(\widehat{\mathbf{C}} \cdot \widehat{\mathbf{D}}) = \frac{(\hat{C} \cdot \hat{D})}{v \cdot w}.$$

It can be $\mathbb{Q}$-bilinearly extended to the *orbital intersection* on the OPM divisor group

$$\mathbf{Div}\, \widehat{X} \times \mathbf{Div}\, \widehat{X} \longrightarrow \mathbb{Q}.$$

For any neat Picard modular congruence subgroup $\Gamma_0 = \Gamma(\mathfrak{a}) \subseteq \Gamma$ the curve intersections of curves on the (smooth) compactified model $\overline{\Gamma_0 \backslash \mathbb{B}}$ have been also well-understood by Cogdell [5]. Locally, everything is clear. We use finite coverings to push down intersection products from neat surface models to arbitrary OPM surfaces. It is a longer procedure through Riemann-Roch, curve and surface singularities to realize this way in termes of geometric local and global Galois theory. For details we refer to [21] and [22].

Important was the construction of direct and inverse images along finite coverings $F : \widehat{Y} \to \widehat{X}$ of PM surfaces

$$\mathbf{F}_\# : \mathbf{Div}\, \widehat{Y} \longrightarrow \mathbf{Div}\, \widehat{X}, \qquad \mathbf{F}^\# : \mathbf{Div}\, \widehat{X} \longrightarrow \mathbf{Div}\, \widehat{Y}.$$

We proved the *Orbital Projection Formula*

$$(\mathbf{F}_\# \mathbf{B} \cdot \mathbf{A}) = (\mathbf{B} \cdot \mathbf{F}^\# \mathbf{A})$$

where $\mathbf{A} \in \mathbf{Div}\, \widehat{X},\ \mathbf{B} \in \mathbf{Div}\, \widehat{Y}$ are cycles on the PM surfaces $\widehat{X}$ or $\widehat{Y}$, respectively.

This is the escalator we need for shifting intersection products from neat PM surfaces to non-neats and vice versa along finite coverings: From well-understood to less-understood intersection products.

3.4. Orbital Heegner Functionals

We consider (rational) functionals

$$\mathbf{f}_{\hat{X}} : \mathbf{Div}\, \widehat{X} \longrightarrow \mathbb{Q}$$

on orbital divisor groups on a PM surface $\hat{X}$.

Definition 57. *We call a set* $\check{\mathbf{f}} = \{\mathbf{f}_{\hat{X}};\ \hat{X} \in \widehat{\mathfrak{D}}_K\}$ *an* orbital functional *on* $\widehat{\mathfrak{D}}_K$ *iff it is compatible with orbital direct images along finite coverings* $F : \hat{Y} \to \hat{X}$. *This means that* $\mathbf{f}_{\hat{X}} \circ F_\# = \mathbf{f}_{\hat{Y}}$ *holds for all finite orbital coverings* $\mathbf{F}$ *in* $\widehat{\mathfrak{D}}_K$.

We repeat the definition of the orbital Heegner functional presented first in [21, 3.4]. First remember to norms $n(\mathbb{D}) \in \mathbb{N}_+$ of K-discs $\mathbb{D}$. It can be find in Subsubsection 3.3, Definition 18, only for $K = \mathbb{Q}(\sqrt{-3})$. But it works also in all cases, when $\mathcal{O}_K$ is an principal domain: $\mathbb{D}$ is the ortho-disc of a primitive $\mathcal{O}$-vector $\mathfrak{a}$ and $n(\mathbb{D}) = n(\mathfrak{a})$. We set also for the image curve $\mathcal{D}_\Gamma$ on $X = X_\Gamma = \Gamma\backslash\mathbb{B}$ (also for smooth and other compactifieed models)

$$n(\widehat{\mathcal{D}}_\Gamma) = n(\mathcal{D}_\Gamma) = n(\Gamma\backslash\mathbb{D}) = n(\mathbb{D}).$$

Definition 58. *For $N \in \mathbb{N}_+$ we call the reduced Weil-divisor*

$$H_N = H_N(\widehat{X}_\Gamma) = \sum_{n(\widehat{\mathcal{D}}_\Gamma)=N} \widehat{\mathcal{D}}_\Gamma \tag{28}$$

the Heegner divisor *of norm N on $\widehat{X}_\Gamma$, (set 0, if the sum is void). Taking the orbitalized curves in the sum, we get the* orbital Heegner divisors $\mathbf{H_N} = \mathbf{H_N}(\widehat{X}_\Gamma)$.

Moreover we introduce on $\widehat{X}_\Gamma$ the *orbital Heegner functionals*

$$\mathbf{h_N} : \mathbf{Div}\,\widehat{X} \longrightarrow \mathbb{Q}, \qquad \widehat{\mathbf{C}} \mapsto \langle \widehat{\mathbf{C}}, \mathbf{H_N} \rangle.$$

We construct from them the formal series (with independent variable q)

$$\mathbf{H}(q) = \sum_{N=1}^{\infty} \mathbf{h^N} \cdot q^N.$$

We apply now simultaneously the functionals to an irreducible orbital curve $\widehat{\mathbf{C}}$ on $\widehat{X}$ to get the formal power series

$$\mathbf{H}_{\widehat{\mathbf{C}}}(q) = \sum_{N=1}^{\infty} \mathbf{h_N}(\widehat{\mathbf{C}}) \cdot q^N \in \mathbb{Q}[[q]]. \tag{29}$$

It is called the *formal orbital Heegner series* for $\widehat{\mathbf{C}}$. In [21] between Definitions 3.5 and 3.6 we included also $\mathbf{h_0}$, but it does not play any role in the actual article. It is componentwise clear that

$$\mathbf{H}_{\widehat{\mathbf{D}}}(q) = [\widehat{\mathbf{D}} : \widehat{\mathbf{C}}] \cdot \mathbf{H}_{\widehat{\mathbf{C}}}(q) \tag{30}$$

for any orbital curve covering $\widehat{\mathbf{D}}$ of $\widehat{\mathbf{C}}$, see also [21, formula (11)].

We fill our formal series with more life substituting q by $\mathrm{e}^{2\pi\mathrm{i}\tau}, \tau \in \mathbb{H}$, the upper half plane.

Definition 59. *The series*

$$\mathrm{Heeg}_{\widehat{\mathbf{C}}}(\tau) = h_0 + \sum_{N=1}^{\infty} \mathbf{h_N}(\widehat{\mathbf{C}}) \cdot q^{2\pi \mathrm{i} N\tau}$$

(*with* h_0 *uniquely defined, see* Remark 62*, below*) *is called the* orbital Heegner series *of the orbital Picard curve* $\widehat{\mathbf{C}}$*.*

3.4.1. Modular forms of Nebentypus

Remember the classical congruence subgroups ($m \in \mathbb{N}_+$)

$$\Gamma_0(m) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{S}L_2(\mathbb{Z});\ c \equiv 0 \mod m \right\}$$

of the modular group $\mathbb{S}L_2(\mathbb{Z})$ acting on the upper half place $\mathbb{H}$. Let, moreover

$$\chi = \chi_K : \mathbb{Z} \to \{0, \pm 1\}, \qquad h \mapsto \left(\frac{-D_{K/\mathbb{Q}}}{h} \right) \text{ (Legendre symbol)}$$

be the Dirichlet character of the imaginary quadratic number field K. It factorizes through the residue ring modulo the discriminant $D_{K/\mathbb{Q}}$.

Definition 60. *A holomorphic function* $f(\tau)$ *on* $\mathbb{H}$ *is called* modular form *of* weight $k \in \mathbb{N}$, level $N \in \mathbb{N}$ *and* Nebentypus χ*, iff it satisfies the following functional equations*

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k \chi(d)^k f(\tau), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \tag{31}$$

(and f has meromorphic extensions to the cusps). If it is 0 at the cusps, we call it a *cusp form*. The space of modular forms of weight k, level N and Dirichlet character χ will be denoted by $M_k(N, \chi)$. It is a finite-dimensional $\mathbb{C}$-vector space. In my paper [22, 7.4] one can find the following

Theorem 61. *With suitable* $h_0 = h_0(\widehat{\mathbf{C}})$ *the Heegner series* $Heeg_{\widehat{\mathbf{C}}}(\tau)$ *of a Picard curve* $\widehat{\mathbf{C}}$ *of the field* K *is a modular form belonging to* $M_k(m, \chi_K)$*.*

Remark 62. *Since a non-zero constant cannot satisfy* (29)*, it does nott belong to* $M_k(m, \chi_K)$*. Therefore the constant* h_0 *is uniquely determined by the modular property of the series. It can be expressed in terms of the orbital Euler number of* $\widehat{\mathbf{C}}$ (Cogdell [5]).

Remark 63. *The above theorem was first proved by Cogdell in* [5] *for neat congruence subgroups. The extension of the result to all Picard modular groups one can find in my papers* [21] *and* [22].

3.5. Hecke's Explicit Construction of Modular Forms of Nebentypus

Hecke's notation of the space of *modular forms* of Γ *of weight* k, where Γ is a subgroup of finite index of $\subseteq \mathbb{S}L_2(\mathbb{Z})$, is

$$M_k(\Gamma) = \left\{ f \in \mathrm{Hol}(\mathbb{H})\,;\; f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau),\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \right\}. \quad (32)$$

Keep in mind the meromorphic condition at cusps, after Definition 60 for all modular forms considered in this paper.
Well-defined in many textbooks about modular forms is the congruence subgroup

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mod N \right\} \subseteq \mathbb{S}L_2(\mathbb{Z}).$$

Obviously, we have the exact sequence of groups

$$1 \longrightarrow \Gamma_1(N) \longrightarrow \Gamma_0(N) \longrightarrow (\mathbb{Z}/N\mathbb{Z})^* \longrightarrow 1 \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \quad d.$$

For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{S}L_2(\mathbb{Z})$ Hecke uses the operation (from the right) $f|\gamma_k :$ $= (c\tau+d)^{-k} f(\gamma(\tau))$, see also [28, § 2.1]. Then the Definition in (32) is extended to modular forms of *Nebentypus* χ

$$M_k(N,\chi) = \left\{ f \in M_k(\Gamma_1(N))\;;\; f|\gamma_k = \chi(d)\cdot f, \gamma \in \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{S}L_2(\mathbb{Z}) \right\}. \quad (33)$$

It is easy to verify that the functional conditions in (31) and (32) are the same. Helpful are the following facts

- ◇ For odd primes $N = p$ and $k \equiv \frac{p-1}{2}$ is $\chi = \left(\frac{\bullet}{p}\right)$ the quadratic residue modulo p ([14, p. 810]). Especially $M_3(3,\chi) = M_3(3, \left(\frac{\bullet}{3}\right))$.
- ◇ $\left(\frac{\bullet}{3}\right) = \left(\frac{3}{\bullet}\right) = \left(\frac{-3}{\bullet}\right)$ (for MAPLE-application "$jacobi(-3,n)$").
- ◇ In $M_3(3)$ there exists no (non-zero) cusp form of Nebentypus $\left(\frac{3}{\bullet}\right)$, see [14, Proposition 10, p. 817].

◇ There are precisely two linearly independent Eisenstein series in $M_3(3)$ of Nebentypus $\left(\frac{3}{\bullet}\right)$, see [14, Proposition 12, p.818], (with $q = \mathrm{e}^{2\pi \mathrm{i}\tau}$ according to [14, equation (20), p. 811]).

$$E_1(\tau) = \sum_{n=1}^{\infty} c_1(n) q^n, \qquad c_1(n) = \sum_{0<d|n} d^2 \chi(n/d)$$

$$E_2(\tau) = A + \sum_{n=1}^{\infty} c_2(n) q^n, \qquad c_2(n) = \sum_{0<d|n} d^2 \chi(d).$$

(We do not need Hecke's explicit determination of $A = -\frac{3^{5/2}}{(2\pi)^3} \cdot 2 \cdot \sum_{n=1}^{\infty} \chi(n) n^{-3}$)

$$E_1 = q + 3q^2 + 9q^3 + O(q^4), \qquad E_2 = A + q - 3q^2 + q^3 + O(q^4).$$

3.6. Heegner Series for Picard-Curves of Eisenstein Type

Let Γ be a Picard modular group of the field $K = \mathbb{Q}(\sqrt{-3})$ of Eisenstein numbers and $\widehat{X}_\Gamma$ the BB-compactified quotient surface $\Gamma\backslash\mathbb{B}$. For simplicity we assume that Γ is a sublattice of $\Gamma(\sqrt{-3})$. Take moreover one of the discs $\mathbb{D}$ of norm 1, e.g. the orthogonal disc of $\mathfrak{d}_{2,4} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ in Definition 12 visualized in Fig. 2 joining the cusps -1 and $+1$. The embedded quotient curve of $\mathbb{D}$ on $\widehat{X}_\Gamma$ is denoted by $\widehat{\mathbf{D}}$, that on $\mathbb{P}^2 = X_{\Gamma(\sqrt{-3})}$ by $\widehat{\mathbf{T}}$.

According to Definition 11.3 we have to calculate the first coefficients

$$\mathbf{h}^{\mathbf{N}}(\widehat{\mathbf{T}}) = \langle \widehat{\mathbf{T}}, \mathbf{H}_{\mathbf{N}} \rangle$$

of the Heegner series. Since $\widehat{T}$ is a line on the projective plane $\mathbb{P}^2$ its intersection with any plane curve B is coincides with the degree $\deg B$. Moreover we know the weight $v(\widehat{\mathbf{T}}) = 3$ (branch index). So

$$\langle \widehat{\mathbf{T}}, \widehat{C} \rangle = \frac{1}{3} \cdot \deg(\widehat{C}). \tag{34}$$

We know all curves of norm one, namely: $\widehat{C}_j, j = 1...6$. These are the lines of the quadrilateral visualized in Fig. (2.2). They are also the components of branch locus of the covering $\widehat{\mathbb{B}} \to \widehat{X}_{\Gamma(\sqrt{-3})}$. For any irreducible plane curve $\widehat{\mathbf{C}}$ one gets

$$\langle \widehat{\mathbf{T}}, \widehat{\mathbf{C}} \rangle = \frac{1}{3} \cdot \frac{\deg(\widehat{C})}{v_{\widehat{\mathbf{C}}}} = \begin{cases} \frac{1}{9}, & \text{if } \widehat{C} = \widehat{C_j}, \quad j = 1 \dots 6 \\ \frac{1}{3} \cdot \deg(\widehat{C}), & \text{else.} \end{cases}$$

So

$$\langle \widehat{\mathbf{T}}, \mathbf{H_1} \rangle = \langle \widehat{\mathbf{T}}, \mathbf{C_1} \rangle + ... + \langle \widehat{\mathbf{T}}, \mathbf{C_6} \rangle = \frac{2}{3}, \qquad \langle \widehat{\mathbf{T}}, \mathbf{H_2} \rangle = \frac{1}{3} \cdot \deg(H_2) \tag{35}$$

because H_2 has no component of the branch locus. To get the Heegner series as explicit linear combination of Hecke's Eisenstein series $\mathrm{Heeg}_{\widehat{\mathbf{T}}}(\tau)$ as linear combination of Hecke's Eisenstein series E_1, E_2 (see the end of Section 3.5) we must know

- A) $\dim M_3(3, \chi) = 2$
- B) H_2 is the sum of three lines (of weight 1).

From B) and (35) it follows that

$$< \widehat{\mathbf{T}}, \mathbf{H_2} > = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1.$$

Now we have to solve

$$x \cdot E_1 + y \cdot E_2 = ... + (x+y)q + (3x - 3y)q^2 + = ... + \frac{2}{3}q + \frac{3}{3}q^2 + ...$$

With the unique solution $x = \frac{3}{2}, y = \frac{1}{2}$ we obtain

$$\begin{aligned} 3 \cdot \mathrm{Heeg}_{\widehat{\mathbf{T}}}(\tau) =: \mathrm{Heeg}_{\widehat{T}}(\tau) \\ &= \frac{3}{2} \cdot E_1 + \frac{1}{2} \cdot E_2 = ... + 2q + 3q^2 + 14q^3 + ... + \langle \hat{T}, \mathbf{H_N} \rangle q^N + ... \\ &= ... + \langle \hat{T}, \mathbf{H_1} \rangle q + \langle \hat{T}, H_2 \rangle q^2 + \langle \hat{T}, \mathbf{H_3} \rangle q^3 + ... \end{aligned}$$

With the explicit knowledge of Hecke's two basic Eisenstein series E_1, E_2 (end of Section 3.5) we can determine precisely each q^N-coefficient ($N > 0$) of the Heegner series. For the first 8 we have calculated [3]

$$\mathrm{Heeg}_{\widehat{T}}(\tau) = a + 2q + 3q^2 + 14q^3 + 26q^4 + 24q^5 + 39q^6 + 100q^7 + 51q^8 + ... \tag{36}$$

Theorem 64. *The Heegner series of the Picard line $\widehat{T} \hookrightarrow \mathbb{P}^2$ (of norm 1) is*

$$\mathrm{Heeg}_{\widehat{T}}(\tau) = a + \sum_{n=1}^{\infty} \left\{ \sum_{0<d|n} d^2 \cdot \left[\left(\frac{d}{3} \right) + \left(\frac{n/d}{3} \right) \right] \right\} \cdot q^n \tag{37}$$

[3] We determined the constant term with the help of Hecke's article [13, §3], $a = -1/18$.

We have to prove the items A) and B). We start with the
Proof of B). The dimension of the vector space $M_3(\Gamma_0(3), \chi)$ will be calculated with a formula of Cohen/Oesterlé [6].

Proposition 65 ([6]). *For $k \in \mathbb{Z}$ it holds that*

$$\dim S_k(\Gamma_0(N,\chi)) - \dim M_{2-k}(\Gamma_0(N,\chi)) = \frac{k-1}{12} N \prod_{p|N}(1+\frac{1}{p}) - \frac{1}{2}\prod_{p|N}\lambda(r_p, s_p, p) + \varepsilon_k \cdot \sum_{\substack{x \mod N \\ x^2 \equiv -1 \mod N}} \chi(x) + \mu_k \cdot \sum_{\substack{x \mod N \\ x^2+x \equiv -1 \mod N}} \chi(x).$$

Thereby is r_p the exponent of p in the prime decomposition of N, and s_p is the p-exponent of the führer $f(\chi)$ of the character χ. Moreover the authors of [6] defined

$$\lambda(r_p, s_p, p) = \begin{cases} p^{r'} + p^{r'-1}, & \text{if } 2s_p \le r_p = 2r' \\ 2p^{r'}, & \text{if } 2s_p \le r_p = 2r' + 1 \\ 2p^{r_p - s_p}, & \text{if } 2s_p > r_p \end{cases}$$

$$\varepsilon_k = \begin{cases} 0, & \text{if } k \text{ odd} \\ \frac{-1}{4}, & \text{if } 2s_p \le r_p = 2r' + 1 \\ \frac{1}{4}, & \text{if } 2s_p > r_p \end{cases}, \quad \mu_k = \begin{cases} 0, & \text{if } k \equiv 1 \mod 3 \\ \frac{-1}{3}, & \text{if } k \equiv -1 \mod 3 \\ \frac{1}{3}, & \text{if } k \equiv 0 \mod 3. \end{cases}$$

Corollary 66. *It follows that* $\dim M_3(\Gamma_0(3), \chi_3) = 2$.

Proof of A)[4]. We have to set $N = 3, k = -1, \chi = \chi_3$. Then

$$f(\chi_3) = 3,\ r_3 = 1,\ s_3 = 1,\ \varepsilon_{-1} = 0,\ \mu_{-1} = \frac{-1}{3} \text{ and } \dim S_{-1}(\Gamma_0(3), \chi_3) = 0.$$

Hence

$$0 - \dim M_3(\Gamma_0(3), \chi_3) = \frac{-2}{12} \cdot 3 \cdot (1 + \frac{1}{3}) - \frac{1}{2}\lambda(1,1,3) + 0 - \frac{1}{3} \cdot \sum_{\substack{x \mod 3 \\ x^2+x \equiv -1 \mod 3}} \chi_3(x)$$

and finally: $-\dim M_3(\Gamma_0(3), \chi_3) = \frac{-2}{3} - \frac{1}{2} \cdot 2 - \frac{1}{3}\chi_3(1) = -2.$ ■

[4] The proof (including reference [6]) was found by my former PhD student Christian Schön.

Proof of B). It is easy to see, that in $\mathbb{F}^{2,1}$ the only vectors of $\mathbb{F}$-norm -1 are

$$\pm\begin{pmatrix}1\\1\\0\end{pmatrix}, \qquad \pm\begin{pmatrix}1\\-1\\0\end{pmatrix}, \qquad \pm\begin{pmatrix}0\\0\\1\end{pmatrix}.$$

Preimages in $\mathcal{O}^{2,1}$ of norm 2 along the residue map are

$$\pm\begin{pmatrix}1\\1\\0\end{pmatrix}, \qquad \pm\begin{pmatrix}1\\-1\\0\end{pmatrix}, \qquad \pm\begin{pmatrix}\sqrt{-3}\\0\\1\end{pmatrix}. \tag{38}$$

All other norm 2 vectors in $\mathcal{O}^{2,1}$ are $\Gamma(\sqrt{-3})$-congruent to one of them. We can draw two of the three orthodiscs of the above vectors (green)[5].

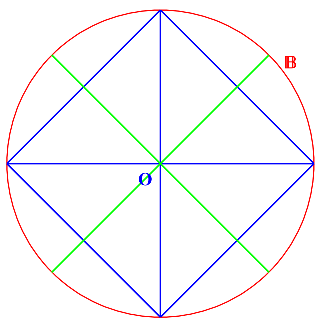


Figure 3. Extended ramifying disc configuration.

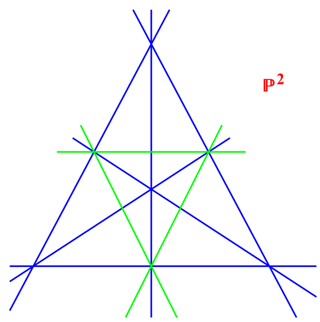


Figure 4. Extended quadrilateral.

The (green) diagonal disc $\mathbb{D}$ is a Γ-reflection disc, pointwise fixed by the reflection

$$\sigma = \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}.$$

The quotient curves of the three orthodiscs of the vectors (38) on the Eisenstein Congruence Surface $\mathbb{P}^2$ are all three visible. They must be S_4-reflection lines on $\mathbb{P}^2$. For instance σ goes down to an effectively on $\mathbb{P}^2$ acting element of order two leaving pointwise fixed the (green) image line L of $\mathbb{D}$. It goes through two of the double points of the quadrilateral (look back to Fig. 1 and also to Proposition 17 at the end of Subsubsection 2.3.2).

Altogether: The Heegner divisor H_2 has precisely three curves on $\mathbb{P}^2$. They do not belong to the $\Gamma(\sqrt{-3})$-branch divisor on $\mathbb{P}^2$. Therefore

$$H_2 = L_1 + L_2 + L_3. \qquad \blacksquare$$

[5] Can be seen only in online version.

References

[1] Barthel G., Hirzebruch F. and Höfer Th., *Geradenkonfigurationen und algebraische Flächen*, Aspects of Mathematics D 4, Vieweg, Braunschweig 1986.

[2] Bianchi L., *Geometrische Darstellung der Gruppen linearer Substitutionen mit ganzen complexen Coefficienten nebst Anwendungen auf die Zahlentheorie*, Math. Ann. **38** (1891) 313-333.

[3] Barth W., Peters C. and van de Ven A., *Compact Complex Surfaces*, Springer, Berlin 1984.

[4] Chevalley C., *Invariants of Finite Groups Generated by Reflections*, Am. J. Math. **77** (1955) 778-782.

[5] Cogdell J., *Arithmetic Cycles on Picard Modular Surfaces and Modular Forms of Nebentypus*, J. f. Math. **357** (1984) 115-137.

[6] Cohen H., Oesterlé J., *Dimensions des espaces de formes modulaires*, International Summer School on Modular Functions, Bonn 1976.

[7] Dieudonné J., *La géométrie des groupes classiques*, Springer, Berlin 1971.

[8] Feustel J.-M., *Kompaktifizierung und Singulaqritäten des Faktorraumes einer arithmetischen diskreten Gruppe, die in der zwei-dimensionalen komplexen Einheitskugel wirkt*, Diplomarbeit, HU Berlin 1975.

[9] Feustel J.-M., *Über die Spitzen von Modulflächen zur zweidimensionalen komplexen Einheitskugel*, Prepr. 13/79, Akad. Wiss. DDR, ZIMM, Berlin 1979.

[10] Feustel J.-M., *Volumina der Fundamentalgebiete der Spiegelscheiben Picardscher Modulgruppen*, Zbl. Math.Mech., Ak.d.Wiss. DDR, Report R-13/79, Berlin 1979.

[11] Hashimoto K., *Elliptic Conjugacy Classes of the Siegel Modular Group and Unimodular Hermitian Forms over the Ring of Cyclotomic Integers*, J. Fac. Sci. Univ. Tokyo **33** (1986) 57-82.

[12] Hecke E., *Die eindeutige Bestimmung der Modulfunktionen q-ter Stufe durch algebraische Eigenschaften*, Math. Ann. **111** (1935) 293-301, also in [14, No.31], pp. 568-576.

[13] Hecke E., *Analytische Arithmetik der positiven quadratischen Formen*, Kgl. Danske Videnskabernes Selskab. Mathematisk-fysiske Meddelelser XIII, 12 (1940), also in: [14, No.41], pp. 789-818.

[14] Hecke E., *Gesammelte Werke*, Vandenhoeck/Ruprecht, Göttingen 1959.

[15] Hirzebruch F., *Topological Methods in Algebraic Geometry*, Grundlehren Math. Wiss. 131, Springer, New York 1966.

[16] Holzapfel R.-P., *Arithmetic Curves on Ball Quotient Surfaces*, Ann. Glob. Anal. Geom. **1** (1983) 21-90.

[17] Holzapfel R.-P., *Geometry and Arithmetic Around Euler Partial Differential Equations*, Dt. Verlag d. Wiss., Berlin, Reidel, Dordrecht 1986.

[18] Holzapfel R.-P., *Ball- and Surface Arithmetics*, Aspekte der Mathematik, Vieweg, Braunschweig 1998.

[19] Holzapfel R.-P., *Zeta Dimension Formula for Picard Modular Cusp Forms of Neat Natural Congruence Subgroups*, HU-Preprint, **97-12**, Abh. Math. Sem. Hamburg, **68** (1998) 169-192.

[20] Holzapfel R.-P., Pinheiro A. and Vladov N., *Picard-Einstein Metrics and Class Fields Connected with Apollonius Cycle*, Preprint-Series 2000-4, Inst. Math. Humboldt-Univ., Berlin 2000.

[21] Holzapfel R.-P., *Enumerative Geometry for Complex Geodesics on Quasihyperbolic 4-Spaces with Cusps*, Geom. Integrability & Quantization **4** (2003) 42-87.

[22] Holzapfel R.-P., *Relative Proportionality on Picard- and Hilbert Modular Surfaces*, see [23, pp 109-162].

[23] Holzapfel R.-P., Uludag A. and Yoshida M. (Eds), *Arithmetic and Geometry Around Hypergeometric Functions*, Lecture Notes of a CIMPA Summer School 2005, Birkhäuser, Basel 2007.

[24] Koblitz N., *Introduction to Elliptic Curves and Modular Forms*, Springer, New York 1984.

[25] Otremba G., *Zur Theorie der hermiteschen Formen in imaginärquadratischen Zahlkoerpern*, J. Reine Angew. Math. **249** (1971) 1-19.

[26] Poincaré H., *Mémoire sur les groupes Kleinéens*, Acta Math. **3** (1883) 49-92.

[27] Shafarevich I. *et al*, *Algebraic Surfaces* (in Russian), Trudi Mat. Inst. Steklov **75**, Nauka, Moskva 1965.

[28] Shimura G., *Introduction to the Arithmetic Theory of Automorphic Functions*, Iwanami Shoten Publ., Tokyo 1971.

[29] Zink Th., *Über die Spitzen einiger arithmetischer Untergruppen unitärer Gruppen*, Math. Nachr. 79 (1979) 315-320.

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