# Global Cyclotomic Picard CM-Jacobians And Their Periods

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Lewittes and Rauch determined in their beautiful paper [LR] the (normalized) period matrix of the Klein curve $X^3Y+Y^3Z+Z^3X=0$. Up to isomorphy, this plane projective complex curve is characterized as compact Riemann surface of genus 3 with the maximal possible number of $168 = 84(g-1)$ automorphisms. In analogy, we want to determine the (normalized) period matrix of the special Picard curve with projective equation $Y^3Z = X^4 - XZ^3$. Moreover, up to isomorphy, we characterize this (complex) curve by the quality of the corresponding Picard moduli point: the only orbitally isolated surface singularity. Most interesting is the characterization of the curve by the endomorphism ring of its Jacobian threefold: the ring of 9-th unit roots. We extend our considerations also to the Picard curve $Y^3Z = X^4 - Z^4$ corresponding to the second singularity of the modular surface of Picard curves. Nowadays the Klein quartics over finite fields are used for the construction of error correcting codes, see [Pr]. This should be done also for special Picard curves.

Let $F$ be a number field and $A$ a complex abelian variety of dimension $g$. We say that $A$ has *$F$-multiplication*, if there is a $\mathbb{Q}$-algebra embedding $\iota$ of $F$ into the endomorphism algebra $End^\circ A = \mathbb{Q}\otimes End\, A$ of $A$. If, moreover, the (total) degree of $F$ is equal to $2g$ and $\iota$ is an isomorphism, then $A$ is called an abelian CM-*variety*. It is well-known in this case that $A$ is simple and $F$ is a CM-field, which is, by definition, a totally imaginary quadratic field extension of a totally real number field, see [La]. A CM-curve is a (smooth complex) projective curve $C$ whose Jacobian variety $J(C)$ is an abelian CM-variety.

A *Picard curve* is the projective closure of an affine plane curve of equation type $Y^3 = p_4(X)$, where $p_4(X)$ is a polynomial of degree 4. We exclude all polynomials $p_4(X)$ with only one zero. So one avoids unstable curves in order to get a compact algebraic moduli space $\hat{M}$ of (isomorphy classes of semistable) Picard curves, which we choose in a very canonical way (see below). Smooth Picard curves have genus 3. They correspond to a Zariski-open part $M^\#$ of $\hat{M}$. Their Jacobians are (principally polarized) abelian threefolds. Via period matrices they are represented by points in the generalized Siegel upper half plane

$$\mathbb{H}_3 = \{\Omega \in Mat_3(\mathbb{C});\ {}^t\Omega = \Omega\,,\ Im\,\Omega\, positive\, definite\},$$

uniquely up to $\mathbb{S}p(6,\mathbb{Z})$-equivalence, where

$$\mathbb{S}p(6,\mathbb{Z}) = \{G \in \mathbb{G}l_6(\mathbb{Z});\ {}^tG \cdot \left(\begin{smallmatrix} O & E_3 \\ -E_3 & O \end{smallmatrix}\right) \cdot G = \left(\begin{smallmatrix} O & E_3 \\ -E_3 & O \end{smallmatrix}\right)\},$$

$E_3 := diag(1,1,1)$, denotes the symplectic group acting on $\mathbb{H}_3$ in the well-known manner. By Torelli's theorem there is a canonical algebraic embedding $M^\# \hookrightarrow \mathfrak{A}_3$ into the moduli space $\mathfrak{A}_3 = \mathbb{H}_3/\mathbb{S}p(6,\mathbb{Z})$ of principally polarized abelian threefolds. Restricting to the Zariski-open subspace $\mathfrak{A}_3^\# \subset \mathfrak{A}_3$ corresponding to Jacobians of smooth genus 3 curves one gets a closed embedding $M^\# \hookrightarrow \mathfrak{A}_3^\#$, which determines $M^\#$ uniquely, up to isomorphy.

Obviously, our special Picard curves

$$C_{a0}\ : Y^3 = X^4 - aX\ ,\ C_{0b} :\ Y^3 = X^4 - b,\ a, b \in \mathbb{C}^*\,,$$

are isomorphic over $\mathbb{C}$ to

$$C_{10} :\ Y^3 = X^4 - X\ ,\ C_{01} :\ Y^3 = X^4 - 1,$$

respectively.

**Proposition 0.1** . *The Picard curves $C_{a0}$ are CM-curves with $\mathbb{Q}(\rho)$-multiplication, where $\rho$ is a primitive 9-th unit root. The endomorphism ring $End\,J(C_{a0})$ is isomorphic to $\mathbb{Z}[\rho]$. Up to isomorphy, $C_{10}$ is the only Picard CM-curve with a cyclotomic maximal order as endomorphism ring.*

**Proposition 0.2** . *The moduli points of $C_{10}$ and of $C_{01}$ are the only surface singularities on $M^\#$ and also on $\hat{M}$.*

**Proposition 0.3** . *Let $\omega = \rho^3$ be the primitive third unit root $e^{2\pi i/3}$. A period matrix of the Jacobian $J(C_{10})$ is:*

$$\Pi = \begin{pmatrix} -\rho+1 & 0 & -2\rho^2-2\rho & -\rho^2-1 & 1 & 2\rho^2+\rho \\ \rho^2-1 & 0 & -\rho^2+2\rho & -\rho^2+\rho+1 & -1 & \rho^2-2\rho \\ -\rho+1 & 0 & -2\rho^2-2\rho & -\rho^2-1 & 1 & 2\rho^2+\rho \end{pmatrix} \cdot \omega$$

$$+ \begin{pmatrix} 2\rho^2+\rho+1 & 1 & -\rho+1 & -2\rho^2-\rho & 0 & \rho^2+\rho-1 \\ -\rho^2+2\rho & 1 & -2\rho^2+2\rho+1 & -\rho+1 & -1 & \rho^2-\rho-1 \\ 2\rho^2+\rho+1 & 1 & -\rho+1 & -2\rho^2-\rho & 0 & \rho^2+\rho-1 \end{pmatrix}.$$

The closed algebraic embedding $M^\# \hookrightarrow \mathfrak{A}_3^\#$ can be *uniformized* in the following sense. In the analytic category there is a commutative *Shimura diagram*

$$(1)\qquad \begin{array}{ccccccc} \mathbb{B} & & & \hookrightarrow & & & \mathbb{H}_3 \\ & \nwarrow & & & & \nearrow & \\ & & \mathbb{B}^\# & \hookrightarrow & \mathbb{H}_3^\# & & \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ & & M^\# & \hookrightarrow & \mathfrak{A}_3^\# & & \\ & \swarrow & & & & \searrow & \\ M & & & \hookrightarrow & & & \mathfrak{A}_3 \end{array}$$

where $\mathbb{B}$ is the two-dimensional complex unit ball

$$(2) \qquad \mathbb{B} = \{z = (z_1, z_2) \in \mathbb{C}^2;\ |z|^2 := |z_1|^2 + |z_2|^2 < 1\},$$

$\mathbb{H}_3 \longrightarrow \mathfrak{A}_3$ the $\mathbb{S}p(6,\mathbb{Z})$-quotient morphism, $\mathbb{H}_3^{\#}$ the preimage of $\mathfrak{A}_3^{\#}$ in $\mathbb{H}_3$, $\mathbb{B} \hookrightarrow \mathbb{H}_3$ is a closed embedding, $\mathbb{B}^{\#} = \mathbb{B} \cap \mathbb{H}_3^{\#}$ and $\mathbb{B} \longrightarrow M$ is the analytic quotient morphism of the arithmetic group

$$(3) \qquad \Gamma = N_{\mathbb{S}p(6,\mathbb{Z})}(\mathbb{B}) := \{G \in \mathbb{S}p(6,\mathbb{Z});\ G(\mathbb{B}) = \mathbb{B}\}$$

acting on $\mathbb{B}$.

The other morphisms in Diagram 1 are open dense embeddings or restrictions, which should be clear. For proofs and more explicit details, which are needed below, we refer to [Ho3].

Identifying for a moment the ball with its image in $\mathbb{H}_3$ we call $\mathbb{B}$ the *period space of Picard curves* and its points are called *Picard period points* (of the family of Picard curves). An element $\gamma \in \Gamma$ is called *elliptic*, iff $\gamma$ has an isolated fixed point $P \in \mathbb{B}$.

Let $\Gamma'$ be a subgroup of $\Gamma$. We call the elliptic element $\gamma$ *purely* $\Gamma'$*-elliptic*, iff all non-trivially on $\mathbb{B}$ acting elements of the stationary group $\Gamma'_P$ are elliptic. The images of purely $\Gamma'$-elliptic points on $\mathbb{B}/\Gamma'$ are isolated (cyclic quotient) singularities. Notice that the fixed point $P$ is uniquely determined by the elliptic element $\gamma$ because the group of biholomorphic automorphisms of $\mathbb{B}$ coincides with $\mathbb{PU}((2,1),\mathbb{C})$, so $\gamma$ has only one negative eigenline in $V = (\mathbb{C}^3, <.,.>)$ with respect to the hermitian metric $<.,.>$ of signature $(2,1)$ on $\mathbb{C}^3$, namely

$$\mathbb{B} = \mathbb{P}V_-\ ,\ V_- = \{\mathfrak{a} \in V;\ <\mathfrak{a},\mathfrak{a}> < 0\}.$$

**Proposition 0.4** . *The set of Picard period points of $C_{10}$ coincides with the set of purely $\Gamma$-elliptic points on $\mathbb{B}$. It coincides with the $\Gamma$-orbit of*

$$P_\rho := (\rho^4 - \rho^2 \ :\ 1\ :\ \rho^5 + \rho^4 - 1) \in \mathbb{B}.$$

**Proposition 0.5** . *The set of $\mathbb{H}_3$-(Siegel-)period points of (Jacobians of) the curves $C_{a0}$ coincides with the $\mathbb{S}p(6,\mathbb{Z})$-orbit of*

$$\begin{pmatrix} \frac{-2rs-1}{3r^2} & \frac{1}{r} & \frac{rs-1}{3r^2} \\ \frac{1}{r} & -1 & 0 \\ \frac{rs-1}{3r^2} & 0 & \frac{-2rs+2}{3r^2} \end{pmatrix} \cdot \omega + \begin{pmatrix} \frac{2rs-2}{3r^2} & \frac{1}{r} & \frac{-rs+1}{3r^2} \\ \frac{1}{r} & -1 & \frac{-1}{r} \\ \frac{-rs+1}{3r^2} & \frac{-1}{r} & \frac{2rs+1}{3r^2} \end{pmatrix}$$

*with*

$$r := -\rho^4 + \rho^3 + 2\rho^2 + \rho + 1\ ,\ s := -(\rho^5 + \rho^3 + 2\rho^2 + \rho)$$

*Proof* of Proposition 0.2. Let $K = \mathbb{Q}(\sqrt{-3}) = \mathbb{Q}(\omega)$ be the field of Eisenstein numbers. The cyclic group $Z_3$ of order 3 acts via $(x,y) \mapsto (x,\omega y)$ on each Picard curve $C$. If $C$ is smooth, we get $\mathbb{P}^1$ as quotient curve $C/Z_3$ with $Z_3$ as

Galois group of $C\,/\,\mathbb{P}^1$. The action of $Z_3$ induces a $K$-multiplication on $J(C)$ of type $(2,1)$, which means that the diagonalized representation group of $Z_3$ on the tangent space $T_0J(C)$ of $J(C)$ is generated by $\left(\begin{smallmatrix}\omega & 0 & 0\\ 0 & \omega & 0\\ 0 & 0 & \bar{\omega}\end{smallmatrix}\right)$. The moduli space of abelian threefolds with $K$-multiplication of type $(2,1)$ is the Shimura surface $\mathbb{B}/\Gamma$, $\Gamma = \mathbb{U}((2,1),\mathfrak{O})$, $\mathfrak{O} = \mathfrak{O}_K = \mathbb{Z}+\mathbb{Z}\omega$ the ring of Eisenstein integers. In [Ho3] we proved that this ball lattice $\Gamma$ coincides with $\Gamma$ defined in (3). We define the congruence subgroup $\Gamma(\sqrt{-3})$ by the exact group sequence

$$1 \longrightarrow \Gamma(\sqrt{-3}) \longrightarrow \Gamma \longrightarrow \mathbb{U}((2,1),\mathfrak{O}/(1-\omega)\mathfrak{O} \longrightarrow 1$$

and know that $\mathbb{U}((2,1),\mathbb{F}_3\,) \cong S_4$, the symmetric group of four letters, see [Ho1]. In the same monograph, see ch. I, Prop. 3.2.3, we proved the following central

**Theorem 0.6** . *The Baily-Borel compactification $\widehat{\mathbb{B}/\Gamma(\sqrt{-3})}$ coincides with the projective plane $\mathbb{P}^2$. The compactifying cusp points are four points $K_1, K_2, K_3, K_4 \in \mathbb{P}^2$ in general position. The open part $\mathbb{P}_2^{\#} \subset \mathbb{P}^2$ coming from smooth Picard curves is precisely the complement of the six projective lines $L_{ij} = L_{ji}$ going through pairs $K_i, K_j$ of different cusp points.*

It turns out that

$$M^{\#} = \mathbb{P}_2^{\#}/S_4\ ,\ \hat{M} = \mathbb{P}^2/S_4\ ,\ M = \mathbb{P}_2^{*}/S_4,$$

where $\mathbb{P}_2^{*} := \mathbb{P}^2 \setminus \{K_1, K_2, K_3, K_4\}$. Now identify $\mathbb{P}^2$ with

$$\mathbb{P}_0^3 = \{(t_1 : t_2 : t_3 : t_4) \in \mathbb{P}^3;\ t_1+t_2+t_3+t_4 = 0\},$$

and introduce projective coordinates such that

$$\begin{aligned}&K_1 = (-3:1:1:1),\ K_2 = (1:-3:1:1),\\ &K_3 = (1:1:-3:1),\ K_4 = (1:1:1:-3).\end{aligned}$$

**Theorem 0.7** *(see [Ho1] I, Prop. 3.4.4). The only singularities of $\hat{M}$ are the image points of $S := (0:1:\omega:\omega^2)$ and $N := (1:i:-1:-i)$, along the $S_4$-quotient morphism.*

This is a simple application of a theorem of Chevalley stating that the singularities of a finite (more generally: locally finite) Galois quotient $X/H$ of a smooth complex manifold X come precisely from points $x \in X$ with isotropy group $G_x$ not generated by reflections at $x$, where reflections at $x$ are defined as elements of $G_x$ acting trivially on a submanifold of $X$ through $x$ of codimension 1. Looking at finite subgroups of $S_4$ and their fixed points on $\mathbb{P}^2$ one finds up to $S_4$-equivalence the points $S$, $N$ as only singular possibilities. The $S_4$-isotropy group of $S$ is generated by the cyclic permutation (234) of order 3. The $S_4$-isotropy group of $N$ is generated by the cyclic permutation (1234) of order 4. The (13)(24)-reflection line on $\mathbb{P}^2$ contains N.

Now we need the precise correspondence of Picard curves with their moduli space $\hat{M} = \mathbb{P}^2/S_4$. Each of them is isomorphic to a *normal form* representative

$$C_{\mathrm{t}}:\ Y^3 = (X-t_1)(X-t_2)(X-t_3)(X-t_4),\ t_1+t_2+t_3+t_4 = 0.$$

The correspondence

$$C_{\mathfrak{t}} \mapsto \mathfrak{t} = (t_1, t_2, t_3, t_4) \mapsto (t_1 : t_2 : t_3 : t_4) \in \mathbb{P}_2^*$$

restricted to $\mathbb{P}_2^{\#}$ and composed with the $S_4$- quotient map yields the precise parametrisation of isomorphy classes, see [Ho1] I, Prop.5.2.3. Especially, the moduli point of

$$C_{10} : Y^3 = X(X-1)(X-\omega)(X-\omega^2)$$

is the image of S. Proposition 2 is proved.

$\square$

*Proof* of Proposition 0.4. For an arbitrary group $G$ we denote by $G_{tor}$ the set of elements of finite order of $G$ (torsion elements), and $G_{k-tor}$ denotes the subset of elements of precise order $k \in \mathbb{N}_+$. $G$ acts on $G_k$ and on $G_{tor}$ via conjugations.

**Lemma 0.8** . *For $\Gamma = \mathbb{U}((2,1), \mathfrak{O})$ the set $\Gamma_{9-tor}$ is not void. It consists of precisely six $\Gamma$-conjugation classes. They are projected onto two $\mathbb{P}\Gamma$-conjugation classes in $(\mathbb{P}\Gamma)_{3-tor}$.*

*Proof.* For the first statement we consider the element

$$\varphi_1 := \begin{pmatrix} -\omega^2 & -1 & \omega^2 \\ \omega & 1 & 1 \\ 1 & -1 & \omega^2 - 1 \end{pmatrix}$$

with

$$det\, \varphi_1 = \omega \ , \ \varphi_1^3 = \omega E_3.$$

found by Feustel in [Feu]. It is easy to check that $\varphi_1$ belongs to $\Gamma$. The eigenvalues are $\rho$, $\rho^4$, $\rho^7$. The powers $\varphi_1^k$, $k = 1, 2, 4, 5, 7, 8$, yield six different conjugation classes in $\Gamma_{9-tor}$ (compare determinants and eigenvalues) and two conjugation classes in $(\mathbb{P}\Gamma)_{3-tor}$.

Now let $\varphi$ be an arbitrary element of $\Gamma_{9-tor}$ with eigenvalues $\rho$, $\rho^j$, $\rho^k$, say. The Galois group of $F := K(\rho)$ over $K$ is generated by $\sigma : \rho \mapsto \rho^4$. The characterisic polynomial $\chi_\varphi(T)$ of $\varphi$ belongs to $K[T]$. Looking at trace and determinant of $\varphi$, which must belong to $K$, it is easy to see, that $\varphi$ has three different eigenvalues. They must be conjugated over $K$, hence $\rho^j = \rho^4 = \sigma(\rho)$, $\rho^k = \rho^7 = \sigma^2(\rho)$. The eigenvectors $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ of $\rho$, $\sigma(\rho)$, $\sigma^2(\rho)$, respectively, can be choosen in $F^3$. They form an orthogonal basis of $F^3$ endowed with our hermitian $(2,1)$-metric because of different eigenvalues. From $\varphi(\mathfrak{a}) = \rho \cdot \mathfrak{a}$ it follows that

$$\sigma(\varphi(\mathfrak{a})) = \sigma(\rho)\sigma(\mathfrak{a}) = \rho^4 \sigma(\mathfrak{a})$$

because $\varphi$ belongs to $Mat_3(K)$. Therefore

(4) $$\mathfrak{a},\ \mathfrak{b}=\sigma(\mathfrak{a}),\ \mathfrak{c}=\sigma^2(\mathfrak{a})\in F^3,$$

satisfying

$$<\mathfrak{a},\mathfrak{a}><0,\ <\mathfrak{b},\mathfrak{b}>>0,\ <\mathfrak{c},\mathfrak{c}>>0,$$

(without loss of generality) is an orthogonal $\varphi$-eigenbasis of $\mathbb{C}^3$ . The elliptic element $\varphi$ has the unique elliptic fixed point $P=\mathbb{P}\mathfrak{a}\in\mathbb{B}$. We show that $P$ is a purely $\Gamma$-elliptic point. With $\Gamma':=\Gamma(\sqrt{-3})$ we have a commutative diagram of quotient morphisms

$$\begin{tikzcd}
\mathbb{B} \arrow[d, "p'"] \arrow[dr, "p"] & \\
\mathbb{B}/\Gamma'=\mathbb{P}_2^* \arrow[r, "\pi"'] & \mathbb{P}_2^*/S_4=\mathbb{B}/\Gamma
\end{tikzcd}$$

In [Ho1] I, Prop. 3.4.4, we listed on $\mathbb{P}_2^*$ the $p'$- images of all $\Gamma$-elliptic points $Q\in\mathbb{B}$ together with their (abstract) isotropy groups $\Gamma_Q$. Our $P$ cannot be an intersection point of two $\Gamma$-reflection discs because the reflections have eigenvalues only in $K$. Otherwise $P\in\mathbb{B}\subset\mathbb{P}^2$ would be the intersection point of two projective lines (the projectivized orthogonal complements of the one-dimensional eigenspaces) defined over $K$. This leads to $\mathbb{P}\mathfrak{a}=P=\mathbb{P}\mathfrak{a}'$, $\mathfrak{a}'\in K^3$, $\sigma(P)=P$, which contradicts to $\sigma(P)\notin\mathbb{B}=\mathbb{P}V_-$, see (4). There are precisely two $\Gamma$-orbits $\Gamma\tilde{N}$, $\Gamma\tilde{S}$ of $\Gamma$-elliptic points whose isotropy groups are not generated by reflections. The projective isotropy groups $\mathbb{P}\Gamma_{\tilde{N}}$ or $\mathbb{P}\Gamma_{\tilde{S}}$ are cyclic of order 4 or 3, respectively. Since $\mathbb{P}\varphi\in\mathbb{P}\Gamma_P$ is elliptic of order 3 the point $P$ must belong to the second orbit. The image $p(\tilde{S})$ coincides with $p'(S)$, which is an orbitally isolated singularity with respect to $\Gamma$. This means that $\tilde{S}$ is a purely $\Gamma$-elliptic point, hence $\mathbb{P}\Gamma_{\tilde{S}}\cong<\mathbb{P}\varphi>$ of order 3.

$\square$

*Proof* of Proposition 0.1. Our special Picard curve $C_{10}:\ Y^3=X(X^3-1)$ has an obvious non-trivial automorphism of 9-th order fixing $\infty=(0:1:0)$:

$$(x,y)\mapsto(\omega x,\ \rho y)\ ,\ (\rho^3=\omega).$$

It extends to an isomorphism of the Jacobian threefold of $C_{10}$. With Theorem 0.9 below we will see that this isomorphism generates a subfield in the endomorphism algebra of the Jacobian. Therefore we get embeddings

(5) $$\mathbb{Z}[\rho]\hookrightarrow End\,J(C_{10})\ ,\ F=\mathbb{Q}(\rho)\hookrightarrow End^\circ J(C_{10}).$$

The representing period point $P_\rho=\mathbb{P}\mathfrak{a}\in\mathbb{B}$ is purely $\Gamma$-elliptic by Proposition 1.4, fixed by $\varphi_1$ of nine-th order. Therefore the ring $End_K(\mathfrak{a},\mathfrak{a}^\perp)$ of

$K$-endomorphisms of $V$ with eigenvector $\mathfrak{a}$ and invariant subspace $\mathfrak{a}^\perp$ is bigger than $K$. Such ball points have been called *exceptional* in [Ho2], Corollary 7.10. Moreover, $\mathfrak{a}$ is eigenvector of a simple eigenvalue of $\varphi_1 \in End_K(\mathfrak{a}, \mathfrak{a}^\perp)$. Therefore $P_\rho$ is an *isolated exceptional* point in the sense of Definition 7.12 of [Ho2]. The $K$-degree $[K(P_\rho) : K]$ of $P_\rho$ is equal to 3. Now apply the following theorem to see that $J(C_{10})$ is a simple CM-threefold with multiplication field $K(\rho)$.

**Theorem 0.9** *(see [Ho2], Theorem 7.20 and the lemmas around). The endomorphism algebra of the Jacobian variety $J_\tau \cong J(C_t)$ of a Picard curve with period point $\tau \in \mathbb{B}$ and moduli point $t = (t_1 : t_2 : t_3 : t_4) \in \mathbb{P}_2^*$ is greater than $K$ if and only if $\tau$ is exceptional. $J_\tau$ splits up to isogeny into abelian CM- subvarieties if and only if $\tau$ is an isolated exceptional point. Thereby Jacobians with CM-field $F$ (of degree 3 over $K$) correspond to isolated exceptional points of $K$-degree* 3 *and $F \cong K(\tau)$. All other isolated exceptional points (of $K$-degree* 2 *or* 1*) ly on $K$-discs on $\mathbb{B}$ (defined as non-empty intersections $L \cap \mathbb{B}$, $L$ projective lines on $\mathbb{P}^2$ defined over $K$). Thereby $\tau \in \mathbb{B}(K)$ if and only if $J_\tau$ splits into $E \times E \times E$. The degree 2 case happens if and only if $J_\tau$ splits into $E \times (E^{'2})$, where $E$ is an elliptic CM-curve with $K$-multiplication and $E'$ elliptic CM with another imaginary quadratic multiplication field.*

□

The endomorphism ring of any abelian CM-variety is an order in the corresponding CM-field. Each order of a number field $L$ is contained in the maximal order, the ring $\mathfrak{O}_L$ of integers in $L$. The maximal order of a cyclotomic field $L = \mathbb{Q}(\zeta)$ is equal to $\mathbb{Z}[\zeta]$, $\zeta$ a generating unit root, see e.g. [Neu], I, Prop. 10.2. So the embeddings (5) must be isomorphisms, especially

$$\mathfrak{O}_F = \mathbb{Z}[\rho] \cong End J(C_{10}) \subseteq End^\circ J(C_{10}) \cong F. \tag{6}$$

The first two parts of Proposition 0.1 are proved because of $C_{a0} \cong C_{10}$. $F$ is the only cyclotomic field of degree 3 over $K$. Therefore the Jacobian threefolds of CM-Picard curves $C$ with cyclotomic endomorphism algebra $End^\circ J(C)$, which must be isomorphic to $F$, have to be isogeneous. There is a bijective correspondence between the ideal classes of $\mathfrak{O}_F$ and the isomorphy classes of principally polarized abelian CM-threefolds A (of same multiplication type) with endomorphism rings $\mathfrak{O}_F$, see e.g. [La], III.2, Cor. 2.7. It is well-known that the class number of $F$ is equal to 1, see e.g. [Has], III, end of 29. Therefore, up to isomorphy, there is only one such $A$. Then, by Torelli's theorem, also the isomorphy class of Picard CM-curves with $End\, J(C) \cong \mathfrak{O}_F$ is uniquely determined. This completes the proof of Proposition 1.

□

**Remark 0.10** . *The* type *of $F$-multiplication is a lift ($F$-extension) from the type* $(2, 1)$ *of $K$-multiplication on $J(C_{10})$. This lifted type is unique by [La], I.3, Theorem 3.6.*

*Proof* of Propositions 0.3 and 0.5. In [Ho3], sections 2.4-2.5, we described the procedure to receive the period matrices starting from the coordinates of the fixed point $P_\rho$. First one has to move the "diagonal ball" (2) $\mathbb{B} \subset \mathbb{P}^2$ by a plane projective linear transformation to the "Picard ball" (Siegel domain)

$\mathbb{B}' \subset \mathbb{P}^2$. This is done by the inverse of

$$M := \begin{pmatrix} \omega & 0 & -1 \\ 0 & 1 & 0 \\ -\omega^2 & 0 & -1 \end{pmatrix},$$

(see [Ho3], p. 28) acting on row-vectors from the right. Let $P' =: (a : b : c) \in \mathbb{B}'$ be the image point of $P_\rho \in \mathbb{B}$. Setting $b = 1$ one gets $a, c \in \mathbb{Z}[\rho]$, explicitly written down in Proposition 0.5. From the vector $(a, 1, c)$ one gets the period matrices of Propositions 0.3 and 0.5 via orthogonal fillings and $*$-procedure coming from Picard period integrals, all described in [Ho3] around Lemma 2.22, see also the example below. The numbers $r$, $s$ appear in the period matrix $\Pi$ (see 0.3) at places $(1, 1)$ or $(1, 4)$, respectively.

□

The period matrices in the Propositions 0.3, 0.5 with coefficients in the ring of 9-th unit roots have been calculated by means of MAPLE. For a better understanding of the procedure we calculate the period matrices of the (generalized Jacobian of the) singular Picard curve

$$C_0 : Y^3 = (X^2 - 1)^2 = (X - 1)(X + 1)(X - 1)(X + 1)$$

of CM-decomposition type

$$J(C_0) \sim E \times E \times E$$

Namely, the moduli point $Q$ of this curve on $\mathbb{P}^2 = \mathbb{P}^3_0$ is equal to $Q = (1 : -1 : 1 : -1)$, up to permutation. It comes from the period point $O = (0 : 0 : 1) \in \mathbb{B} \subset \mathbb{P}^2$ via the quotient map $\mathbb{B} \longrightarrow \mathbb{B}/\Gamma(\sqrt{-3})$. This has been proved in [Ho1], I, Proposition 3.4.4. This period point has obviously coordinates in K. Therefore, by Theorem 0.9, we get the above splitting.

We found the period vector $\mathfrak{a} = (0, 0, 1) \in V_-$ of $C_0$. Now one has to transform the $\mathbb{B}$-period point $\mathbb{P}\mathfrak{a}$ to the Picard period point $\mathbb{P}\mathfrak{a}' \in \mathbb{B}' = \mathbb{P}V'_-$. This is generally done by sending $\mathfrak{a}$ to

$$\mathfrak{a}' := \mathfrak{a} \cdot M^{-1}, \; M^{-1} = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ \omega^2 & 0 & \omega \end{pmatrix}$$

This transformation is an isometry from $V$ to $V'$, the latter space endowed with the hermitian form represented by

$$H := M \cdot diag(1, 1, -1) \cdot {}^t\bar{M} = \begin{pmatrix} 0 & 0 & \omega \\ 0 & 1 & 0 \\ \omega^2 & 0 & 0 \end{pmatrix}.$$

The first row of the period matrix we look for is the vector

$$*\mathfrak{a}' := (\mathfrak{a}', \mathfrak{a}' \cdot diag(-\omega, \omega, \omega^2))(3, 4) \in \mathbb{C}^6,$$

where $(3, 4)$ indicates the transposition of the 3-rd and 4-th coefficient.

For finding the other vectors one takes any basis $\mathfrak{b}$, $\mathfrak{c}$ of the orthogonal complement of $\mathfrak{a}$ in $V$, their images $\mathfrak{b}'$, $\mathfrak{c}'$ in $V$, and finally the conjugates (componentwise) of $*\mathfrak{b}'$, $*\mathfrak{c}'$. Then the $3 \times 6$ matrix

$$\Pi = \begin{pmatrix} *\mathfrak{a}' \\ \overline{*\mathfrak{b}'} \\ \overline{*\mathfrak{c}'} \end{pmatrix}$$

is the period matrix of the period point $\mathbb{P}\mathfrak{a} \in \mathbb{B}$ we started with. It is really a period matrix of the Picard curve corresponding to the moduli point $\pi(\mathbb{P}\mathfrak{a})$, see [BHP], II.

In our special case we take

$$\mathfrak{a} = (0,0,1),\ \mathfrak{b} = (0,1,0),\ \mathfrak{c} = (1,0,0)$$

and get

$$\begin{aligned} \mathfrak{a}' &= (\omega^2, 0, \omega), \\ \mathfrak{b}' &= (0,1,0),\ \mathfrak{c}' = (-1,0,1), \\ *\mathfrak{a}' &= (\omega^2, 0, \omega; -1, 0, 1)(3,4), \\ *\mathfrak{b}' &= (0,1,0;0,\omega,0)(3,4), \\ *\mathfrak{c}' &= (-1,0,1;\omega,0,\omega^2)(3,4), \end{aligned}$$

and finally the period matrix

$$\Pi(C_{01}) = \begin{pmatrix} \omega^2 & 0 & -1 & \omega & 0 & 1 \\ 0 & 1 & 0 & 0 & \omega^2 & 0 \\ -1 & 0 & \omega^2 & 1 & 0 & \omega \end{pmatrix}.$$

Multiplying the second $3 \times 3$-submatrix of $\Pi$ from left with the inverse of the first one gets the corresponding period point in the Siegel domain $\mathbb{H}_3$.

**Proposition 0.11** . *The set of period points of the isomorphic cyclotomic DCM-Picard curves $C_0 : Y^3 = (X^2-1)^2$ coincides with the $\mathbb{S}p(6,\mathbb{Z})$-orbit of*

$$\frac{1}{1-\omega}\begin{pmatrix} -2 & 0 & 1 \\ 0 & \omega^2 - 1 & 0 \\ 1 & 0 & -2 \end{pmatrix}.$$

$\square$

For the sake of completion we give a hint for finding the period matrices of $C_{01} : Y^3 = X^4 - 1$ in the same manner. The moduli point is $M = (1 : i : -1 : i)$. The corresponding period points on $\mathbb{B}$ are $\Gamma(\sqrt{-3}$-elliptic points on $\mathbb{B}$, because the permutation $(1234) \in S_4 \cong \Gamma/\Gamma(\sqrt{-3})$ fixes $M$. In [Ho1], I, Proposition 3 we proved that the $\Gamma(\sqrt{-3})$-orbit of $\tilde{M}$ can be represented by points $\tilde{M}$ on the diagonal disc $\mathbb{D} = \{(z,z) \in \mathbb{C}^2;\, 2|z|^2 < 1\} \cap \mathbb{B}$, which is a $K$-disc. Moreover, the projective isotropy group $\mathbb{P}\Gamma(\sqrt{-3})_{\tilde{M}}$ is cyclic of order 4. Therefore the coordinates of $\tilde{M} = \mathbb{P}\mathfrak{a}$ belong to $K(i) \setminus K$. Each elliptic point is obviously an isolated exceptional point. Now Theorem 0.9 shows that the Jacobians of $J(C_{0b}) \cong J(C_{01})$ have splitting type $E \times E' \times E'$, $E'$ a CM-elliptic curve not isogeneous to $E$. But it is easy to see that the elliptic curve $E' : T^2 = Y^3 + 1$ is an isogeny factor of $C_{01}$ because of the double covering $C_{01} \longrightarrow E'$, $(x, y) \mapsto (x^2, y)$. We proved the first part of

**Proposition 0.12** . *The Jacobians of the Picard curves*

$$C_{0b} : Y^3 = X^4 - b,\ b \neq 0,$$

*split into* $E \times E' \times E'$, *where* $E$, $E'$ *are elliptic CM-curves with fields* $K = \mathbb{Q}(\sqrt{-3})$ *or* $\mathbb{Q}(i)$, *respectively. A representing period point is* $(\sqrt{-3} - i\,,\, \sqrt{-3} - i) \in \mathbb{B} \subset \mathbb{C}^2$.

The second statement has been proved by Feustel [Feu].

$\square$

The calculation of the corresponding period matrices is left to the reader.

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