CHAPTER VI

Main Comparison

1. Notations and conventions

1.1. Let F be a number field. The results of this chapter also hold –we believe– for the function field of a smooth projective absolutely irreducible curve over a finite field, once the invariant trace formula–established for number fields in Chapters III-V–is established for such a function field. Let G be a connected reductive linear algebraic group over F. Let M_0 be a fixed minimal Levi subgroup of G over F. Let $\mathcal{L}$ denote the set of Levi subgroups of G over F that contain M_0. It is a finite set. Usually we will let M denote an element of $\mathcal{L}$. Let $\mathcal{L}^M$ be the set of Levi subgroups over F contained in M (so that $\mathcal{L} = \mathcal{L}^G$), $\mathcal{L}(M)$ the set of Levi subgroups of G over F that contain M (so that $\mathcal{L} = \mathcal{L}(M_0)$), $\mathcal{P}(M)$ (resp. $\mathcal{F}(M)$) the set of parabolic subgroups of G over F with Levi component M (resp. containing) (resp., or in $\mathcal{L}(M)$) and A_M the maximal split torus in the center of M.

1.2. Write F_v for the completion of F at the place v. For for each valuation v of F, we fix a maximal compact subgroup K_v of the group $G(F_v)$ of F_v-valued points of G. We ~~can~~ choose K_v to be hyperspecial for all v where G is unramified, namely quasi-split over F and split over an unramified extension of F_v, thus for almost all v. See [**Ti79**] for the definition of hyperspecial. For each place v of F we write $\mathbb{H}(G(F_v))$ for the algebra of K_v-finite functions in the convolution algebra $C_c^\infty(G(F_v))$ of compactly supported smooth functions on the group $G(F_v)$. It will be referred to as the *Hecke algebra* of $G(F_v)$. A choice of Haar measure dx_v on $G(F_v)$ is implicit. If S is a finite set of places of F, put $F_S = \prod_v F_v$ and $G(F_S) = \prod_v G(F_v)$.

We let $\mathbb{A}$ denote the adèle ring of F. Put $\mathbb{H}(G(F_S)) = \bigotimes_{v \in S} \mathbb{H}(G(F_v))$, the Hecke algebra of compactly supported smooth K_S-finite functions on $G(F_S)$, where $K_S = \prod_{v \in S} K_v$. Multiplying $f \in \mathbb{H}(G(F_S))$ by the characteristic function 1_{K^S} of $K^S = \prod_{v \notin S} K_v$ we obtain a C_c^∞-function on $G(\mathbb{A})$, the group of $\mathbb{A}$-valued points of G. Put $\mathbb{H}(G(\mathbb{A}))$ for the union of all $\mathbb{H}(G(F_S)) \otimes 1_{K^S}$ over all S such that K_v is hyperspecial for all $v \notin S$.

We shall relate objects associated with ~~the~~ an inner form G of G' to analagous objects on $G' = \mathrm{GL}(n)$. Let us recall the definition.

1.3. Definition. The group G (as in 1.1) is an *inner form* of G' ~~$= \mathrm{GL}(n)$~~ over F if there is an isomorphism $\eta : G \to G'$ over an algebraic extension of F such that, for every $\theta \in \mathrm{Gal}(\overline{F}/F)$, the composition $\eta_\theta = \eta^{-1} \circ \theta \circ \eta$ equals conjugation, $\mathrm{Int}(a_\theta)$, by an element a_θ in G. If $G' = \mathrm{GL}(n)$ then the group G is ~~then~~ the multiplicative group of a central simple algebra over F.

We can choose η such that $\eta(M_0)$ contains the standard (diagonal) minimal Levi subgroup of G'. Furthermore, we may assume that the restriction of η to A_{M_0} is defined over F.

1.4. Our aim is to compare the trace formula of G with the trace formula of G', where $G' = \mathrm{GL}(n)$, for matching test functions $f \in \mathbb{H}(G(\mathbb{A}))$ and $f' \in \mathbb{H}(G'(\mathbb{A}))$. It suffices to consider $f = \otimes_v f_v$ and $f' = \otimes_v f'_v$. We take $f_v = f'_v$ under the isomorphism $G(F_v) \simeq G'(F_v)$ for all places $v \notin S_{\mathrm{ram}} = \{$places where G is not split$\}$. Denote the local correspondence of conjugacy classes by $\gamma \mapsto \gamma'$ (thus if γ, γ' are semisimple, their characteristic polynomials are equal). More precisely, if $\{\gamma\}$ is a conjugacy class, or a $G(F)$-orbit in $G(F)$, then $\{\eta(\gamma)\}$ is a $G'(F)$-conjugacy class. It is this class that is represented